SAMPLE 8.33 *Flying dish and the solar panel.* A uniform rectangular plate of dimensions $2a = 2\,\text{m}$ and $2b = 1\,\text{m}$ and mass $m_P = 2\,\text{kg}$ drifts in space at a uniform speed $v_p = 10\,\text{m/s}$ (in a local Newtonian reference frame) in the direction shown in the figure. Another circular disk of radius $R = 0.25\,\text{m}$ and mass $m_D = 1\,\text{kg}$ is heading towards the plate at a linear speed $v_D = 1\,\text{m/s}$ directed normal to the facing edge of the plate. In addition, the disk is spinning at $\omega_D = 5\,\text{rad/s}$ in the clockwise direction. The plate and the disk collide at point A of the plate, located at $d = 0.8\,\text{m}$ from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

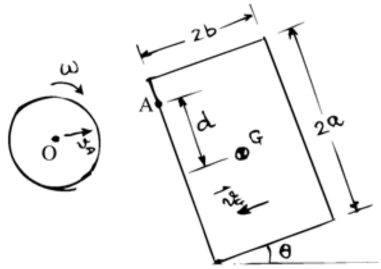

Figure 8.87: (Filename:sfig9.5.diskplate)

Solution To find the linear as well as the angular velocities of the disk and the plate, we will have to use linear and angular momentum-impulse relations. In total, we have 7 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components), 2 for the two angular velocities, and 1 for the collision impulse. Naturally, we need 7 independent equations. We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution e (=1 for elastic collision). Thus we have enough equations. Let us set up all the required equations. We can then solve the equations using a computer.

The free body diagrams of the disk and the plate together and the two separately are shown in Fig. 8.88 and 8.89, respectively. Using an xy coordinate system oriented as shown in Fig. 8.88, we can write

$$\begin{aligned}
\text{LMB for disk:} &\quad m_D(\vec{\boldsymbol{v}}_D^+ - \vec{\boldsymbol{v}}_D^-) &=& \; -P\hat{\boldsymbol{\imath}} \\
\text{LMB for plate:} &\quad m_P(\vec{\boldsymbol{v}}_P^+ - \vec{\boldsymbol{v}}_P^-) &=& \; P\hat{\boldsymbol{\imath}} \\
\text{AMB for disk:} &\quad I_D^{\text{cm}}(\vec{\boldsymbol{\omega}}_D^+ - \vec{\boldsymbol{\omega}}_D^-) &=& \; \vec{\mathbf{0}} \\
\text{AMB for plate:} &\quad I_P^{\text{cm}}(\vec{\boldsymbol{\omega}}_P^+ - \vec{\boldsymbol{\omega}}_P^-) &=& \; \vec{\boldsymbol{r}}_{A/G} \times P\hat{\boldsymbol{\imath}} \\
\text{kinematics:} &\quad \hat{\boldsymbol{\imath}} \cdot \{\vec{\boldsymbol{v}}_{A_D}^+ - \vec{\boldsymbol{v}}_{A_P}^+ &=& \; e(\vec{\boldsymbol{v}}_{A_P}^- - \vec{\boldsymbol{v}}_{A_D}^-)\}
\end{aligned}$$

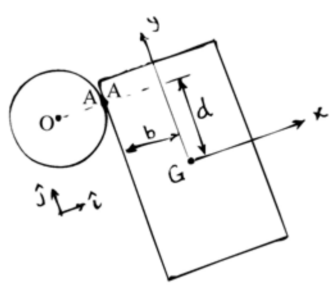

Figure 8.88: The free body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free body diagram . (Filename:sfig9.5.diskplate.a)

where, in the last equation $\vec{\boldsymbol{v}}_{A_D}$ and $\vec{\boldsymbol{v}}_{A_P}$ refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center of mass of the corresponding bodies. We are given that $\vec{\boldsymbol{v}}_D^- = v_D\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}}_P^- = -v_P\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{\omega}}_D^- = -\omega_D\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{\omega}}_P^- = \vec{\mathbf{0}}$. Let us assume that $\vec{\boldsymbol{\omega}}_D^+ = \omega_D\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_P^+ = \omega_P\hat{\boldsymbol{k}}$, $\vec{\mathbf{v}}_D^+ = v_{D_x}^+\hat{\boldsymbol{\imath}} + v_{D_y}^+\hat{\boldsymbol{\jmath}}$, and similarly, $\vec{\mathbf{v}}_P^+ = v_{P_x}^+\hat{\boldsymbol{\imath}} + v_{P_y}^+\hat{\boldsymbol{\jmath}}$. Then,

