MAE 5730
HW6 10/7/13

① Handout 9

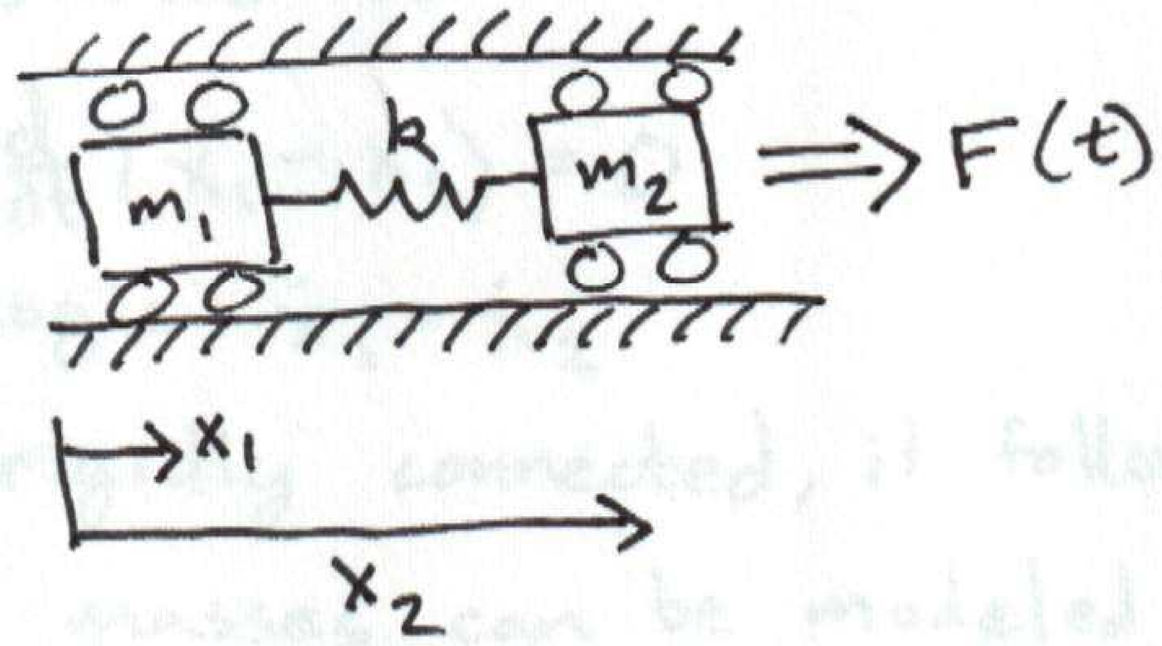
Two masses (m_1, m_2) constrained to the x-axis

Initial conditions

$x_1(0) = \dot{x}_1(0) = \ddot{x}_1(0) = 0$

$x_2(0) = l_0$

$\dot{x}_2(0) = \ddot{x}_2(0) = 0$



l_0 = free length of spring

$F(t) = F_0 H(t) = \begin{cases} 0 & t < 0 \\ F_0 & t \geq 0 \end{cases}$

a) MATLAB plots of motion attached for arbitrary parameters

b) $F = (m_1 + m_2) a_G$ where $x_G = \dfrac{(x_1 m_1 + x_2 m_2)}{(m_1 + m_2)}$ should ALWAYS be TRUE

A proof was shown in HW4, Handout Problem #8:

$$\Sigma \vec{F}_i = \Sigma m_i \vec{a}_i$$

$$\Sigma(\vec{F}_i^{\,int} + \vec{F}_i^{\,ext}) = \frac{\partial}{\partial t} \underbrace{\Sigma m_i \vec{v}_i}_{\vec{L}} = \dot{\vec{L}}$$

For classical mechanics, assume $\begin{cases} \Sigma \vec{F}^{\,int} = 0 \\ \Sigma \vec{F}^{\,ext} = \dot{\vec{L}} \end{cases}$

$$\Sigma \vec{F}_i^{\,ext} = \frac{\partial}{\partial t}(m_{tot} \vec{v}_G)$$ since it was shown in ⑧a) that $\Sigma m_i \vec{v}_i = \vec{v}_G m_{tot}$

$$\Sigma \vec{F}_i^{\,ext} = m_{tot} \vec{a}_G$$

For this problem, the only external force is F(t). (The spring force is internal to the system). Also, $m_{tot} = m_1 + m_2$

$$\Rightarrow F = (m_1 + m_2) a_G$$

This is also verified numerically in an attached plot ✓

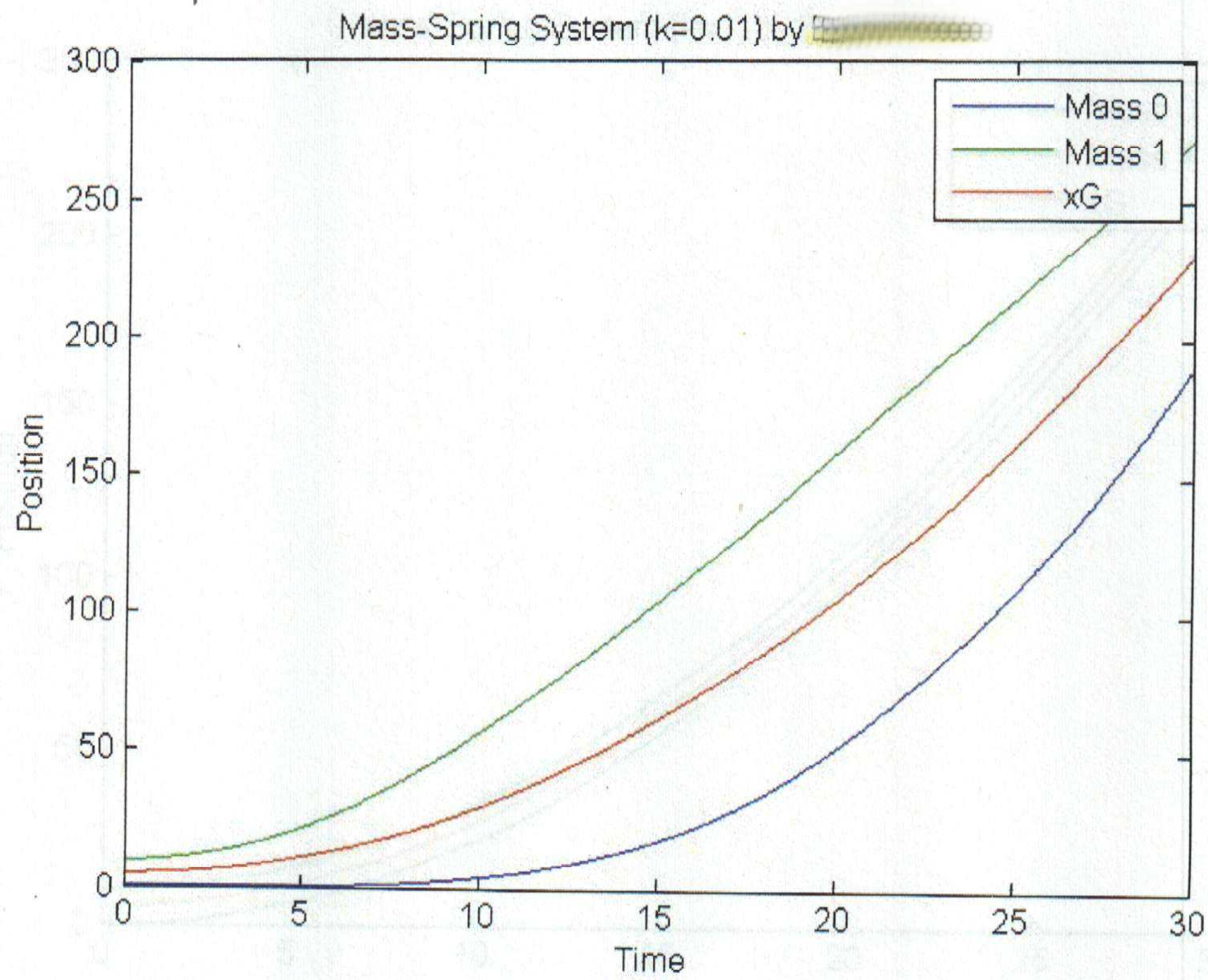
c) MATLAB plots for a variety of k are attached

