Your TA, Section # and Section time:

Your name:

Cornell TAM/ENGRD 2030

Prelim 3

April 21, 2014

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

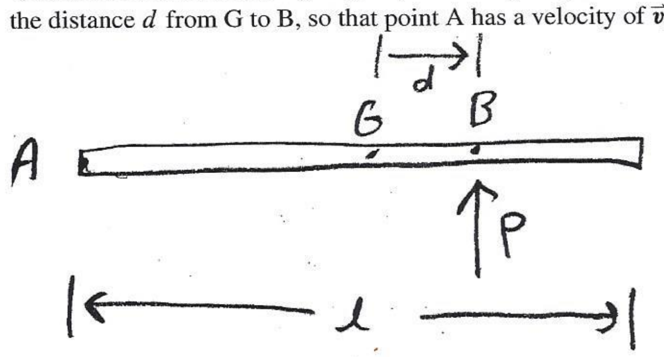
- : Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- $\vec{\bullet}$: Use correct **vector notation.**
- A+ : Be (I) neat, (II) clear and (III) well organized.
- □ : TIDILY REDUCE and box in your answers (Don't leave simplifiable algebraic expressions).
- >> : Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- : Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| : **Justify** your results so a grader can distinguish an informed answer from a guess.
- : If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ : Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 7: ____ /25

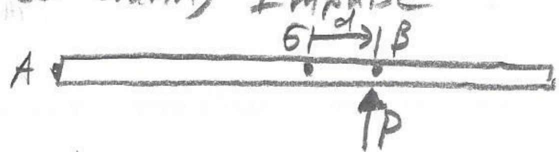
Problem 8: ____ /25

Problem 9: ____ /25

7) A uniform stick (mass m, length ℓ) is floating in space at rest. It is then hit with an impulsive force at B. Find the distance d from G to B, so that point A has a velocity of $\vec{v}_A = \vec{0}$ immediately after the impulse.



FBD during Impulse



FBD after Impulse

Kinematics

$\vec{0} = \vec{V_A^+} = \vec{V_G^+} + \vec{\omega}^+ \times \vec{r_{A/G}}$ ①

LMB

$\Sigma \int F dt = \int m \vec{a_G} dt$

$\vec{P} = m \Delta \vec{V_G} \Rightarrow \Delta \vec{V_G} = \vec{P}/m$ ②

AMB/G

$\Sigma \int M dt = \int I_G \alpha dt$

$\vec{r_{B/G}} \times \vec{P} = I \Delta \vec{\omega} \Rightarrow \Delta \vec{\omega} = \dfrac{\vec{r_{B/G}} \times \vec{P}}{I_G}$ ③

ICs

$\vec{\omega}(0^-) = \vec{0}$, $\vec{V_G}(0^-) = \vec{0}$ $\Rightarrow$ $\Delta \vec{\omega} = \vec{\omega}(0^+)$, $\Delta \vec{V_G} = \vec{V_G}(0^+)$ ④

substitute 2,3 into 1 $\Rightarrow \vec{0} = \vec{P}/m + \dfrac{\vec{r_{B/G}} \times \vec{P}}{I_G} \times \vec{r_{A/G}}$

$\vec{P} = P\hat{\jmath}$, $\vec{r_{B/G}} = d\hat{\imath}$, $\vec{r_{A/G}} = -\ell/2\, \hat{\imath}$

so $\vec{0} = \frac{P}{m}\hat{\jmath} + \left(\frac{dP}{I_G}\right)\hat{k} \times \left(\frac{-\ell}{2}\right)\hat{\imath} = \frac{P}{m}\hat{\jmath} - \frac{dP\ell}{2I_G}\hat{\jmath}$

$\{\}\cdot\hat{\jmath} \rightarrow 0 = \frac{P}{m} - \frac{dP\ell}{2I_G} \rightarrow d = \frac{2I_G}{m\ell}$