$$\begin{aligned}
\vec{\boldsymbol{v}}_{A_D}^- &= \vec{\boldsymbol{v}}_D^- + \vec{\boldsymbol{\omega}}_D^- \times \vec{\boldsymbol{r}}_{A/O} = v_D\hat{\boldsymbol{\imath}} - \omega_D R\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_{A_D}^+ &= \vec{\boldsymbol{v}}_D^+ + \vec{\boldsymbol{\omega}}_D^+ \times \vec{\boldsymbol{r}}_{A/O} = v_{D_x}^+\hat{\boldsymbol{\imath}} + (v_{D_y}^+ + \omega_D^+ R)\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_{A_P}^- &= \vec{\boldsymbol{v}}_P^- = -v_P\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{v}}_{A_P}^+ &= \vec{\boldsymbol{v}}_P^+ + \vec{\boldsymbol{\omega}}_P^+ \times \vec{\boldsymbol{r}}_{A/G} = (v_{P_x}^+ - \omega_P^+ d)\hat{\boldsymbol{\imath}} + (v_{P_y}^+ - \omega_P^+ d)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

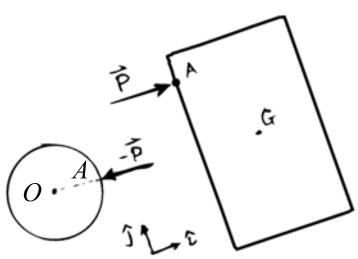

Figure 8.89: Separate free body diagrams of the disk and the plate during collision. (Filename:sfig9.5.diskplate.b)

Substituting these quantities in the kinematics equation above and dotting with the normal direction at A, $\hat{\boldsymbol{\imath}}$, we get

$$v_{D_x}^+ - v_{P_x}^+ + \omega_P^+ d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \tag{8.59}$$

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_D(v_{D_x}^+ - v_D) = -P \tag{8.60}$$

$$m_D v_{D_y}^+ = 0 \tag{8.61}$$

Dotting LMB for the plate with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_P(v_{P_x}^+ + v_P) = P \tag{8.62}$$

$$m_P v_{P_y}^+ = 0 \tag{8.63}$$

Dotting AMB for the disk and the plate with $\hat{k}$, we get

$$I_D^{cm}(\omega_D^+ + \omega_D) = 0 \tag{8.64}$$

$$I_P^{cm}\omega_P^+ = Pd \tag{8.65}$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (8.60)–(8.65), and then eqn. (8.59) as

$$\begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_D^{cm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_P^{cm} & -d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{bmatrix} \begin{Bmatrix} v_{D_x}^+ \\ v_{D_y}^+ \\ v_{P_x}^+ \\ v_{P_y}^+ \\ \omega_D^+ \\ \omega_P^+ \\ P \end{Bmatrix} = \begin{Bmatrix} m_D v_D \\ 0 \\ -m_P v_P \\ 0 \\ -I_D^{cm}\omega_D \\ 0 \\ -v_P - v_D \end{Bmatrix}$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia, $I_D^{cm} = (1/2)m_D R^2$ and $I_P^{cm} = (1/12)m_P(4a^2 + 4b^2)$, and then solving the matrix equation on a computer, we get,

$$\begin{aligned} \vec{v}_D^+ &= -12.88\,\text{m/s}\hat{\imath}, & \vec{v}_P^+ &= -3.06\ \text{m/s}\hat{\imath} \\ \vec{\omega}_D^+ &= -5\,\text{rad/s}\hat{k}, & \vec{\omega}_P^+ &= -1.48\ \text{rad/s}\hat{k} \\ P &= -\ 13.88\,\text{kg}\cdot\text{m/s} \end{aligned}$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

Comments: In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (8.61), (8.63), and (8.64) are trivial to solve and immediately give, $v_{D_y}^+ = 0$, $v_{P_y}^+ = 0$, and $\omega_D^+ = \omega_D$ = -5 rad/s. Rest of the equations can be solved by usual eliminations and substitutions, etc. However, it is important to learn how to set up these equations in matrix form so that no matter how complicated the equations are, they can be easily solved on a computer. What really counts is do you have 7 linear independent equations for the 7 unknowns. If you do, you are home.