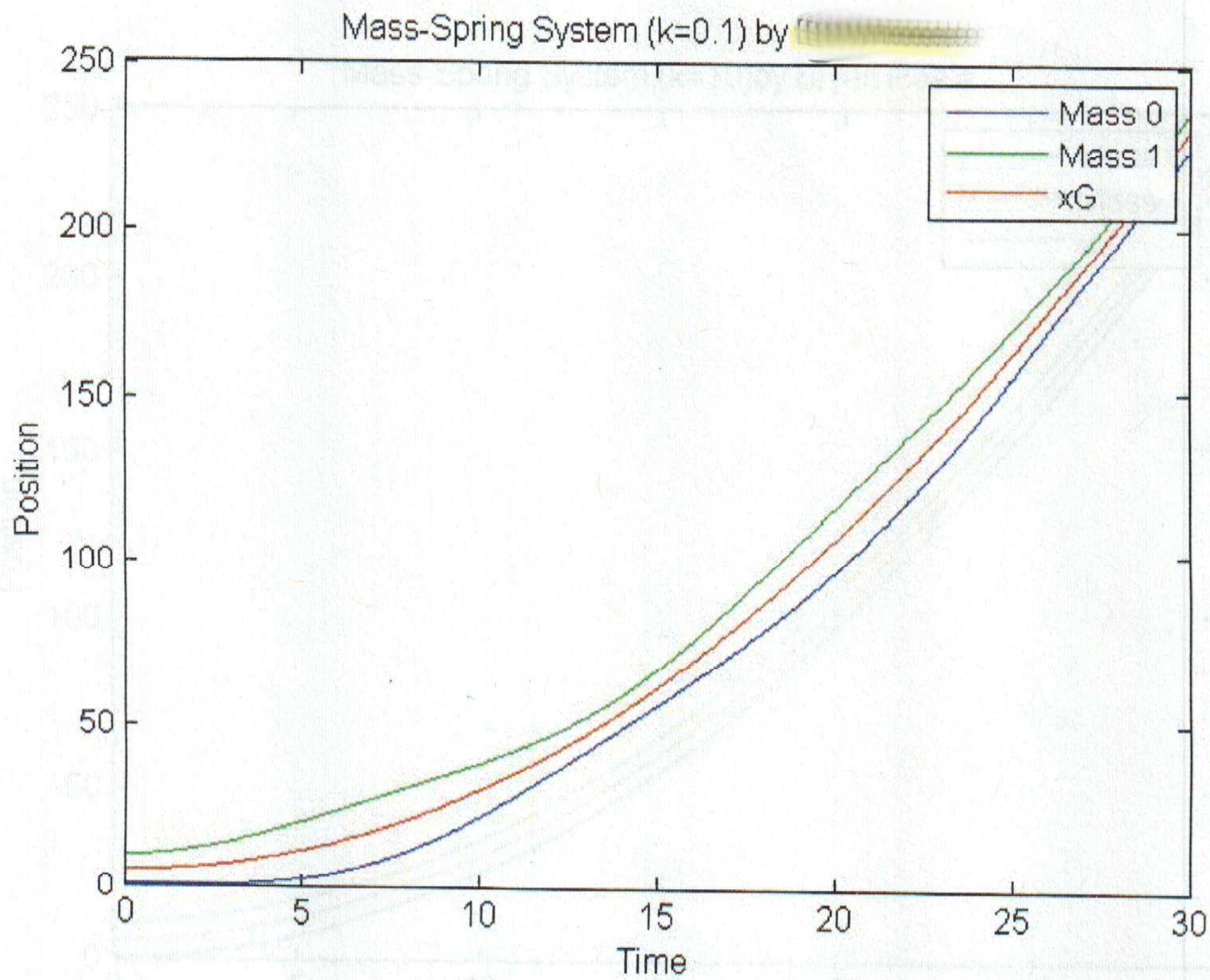
d) Make the following statement as precise as possible:

"For high values of k the system nearly behaves like a single mass"

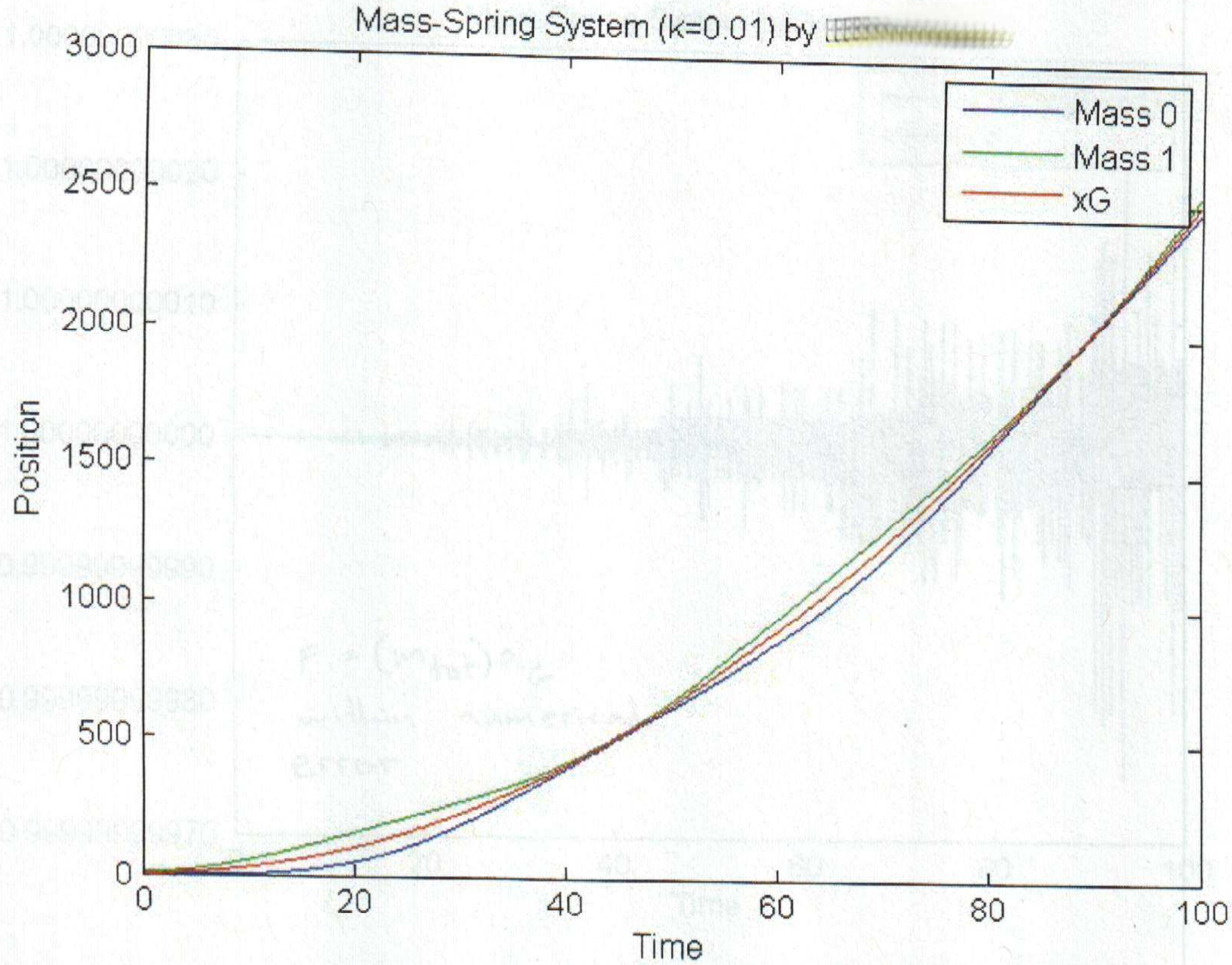
First, let us think about what it means for the system to behave like a single mass. If the two masses were rigidly attached, we would notice several physical characteristics