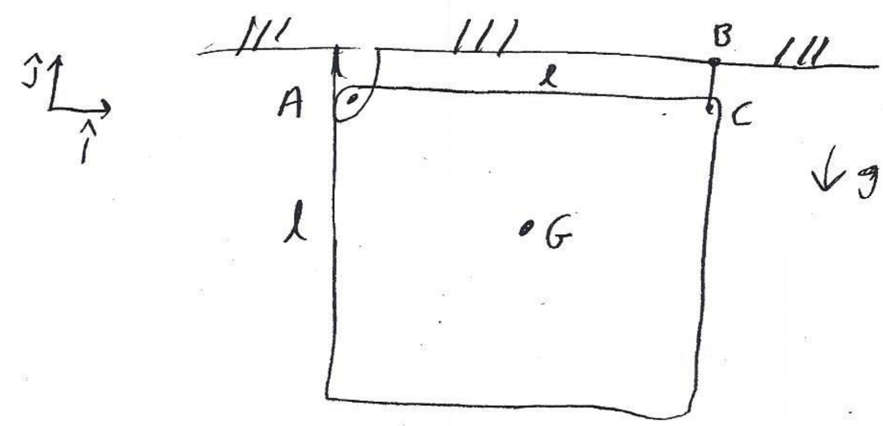
for a rod $I_G = \frac{m\ell^2}{12} \Rightarrow d = \frac{2}{m\ell}\left(\frac{m\ell^2}{12}\right) = \frac{\ell}{6} \Rightarrow \boxed{d = \frac{\ell}{6}}$

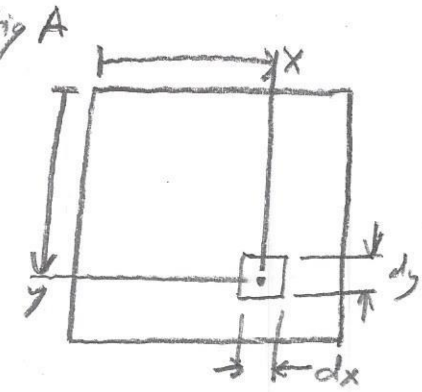
8) 2D. A uniform square plate of mass m and sides ℓ is suspended by a frictionless hinge at A and a string at C from a flat horizontal ceiling. At $t = 0$ the string BC is cut.

a) Calculate the moment of inertia I^G of the block about its center of mass in terms of m and ℓ.

b) Immediately after the string is cut, at $t = 0^+$, what is the acceleration of G, $\vec{a}_G$?
(Answer in terms of ℓ, m, g, I^G and the base vectors shown. Do not use your I^G from part a.)

c) Immediately after the string is cut, at $t = 0^+$, the vertical reaction force at A, $\vec{F}_A \cdot \hat{\jmath} = mg/2$, or not? Please give a clear reasoning for a yes or no answer without any long calculations. Full credit if there is precise reasoning with no algebra.





a) $I_G = \int r^2 dm$

$dm = \rho\, dx\, dy = \frac{m}{\ell^2} dx\, dy$

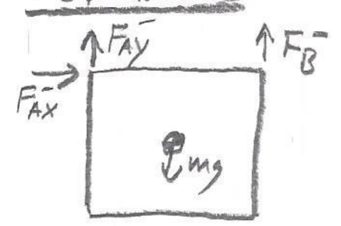
$r^2 = x^2 + y^2$ by figure A

$I_G = \int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2} (x^2+y^2)\left(\frac{m}{\ell^2} dx\, dy\right) = \frac{m}{\ell^2} \iint x^2+y^2\, dx\, dy$

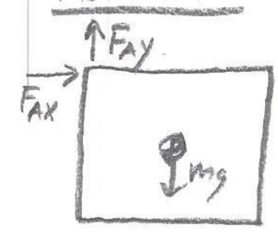
$\iint x^2+y^2\, dx\, dy = \int \left(\frac{x^3}{3} + y^2 x \Big|_{-\ell/2}^{\ell/2}\right) dy = \int \frac{\ell^3}{12} + \ell y^2\, dy = \frac{\ell^3 y}{12} + \frac{\ell y^3}{3}\Big|_{-\ell/2}^{\ell/2} = \frac{\ell^4}{12} + \frac{\ell^4}{12} = \frac{\ell^4}{6}$