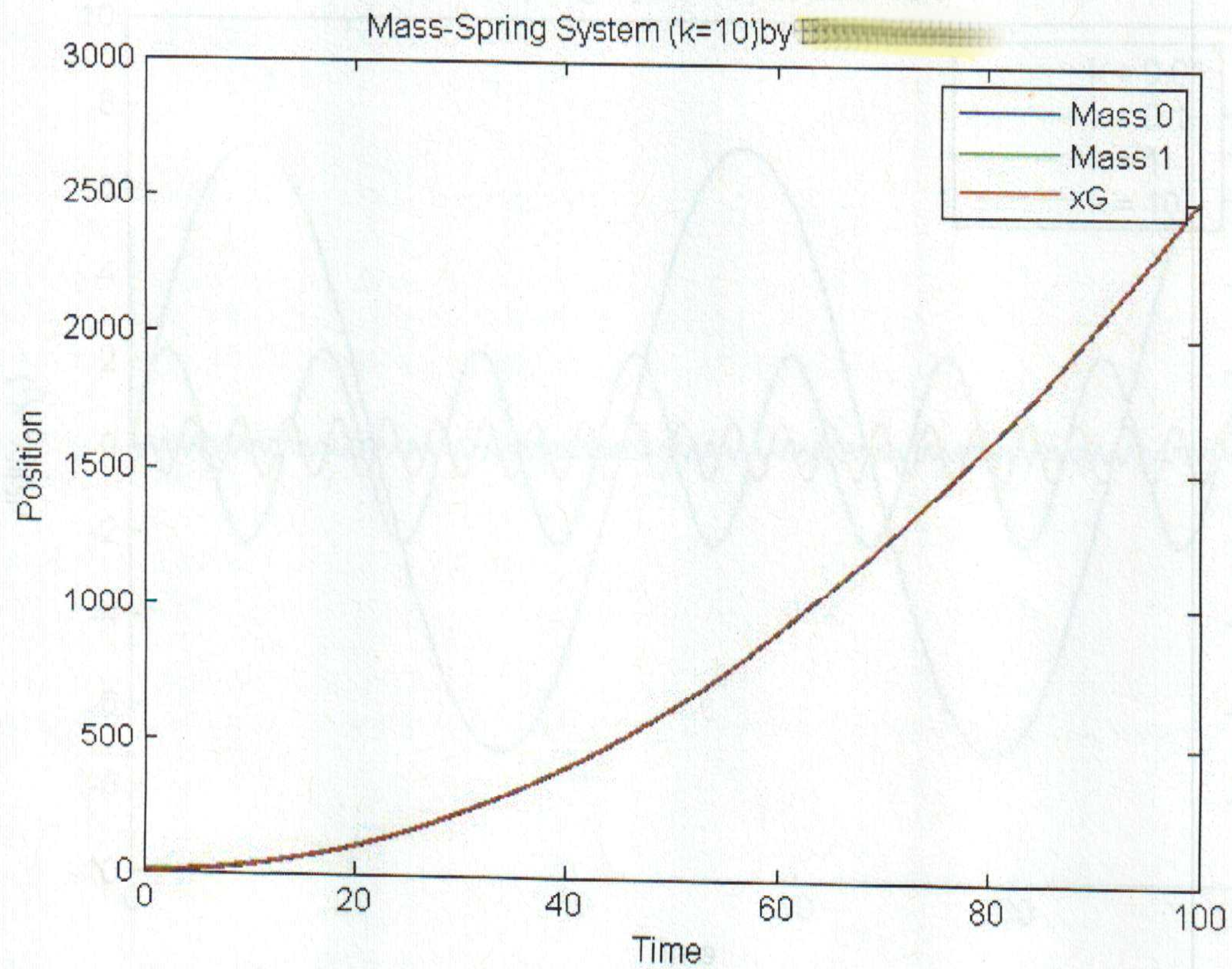
As k increases, motion of the masses approximates the motion of the center of mass

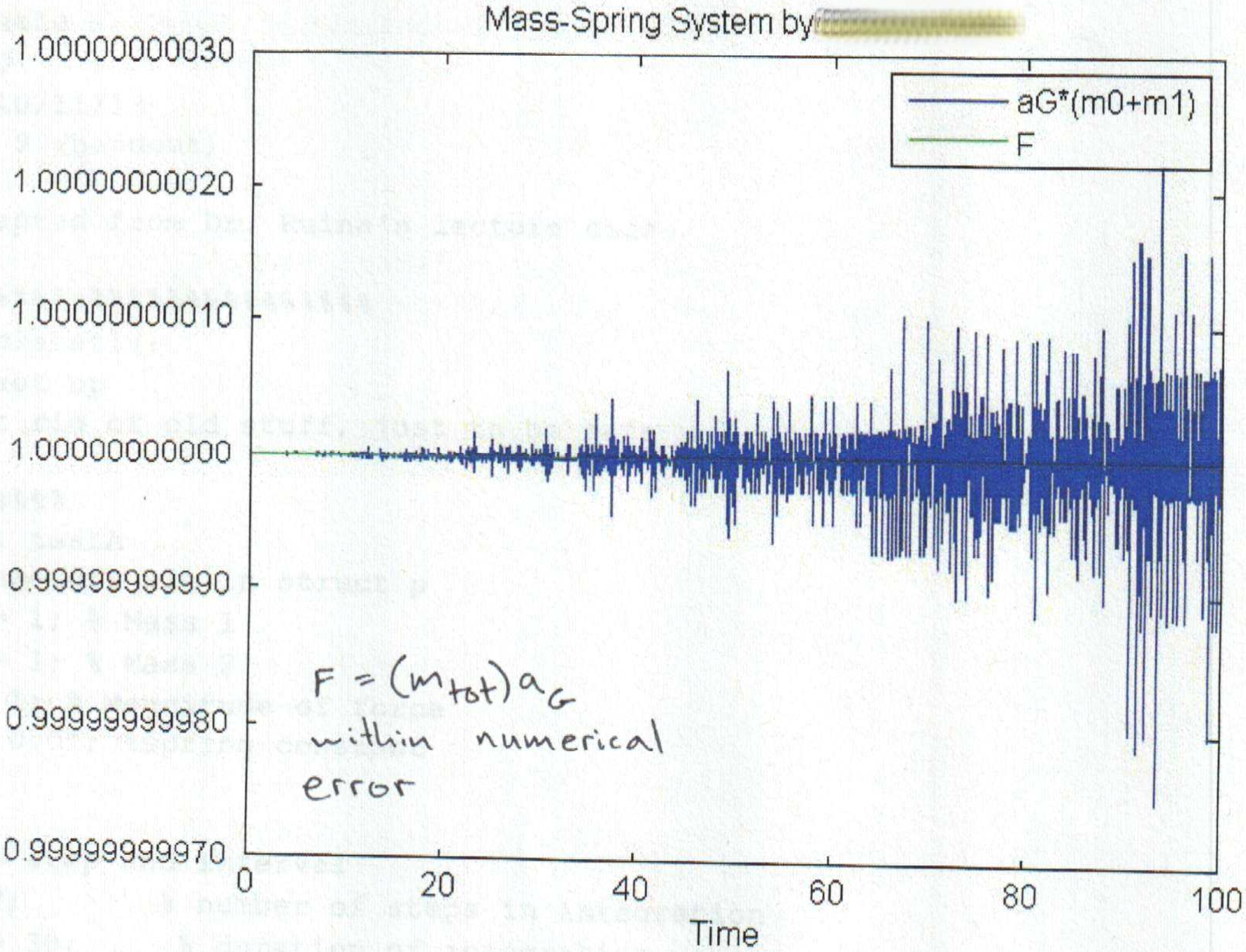




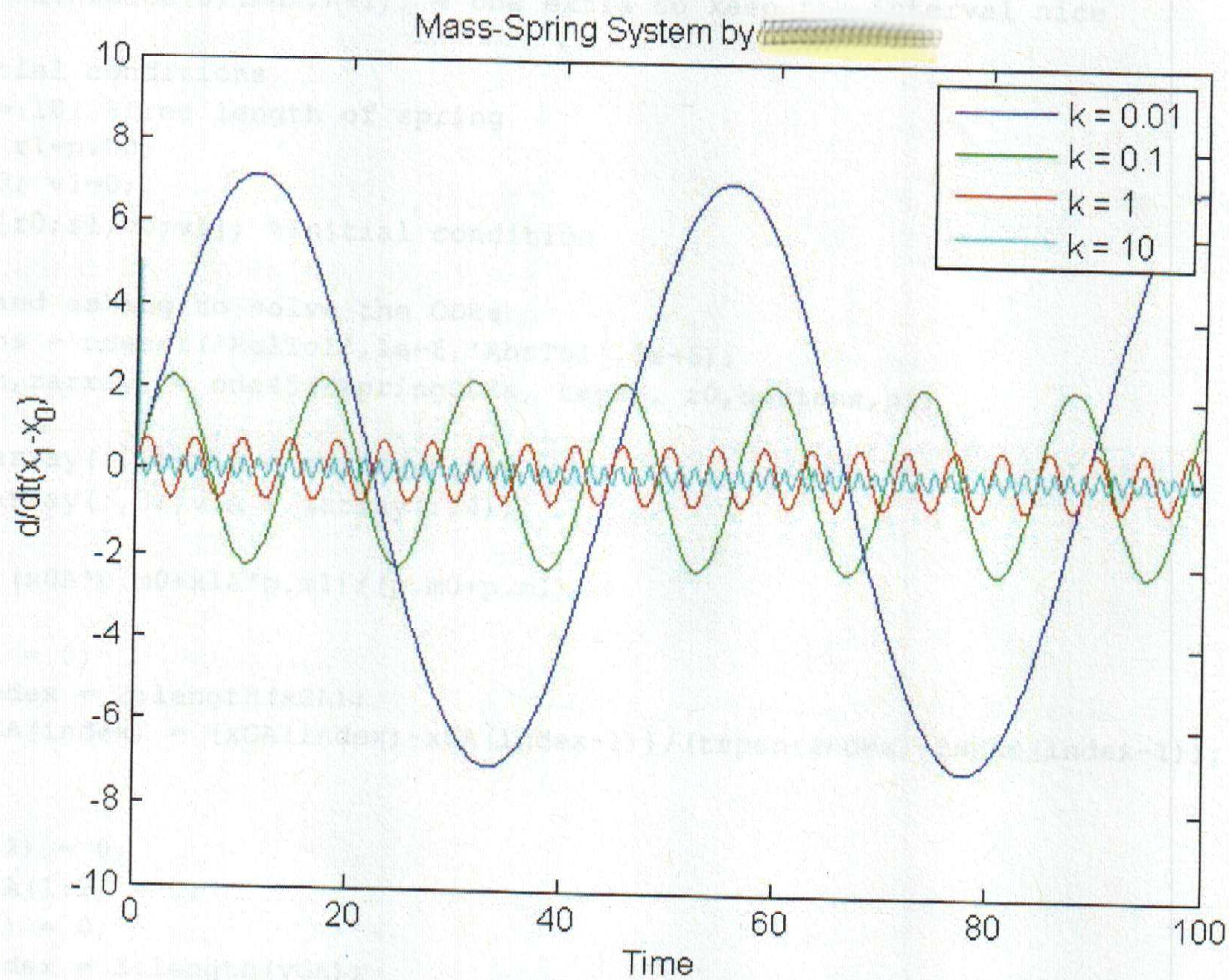
As k ↑, each masses motion becomes closer to the motion of the center of mass







$F = (m_{tot}) a_G$ within numerical error



As k increases, $\frac{d}{dt}(x_1 - x_0)$ decreases and frequency increases

d) One way to measure 1 DoF-ness is to compare the movement of the CoM to the movement of the masses w.r.t. the CoM. When that ratio is small then you can claim 1 DoF. ✓

From the diff eqs.

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = \cancel{k(x_2 - x_1 - l_0)} + F_0 - \cancel{k(x_2 - x_1 - l_0)}$$

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = F_0$$

$$\frac{m_1\ddot{x}_1 + m_2\ddot{x}_2}{m_1+m_2} = \frac{F_0}{m_1+m_2} \Rightarrow a_G = \frac{F_0}{m_1+m_2}$$

proves part b) as well...

$$\Rightarrow x_G = \frac{F_0 t^2}{2(m_1+m_2)}$$

$$m_1\ddot{x}_1 - m_2\ddot{x}_2 = 2k(x_2 - x_1 - l_0) - F_0$$

For simplicity, let $m_1 = m_2 = m$ and then $x_m = \frac{1}{2}(x_2 - x_1)$

$$2\ddot{x}_m = -\frac{2k}{m}(2x_m - l_0) + F_0/m$$

$$\ddot{x}_m + \frac{2k}{m}x_m = \frac{2kl_0 + F_0}{2m} \Rightarrow x_m = A\cos\left(\sqrt{\tfrac{2k}{m}}\,t\right) + \left(\frac{l_0}{2} + \frac{F_0}{4k}\right)$$

$$x_m(0) = \frac{l_0}{2} \Rightarrow A = -\frac{F_0}{4k} \Rightarrow \text{max value of } |x_m| = \frac{l_0}{2} + \frac{F_0}{2k}$$

$$\Rightarrow \text{max disp. of masses from eq.} = \frac{F_0}{2k}$$

$$\Rightarrow \sim 1 \text{ DOF when } \left(\frac{F_0}{2k}\right)\Big/\left(\frac{F_0 t^2}{4m}\right) = \frac{2m}{kt^2} \text{ } \underline{\text{small}} \text{ } (\leq .01 \text{ maybe?})$$

$$\Rightarrow 1 \text{ DoF when } k, t\uparrow, m\downarrow$$

Very Nice!