$I_G = \frac{m}{\ell^2}\left(\frac{\ell^4}{6}\right) = \frac{m\ell^2}{6} \rightarrow \boxed{I_G = \frac{m\ell^2}{6}}$

b) FBD before



FBD after



$F_{AX} = 0$ by statics

AMB/A

$\Sigma \vec{M}_{/A} = I_A \vec{\alpha} \rightarrow \Sigma M_{/A} = \frac{-mgl}{2}\hat{k} = I_A \alpha \hat{k}$

$\alpha = \frac{-mgl}{2I_A}$ (1)

Parallel Axis Thm

$I_A = I_G + m|\vec{r}_{A/G}|^2 = I_G + m\left(\left(\frac{l}{2}\right)^2+\left(\frac{l}{2}\right)^2\right) = I_G + \frac{ml^2}{2}$ (2)

combining 1 and 2 → $\alpha = \frac{-mgl}{2I_G + ml^2}$

Rigid body $\vec{a}_A = \vec{\alpha}\times\vec{r}_{A/G} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/G}) + \vec{a}_G$

$\vec{\omega}=\vec{0}$ at instant rope is cut → $\vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/G}) = \vec{0}$

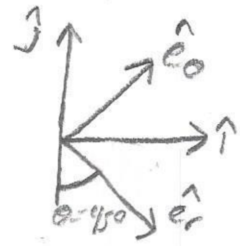
point A is hinge → $\vec{a}_A = \vec{0}$

∴ $\vec{a}_G + \vec{\alpha}\times\vec{r}_{A/G} = 0 \rightarrow \vec{a}_G = -\vec{\alpha}\times\vec{r}_{A/G}$ (3)

sub 2 into 3 → $\vec{a}_G = -\left(\frac{-mgl}{2I_G+ml^2}\right)\hat{k}\times\left(\frac{-l}{2}\hat{\imath} + \frac{l}{2}\hat{\jmath}\right)$

$$\boxed{\vec{a}_G = \frac{-mgl^2}{4I_G+2ml^2}(\hat{\jmath}+\hat{\imath})}$$

note: if used $\hat{e}_r, \hat{e}_\theta$ → $\vec{a}_G = \frac{-mgl^2\sqrt{2}}{4I_G+2ml^2}\hat{e}_\theta$



c) Realize the force at A, the angular and linear acceleration and the force at B all change instantaneously when B is cut.

This means you cannot argue $F_{AY}^- = \frac{mg}{2} \rightarrow F_{AY}^+ = \frac{mg}{2}$

Mathematically

LMB: $\Sigma\vec{F} = F_{AX}\hat{\imath} + F_{AY}\hat{\jmath} - mg\hat{\jmath} = m\vec{a}_G$ (see FBD "after" on previous page)

$\{3\}\cdot\hat{\jmath} \rightarrow F_{AY} = m\vec{a}_G\cdot\hat{\jmath} + mg = \frac{-mgl^2}{4I_G+2ml^2} + mg = mg\left(1-\frac{ml^2}{4I_G+2ml^2}\right)$

from part a: $I_G = \frac{ml^2}{6}$

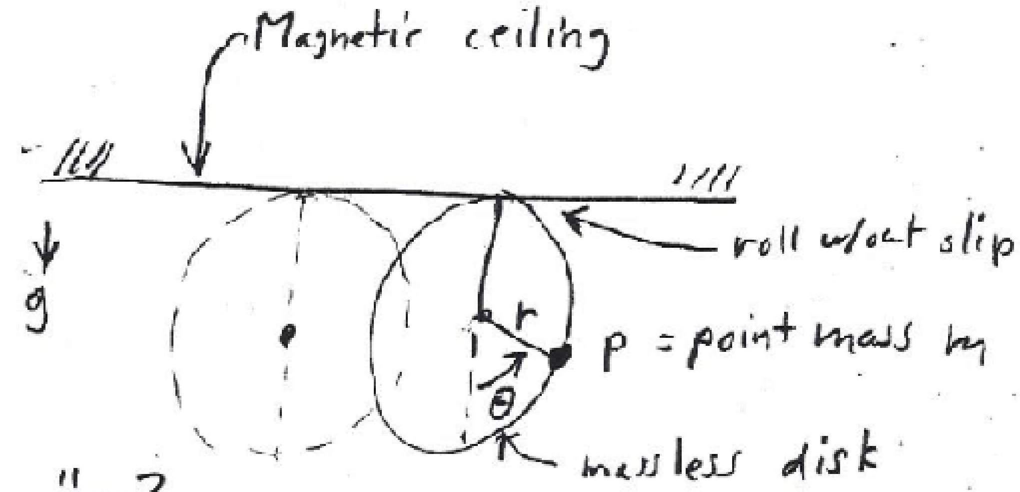
$F_{AY} = \left(1-\frac{ml^2}{4\left(\frac{ml^2}{6}\right)+2ml^2}\right)mg = \left(1-\frac{1}{2/3+2}\right)mg = \frac{5}{8}mg \Rightarrow \boxed{F_{AY} = \frac{5}{8}mg \neq \frac{mg}{2}}$