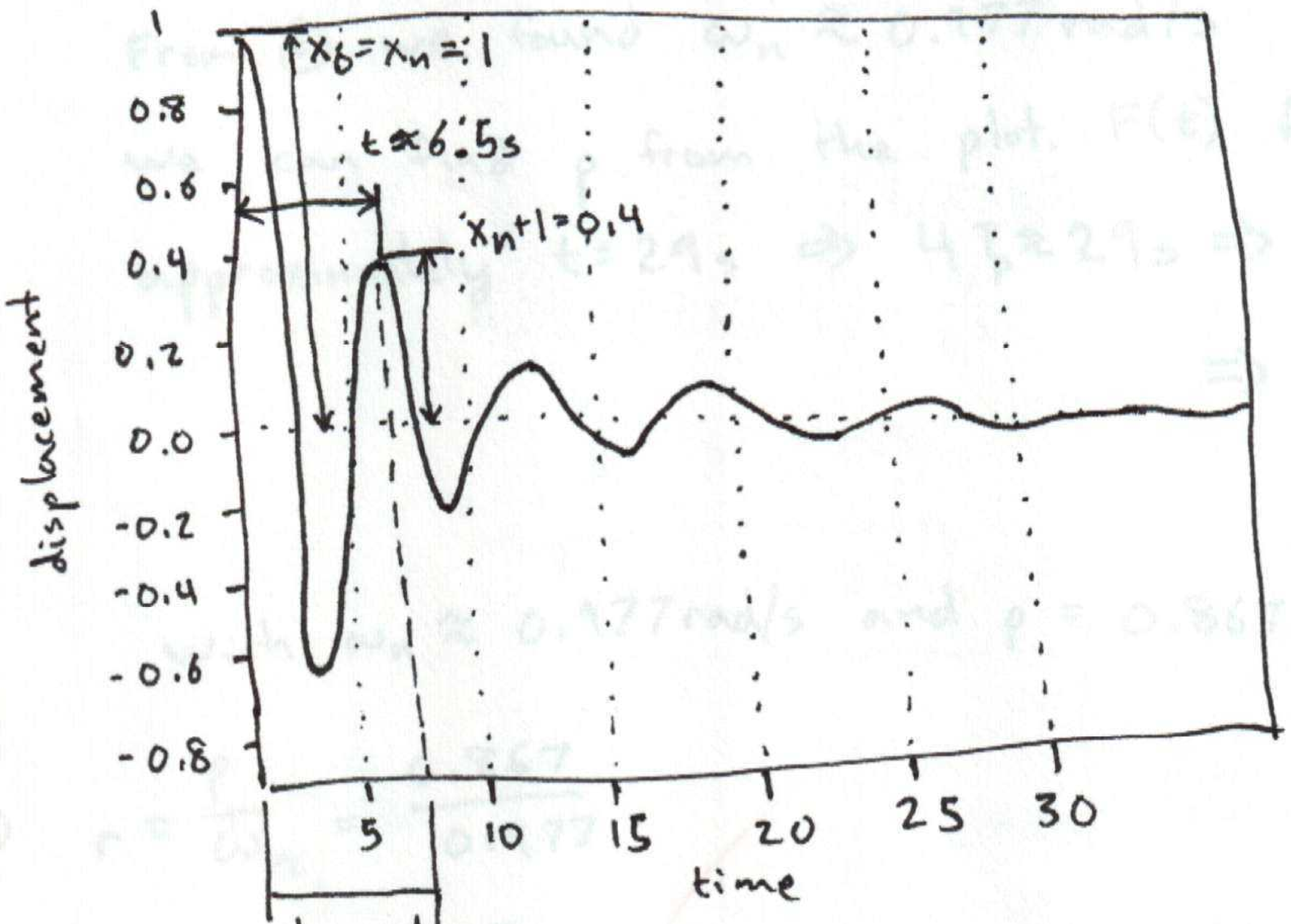
② RP 10.2.4

SDOF system is tested to find m, k, and c

FIRST : transient test, machine disturbed from equilibrium and allowed to settle
displacement v. time is plotted

SECOND sinusoidal forcing with amplitude F_0 and angular freq p is applied
steady-state response, along with forcing function are plotted

a) MARK relevant points on transient response plot, EXPLAIN what parameters can be determined from this plot



$T_d = \frac{2\pi}{\omega_d}$

ω_d can be found from the period of damped oscillations

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \frac{2\pi}{T_d} = \frac{2\pi}{6.5s}$$

$$\omega_n\sqrt{1-(0.1458)^2} = \frac{2\pi}{6.5s}$$

$$\Rightarrow \omega_n \approx 0.977 \text{ rad/s} = \sqrt{k/m}$$

Logarithmic Decrement

$$D = \ln\left(\frac{x_n}{x_{n+1}}\right) = \ln\left(\frac{1}{0.4}\right) = 0.9163$$

$$D = \frac{cT}{2m} = 2\pi\xi$$

$$\xi = \frac{D}{2\pi} = 0.1458$$

Logarithmic decay is used to find ξ by looking at the change in aplitude over one period

$c_{crit} = 2\sqrt{km}$ and $\xi = \frac{c}{c_{crit}}$

$$\Rightarrow c = 2\xi\left(\sqrt{km}\right)$$

Using just this plot, we can find $\xi = 0.1458$ and $\omega_n \approx 0.977$ rad/s
However, while we can find relationships between m, c + k we do not have enough information to solve for any of them directly