9) 2D. A massless disk with radius r rolls without slip on the ceiling, held up by magnetic forces. On the periphery is attached a point mass m. Gravity g acts. An equilibrium position is with $\theta = 0$ and the mass hanging a distance $2r$ from the ceiling.

a) Given r, θ and g find $\ddot{\theta}$.

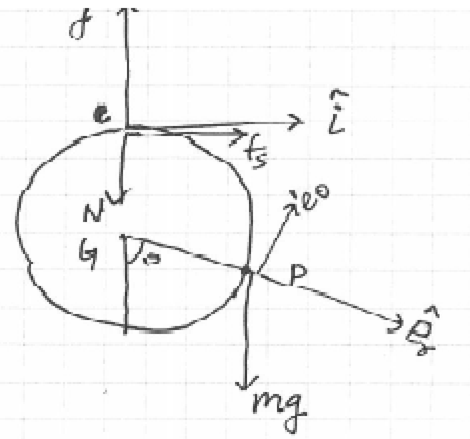
b) Find the period of small oscillation?

c) Does the period depend on amplitude? (genuinely hard question)



$\ddot{\theta} = ?$

FBD



$C \equiv (0,0)$

$G \equiv (0, -R)$

$P \equiv (R\sin\theta, -R(1+\cos\theta))$

$\vec{r}_{PC} = R\sin\theta\, \hat{i} - R(1+\cos\theta)\, \hat{j}$

$\vec{a}_P = \vec{a}_G + \vec{a}_{PG}$

$\vec{a}_G = R\ddot{\theta}\, \hat{i}$ ROLLING

$\vec{a}_{PG} = -\dot{\theta}^2 R\, \hat{e}_r + \ddot{\theta} R\, \hat{e}_\theta$

$\hat{e}_r = \sin\theta\, \hat{i} - \cos\theta\, \hat{j}$

$\hat{e}_\theta = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$

$\vec{a}_P = (-\dot{\theta}^2 R\sin\theta + \ddot{\theta}R(1+\cos\theta))\hat{i} + (\dot{\theta}^2 R\cos\theta + \ddot{\theta}R\sin\theta)\hat{j}$

AMB

$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$\sum \vec{M}_{/C} = (R\sin\theta\, \hat{i} - R(1+\cos\theta)\hat{j}) \times (-mg\,\hat{j})$

$= -mgR\sin\theta\, \hat{k}$

$\dot{\vec{H}}_{/C} = m\, \vec{r}_{PC} \times \vec{a}_{cm} + \underbrace{I_{cm}}_{=0}\vec{\alpha}$ POINT MASS

$= m\, \vec{r}_{PC} \times \vec{a}_P$

$= m(R\sin\theta\,\hat{i} - R(1+\cos\theta)\hat{j}) \times \Big[(\ddot{\theta}R(1+\cos\theta) - \dot{\theta}^2 R\sin\theta)\hat{i}$