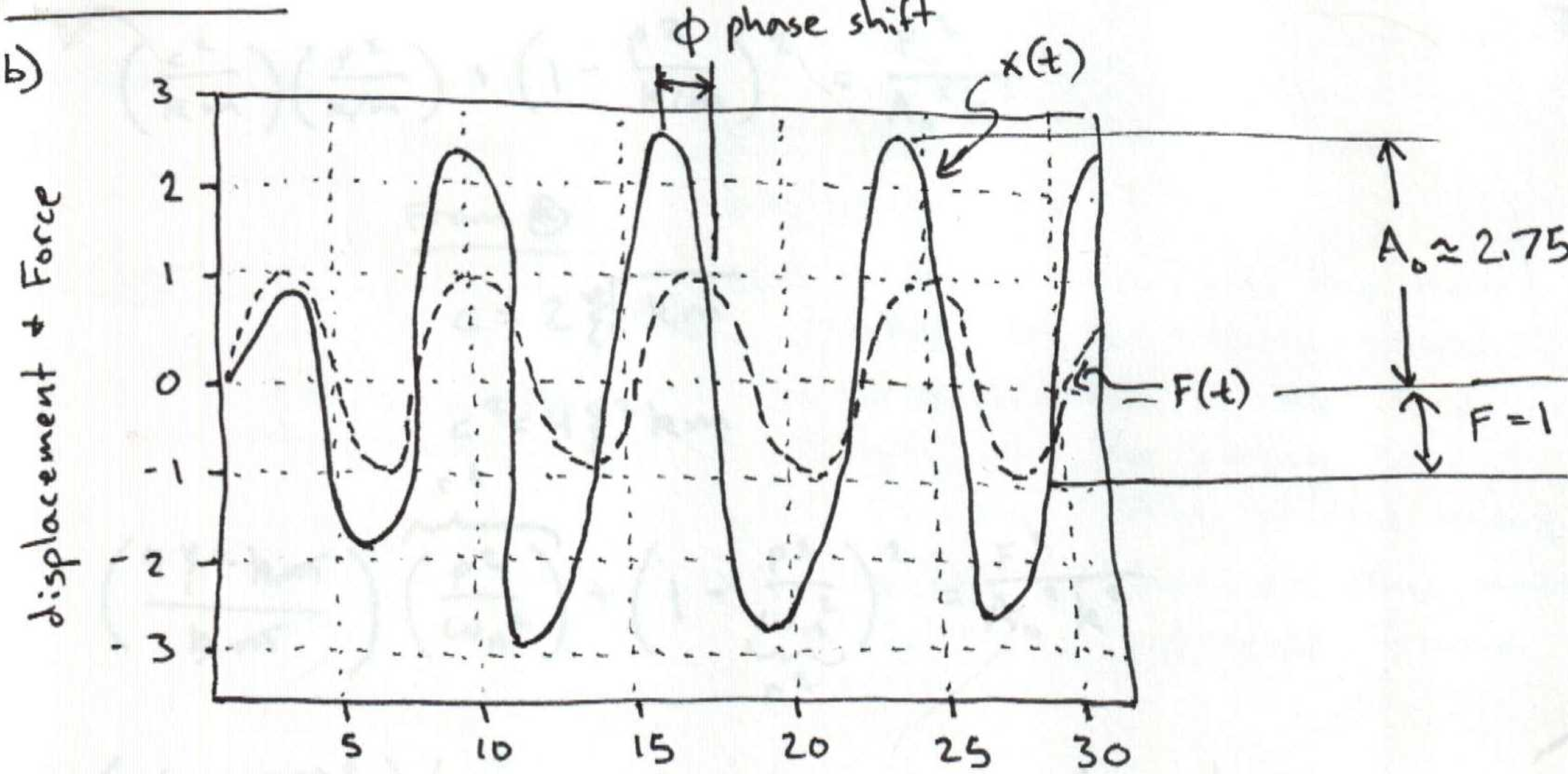
$\omega_n = \sqrt{k/m}$

from $D = \frac{cT_d}{2m} \Rightarrow m = \frac{cT_d}{2D}$

$k = \omega_n^2 m$

RP 10.2.4

b)



From ⓐ, we found $\omega_n \approx 0.977$ rad/s

we can find p from the plot. $F(t)$ finishes 4 periods at approximately $t = 29s \Rightarrow 4T_p \approx 29s \Rightarrow T_p \approx 7.25s$

$$\Rightarrow p = \frac{2\pi}{T_p} = \frac{2\pi}{7.25}$$

$$p = 0.867 \text{ rad/s}$$

with $\omega_n \approx 0.977$ rad/s and $p = 0.867$ rad/s, $\boxed{p < \omega_n}$

c) $r = \frac{p}{\omega_n} = \frac{0.867}{0.977}$

$$\boxed{r = 0.8874}$$

d) From the steady-state response, we find the amplitude of the steady state displacement, $x(t)$

$A_0 \approx 2.75$

From Eqn 10.35

$$A_0 = \frac{F/k}{\sqrt{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{(k/m)}\right) + \left(1 - \frac{p^2}{k/m}\right)^2}}$$

$$A_0^2\left[\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{k/m}\right)^2\right] = \frac{F^2}{k^2}$$

b) RP 10.2.4

$$\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{k/m}\right)^2 = \frac{F^2}{A_0^2 k^2}$$

From ⓐ

$$c = 2\zeta\sqrt{km}$$

$$c^2 = 4\zeta^2 km$$

$$\left(\frac{4\zeta^2 \cancel{km}}{\cancel{km}}\right)\overbrace{\left(\frac{p^2}{\omega_n^2}\right)}^{r^2} + \left(1 - \underbrace{\frac{p^2}{\omega_n^2}}_{r^2}\right)^2 = \frac{F^2}{A_0^2 k^2}$$

$$\left(\frac{4(0.1458)^2}{1}\right)(0.8874)^2 + \left(1 - (0.8874)^2\right)^2 = \frac{1}{(2.75)^2 k^2}$$

$$0.067 + 0.0452 = \frac{1}{7.5625 k^2}$$

$$(0.1122)(7.5625) = \frac{1}{k^2}$$

$$k^2 = 1.1785$$

$$\boxed{k \approx 1.086}$$

$$\omega_n^2 = \frac{k}{m}$$

$$m = \frac{k}{\omega_n^2}$$

$$m = \frac{1.086}{(0.977)^2}$$

$$\boxed{m \approx 1.14}$$

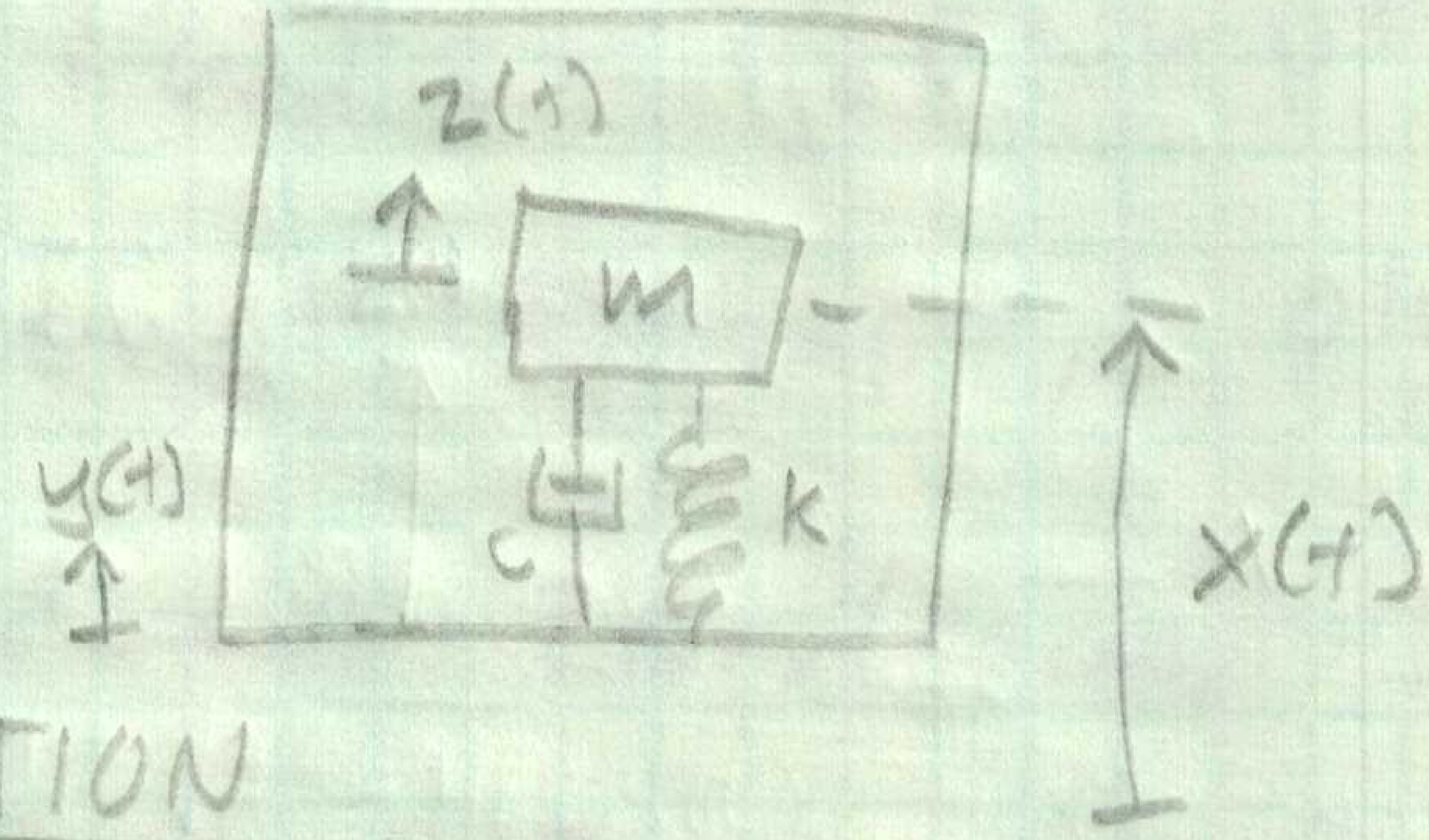
$$c = 2\zeta\sqrt{km}$$

$$c = 2(0.1458)\sqrt{(1.086)(1.14)}$$

$$\boxed{c \approx 0.3241}$$

RP10.2.11) PROBLEM

Accelerometer stuff
(I'm not writing a) -> f))

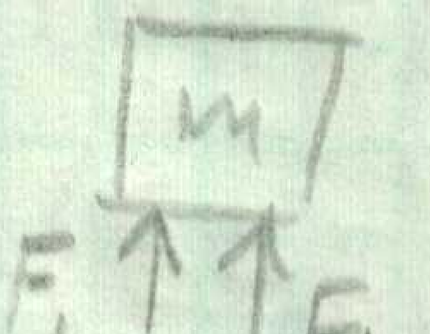


SOLUTION

a) Let $x(t)$ be absolute position of m

$$\boxed{\ddot{x} = \ddot{y} + \ddot{z}}$$

b) FBD



Const. Equs

$$F_k = -kz$$

$$F_d = -c\dot{z}$$

LMB

$$m\ddot{x} = F_d + F_k$$

$$\boxed{m\ddot{x} = -c\dot{z} - kz} \Longleftrightarrow -\ddot{y} = \ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z$$

c) $y(t) = y_0 \sin pt$

$\dot{y}(t) = y_0 p \cos pt$

$\ddot{y}(t) = -y_0 p^2 \sin pt$

$$\boxed{|\ddot{y}| = y_0 p^2}$$

d) $\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z = -\ddot{y} = y_0 p^2 \sin pt$

Let $\frac{c}{m} = 2\zeta\omega_n$, $\frac{k}{m} = \omega_n^2$;

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = y_0 p^2 \sin pt$$

Guess $z = A\sin pt + B\cos pt$, plug in:

$$-Ap^2\sin pt - Bp^2\cos pt + 2\zeta\omega_n(Ap\cos pt - Bp\sin pt)$$
$$+ A\omega_n^2 \sin pt + B\omega_n^2 \cos pt = y_0 p^2 \sin pt$$

$$\Rightarrow \quad -Ap^2 - 2\zeta\omega_n Bp + A\omega_n^2 = y_0 p^2$$
$$-Bp^2 + 2\zeta\omega_n Ap + B\omega_n^2 = 0$$

$$\Rightarrow A(\omega_n^2 - p^2) - B(2\zeta\omega_n p) = y_0 p^2$$
$$\left\{ A(2\zeta\omega_n p) + B(\omega_n^2 - p^2) = 0 \right.$$

$\rightarrow$ solve for A $\Rightarrow A = -\frac{(\omega_n^2 - p^2)}{2\zeta\omega_n p} B$

plug in A

$$\Rightarrow -\frac{(\omega_n^2 - p^2)^2}{2\zeta\omega_n p} B - 2\zeta\omega_n p B = y_0 p^2$$

solve for B $\Rightarrow B = -\frac{2\zeta\omega_n p^3 y_0}{(\omega_n^2 - p^2)^2 - (2\zeta\omega_n p)^2}$

$$\Rightarrow A = \frac{(\omega_n^2 - p^2)\omega^2 y_0}{(\omega_n^2 - p^2)^2 - (2\zeta\omega_n p)^2}$$

$$|z(t)| = \sqrt{A^2 + B^2}$$

ugly as it looks, this corresponds exactly to MATLAB simulations I ran

$$= \boxed{\frac{\left[(\omega_n^2 - p^2)^2 p^4 y_0^2 + 4\zeta^2 \omega_n^2 p^6 y_0^2\right]^{1/2}}{(\omega_n^2 - p^2)^2 - (2\zeta\omega_n p)^2}}$$

Plot of this shown on next page.

e) If $p \ll \omega_n$ then $(\omega_n^2 - p^2) \approx \omega_n^2$

$$|z(t)|_{p \ll \omega_n} = \frac{\left[\omega_n^4 p^4 y_0^2 + 4\zeta^2 \omega_n^2 p^6 y_0^2\right]^{1/2}}{\omega_n^4 - 4\zeta^2 \omega_n^2 p^2}$$

$$= \frac{\omega_n^2}{\omega_n^2} \cdot \frac{\left[p^4 y_0^2 + \underbrace{\cancel{4\zeta^2 p^4 y_0^2 \left(\frac{p}{\omega_n}\right)^2}}_{0,\ (p/\omega_n)^2 \ll 1}\right]^{1/2}}{\omega_n^2 - 4\zeta^2 p^2}$$

$(p/\omega_n)^2 \ll 1$

$$\Rightarrow \omega_n^2 - 4\zeta^2 p^2 = \omega_n^2 \left(1 - \cancel{4\zeta^2 \left(\frac{p}{\omega_n}\right)^2}\right) \approx \omega_n^2$$

$$\Rightarrow |z(t)|_{p \ll \omega_n} \approx \frac{p^2 y_0}{\omega_n^2}$$, <u>note</u>: $p^2 y_0 = |\ddot{y}|$

$$\approx \boxed{\frac{|\ddot{y}|}{\omega_n^2}}$$

✓

Ta-dah!