$+ (\dot{\theta}^2 R\cos\theta + \ddot{\theta}R\sin\theta)\hat{j}\Big]$

$$= mR^2\left[\sin\theta\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) + (1+\cos\theta)\left(\ddot{\theta}(1+\cos\theta) - \dot{\theta}^2\sin\theta\right)\right]\hat{k}$$

$$= mR^2\left[\ddot{\theta}\left(\sin^2\theta + (1+\cos\theta)^2\right) + \dot{\theta}^2\left(\sin\theta\cos\theta - \sin\theta(1+\cos\theta)\right)\right]\hat{k}$$

$$= mR^2\left[\ddot{\theta}(2+2\cos\theta) + \dot{\theta}^2(-\sin\theta)\right]\hat{k}$$

$$\Rightarrow \quad -mgR\sin\theta = mR^2\left[2\ddot{\theta}(1+\cos\theta) - \sin\theta\,\dot{\theta}^2\right]$$

$$\Rightarrow \quad \frac{g}{R}\sin\theta + 2\ddot{\theta}(1+\cos\theta) - \dot{\theta}^2\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} - \dot{\theta}^2\frac{\sin\theta}{2(1+\cos\theta)} + \frac{\sin\theta}{2(1+\cos\theta)}\frac{g}{R} = 0$$

$$\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

(a)

$$\Rightarrow \quad \boxed{\ddot{\theta} - \frac{1}{2}\tan\frac{\theta}{2}\,\dot{\theta}^2 + \frac{1}{2}\tan\frac{\theta}{2}\,\frac{g}{R} = 0}$$

(b) SMALL ANGLE APPROXIMATION:

$$|\theta| \ll 1 \equiv \tan\frac{\theta}{2} \approx \frac{\theta}{2}$$

EQUATION (a) BECOMES:

$$\ddot{\theta} - \frac{\theta}{4}\dot{\theta}^2 + \frac{g}{4R}\theta = 0$$

IGNORING $\dot{\theta}^2$ TERM ALSO,

$$\boxed{\ddot{\theta} + \frac{g}{4R}\theta = 0}$$ EQUATION OF MOTION

TIME PERIOD $= 2\pi\sqrt{\frac{4R}{g}}$

CHECK IF APPROXIMATION IS FAIR :

$$\dot{\theta} \sim \frac{\theta}{T} = \frac{\theta}{2\pi}\sqrt{\frac{g}{4R}}$$

$$\ddot{\theta} \sim \frac{\theta}{T^2} = \frac{\theta}{4\pi^2}\left(\frac{g}{4R}\right)$$

WE FIND THAT

$$\left|\frac{\theta}{4}\dot{\theta}^2\right| \sim \left|\frac{1}{4}\frac{\theta^3}{4\pi^2}\left(\frac{g}{4R}\right)\right| \ll |\ddot{\theta}|, \left|\frac{g}{4R}\theta\right|$$

SO OUR APPROXIMATION IS OK!

(c) EQUATION FROM (a) CAN BE REWRITTEN AS

$$\cos\frac{\theta}{2}\,\ddot{\theta} - \frac{1}{2}\sin\frac{\theta}{2}\,\dot{\theta}^2 + \frac{g}{2R}\sin\frac{\theta}{2} = 0 \quad \text{— (1)}$$

SUBSTITUTE $\alpha = R\sin\frac{\theta}{2}$

$$\dot{\alpha} = \frac{R}{2}\cos\frac{\theta}{2}\,\dot{\theta}$$

$$\ddot{\alpha} = -\frac{R}{4}\sin\frac{\theta}{2}\,\dot{\theta}^2 + \frac{R}{2}\cos\frac{\theta}{2}\,\ddot{\theta}$$

SO EQUATION (1) BECOMES

$$\frac{2\ddot{\alpha}}{R} + \frac{g\alpha}{2R^2} = 0$$

$$\Rightarrow \left.\ddot{\alpha} + \frac{g}{4R}\alpha = 0\right\} \text{ SHM WITH } T = 2\pi\sqrt{\frac{4R}{g}}$$

WHICH HAPPENS TO BE EQUATION OF SIMPLE HARMONIC MOTION, WHICH SHOWS INDEPENDENCE BETWEEN TIME PERIOD & AMPLITUDE. ALSO THIS TIME PERIOD IS SAME AS FOR SMALL ANGLE APPROXIMATION!