f) If $\omega_n << p$ then $(\omega_n^2 - p^2) \approx -p^2$

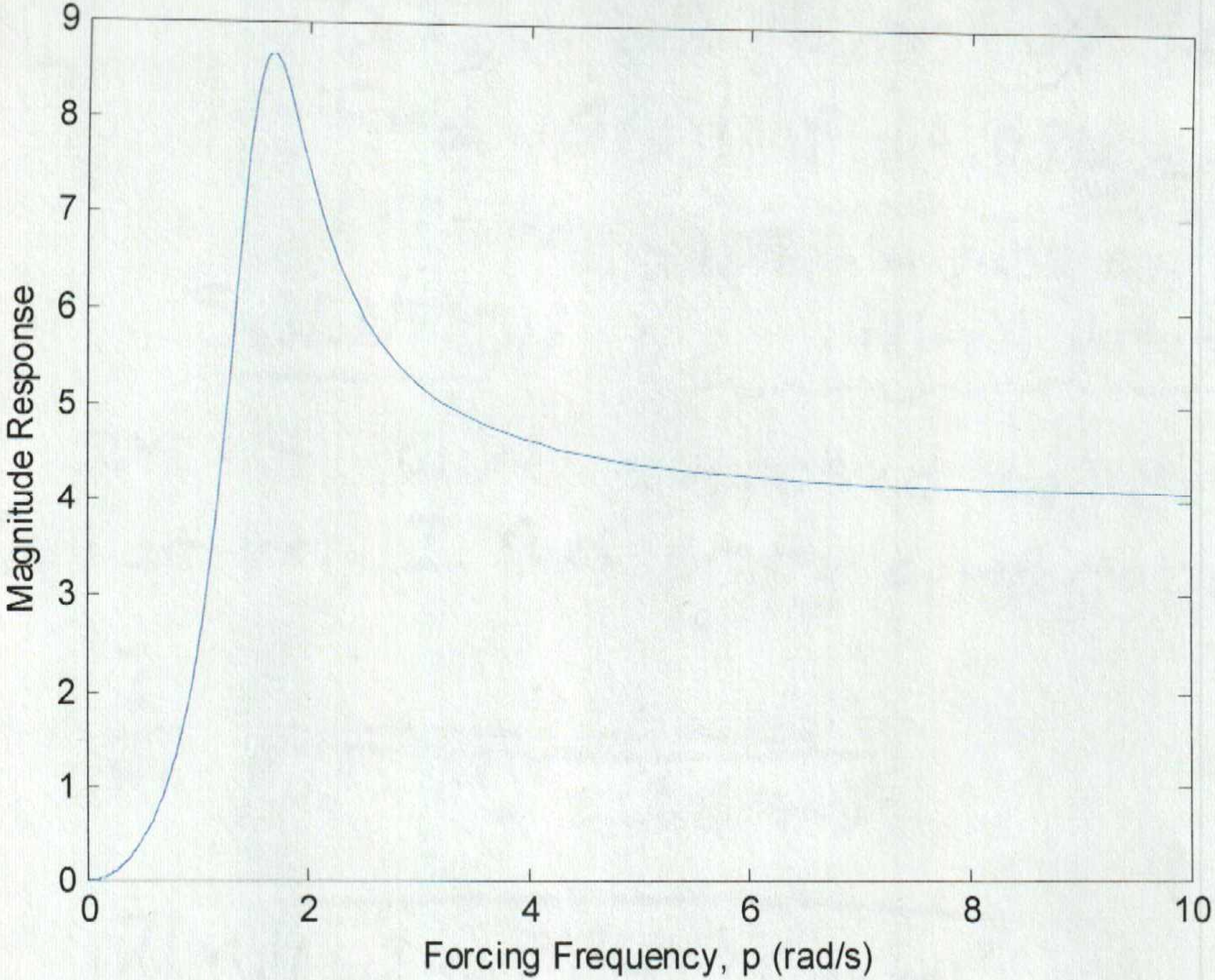
$$|z(t)|_{\omega_n << p} \approx \frac{\left[p^8 y_0^2 + 4\zeta^2 \omega_n^2 p^6 y_0^2 \right]^{1/2}}{p^4 - 4\zeta^2 \omega_n^2 p^2}$$

$$\approx \frac{\cancel{p^4}}{\cancel{p^4}} \cdot \frac{\left[y_0^2 + \cancel{4\zeta^2 y_0^2 \left(\frac{\omega_n}{p}\right)^2} \right]^{1/2}}{1 - \cancel{4\zeta^2 \left(\frac{\omega_n}{p}\right)^2}}$$

$0, \left(\frac{\omega_n}{p}\right)^2 << 1$

$\rightarrow 0, \left(\frac{\omega_n}{p}\right)^2 << 1$

$\approx \boxed{y_0}$ And my plot agrees!



$m = 2$

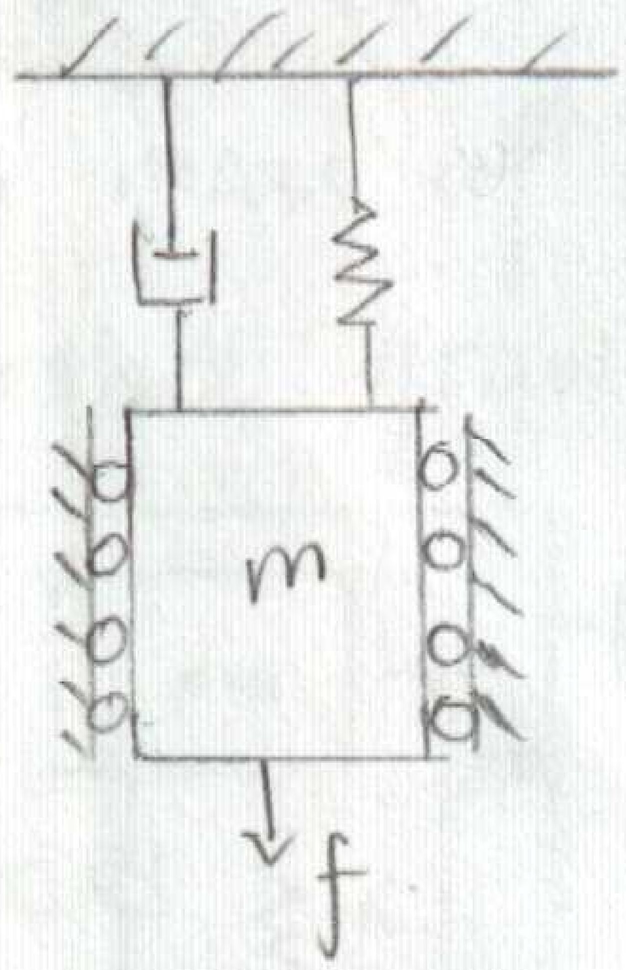
$k = 5$

$c = 1.5$

$y_0 = 4$

$\omega_n = 1.58 \text{ rad/s}$

2.79. The peak amplitude response $|g(\omega_p)|$ is equal to 5 cm for a direct force excited, damped SDOF system. $m = 0.03\,\text{kg}$, $c = 0.048\,\text{N}\cdot\text{s/m}$, $k = 12\,\text{N/m}$. Determine $|g(0)|$ MAE5130



SOLUTION:

$$\omega_p = \omega_n\sqrt{1-2\zeta^2} \qquad |g(\omega)| = \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}$$ from the book.

$$|g(\omega_p)| = \frac{1}{m\sqrt{(\omega_n^2 2\zeta^2)^2+(2\zeta\,\omega_n^2\sqrt{1-2\zeta^2})^2}} = \frac{1}{m\omega_n^2\sqrt{4\zeta^4+4\zeta^2(1-2\zeta^2)}} = \frac{1}{2\zeta m\omega_n^2\sqrt{1-\zeta^2}}$$

$$|g(0)| = \frac{1}{m\omega_n^2}$$

$$\left|\frac{g(\omega_p)}{g(0)}\right| = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \zeta = \frac{c}{2m\omega_0} = \frac{c}{2m\sqrt{\frac{k}{m}}} = \frac{0.048}{2\times0.03\times\sqrt{\frac{12}{0.03}}} = 0.04.$$

$$|g(0)| = |g(\omega_p)|\cdot 2\zeta\sqrt{1-\zeta^2} = 5\cdot 2\times0.04\times\sqrt{1-0.04^2} = \boxed{0.4\ \text{cm}}$$