EVOLVING DOCUMENT
TAM 4730/5730 Intermediate Dynamics and Vibrations
Homework

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Hyperlinks look like this. Dates in headings below are the dates of the associated lecture.

Wed. Aug 26

1. **Simplest dynamics with Polar coordinates.** This is the simplest dynamics problem, but posed in polar coordinates. Assume a particle is on a plane with no force on it. So, you know it moves at constant speed in a constant direction. Now write the differential equations
$$\vec{a} = \vec{0}$$
in polar coordinates. a) Solve them numerically for various initial conditions. b) Plot the solution and check that the motion is a straight line at constant speed. c) Using your numerical result, pick a way to measure how straight the path is, and see how straight a line your polar coordinate solution gives. d) Is the path more straight when you lower the numerical tolerances.

Wed. Sept 3

2. **Canon ball.** A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a viscous drag with magnitude $|cv|$.
   (a) Find position vs time analytically.
   (b) Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,\mathrm{m/s}$, $g = 1\,\mathrm{m/s^2}$, $m = 1\,\mathrm{kg}$.
   (c) Compare the numeric and analytic solutions. At $t = 2$ how big is the error? How does the error depend on specified tolerances or step sizes?
   (d) Use larger and larger values of v_0 and for each trajectory choose a time interval so the canon at least gets back to the ground. Plot the trajectories (using equal scale for the x and y axis. As $v \to \infty$ what is the eventual shape? [Hint: the answer is simple and interesting.]
   (e) For any given v_0 there is a best launch angle θ^* for maximizing the range. As $v_0 \to \infty$ to what angle does θ^* tend? Justify your answer as best you can with careful numerics, analytical work, or both.

3. **Mass hanging from spring.** Consider a point mass hanging from a zero-rest-length linear spring in a constant gravitational field.
   (a) Set up equations. Set up for numerical solution. Plot 2D projection of 3D trajectories.
   (b) By playing around with initial conditions, find the most wild motion you can find (wild means most wiggles, or most complicated). Make one or more revealing plots. [Hint: Make sure the features you observe are properties of the system

and not due to numerical errors. That is, check that the features do not change when the numerics is refined.]

(c) Using analytical methods justify your answer to part (b).

4. **Central force.** These two problems are both about central forces. One does not follow from the other.

   (a) Find a central force law so that for circular orbits the speed is independent of radius.

   (b) By numerical experiments, and trial and error, try to find a period motion that is *neither* circular nor a straight line for some central force *besides* $F = -kr$ or $F = -GmM/r^2$. In your failed searches, before you find a periodic motion, do the motions always have regular patterns or are they sometimes chaotic looking (include some pretty pictures)?

   To do this properly you probably need to do numerical root finding. Once you have your system you can define a function whose input is the initial conditions and the time of integration and whose output is the difference between the initial state and the final state. You want to find that input which makes the output the zero vector.

Wed. Sept 10

5. **Canon ball.** A cannon ball m is launched at angle θ and speed v_0. It is acted on by gravity g and a quadratic drag with magnitude $|cv^2|$.

   (a) Find a numerical solution using $\theta = \pi/4$, $v_0 = 1\,\mathrm{m/s}$, $g = 1\,\mathrm{m/s^2}$, $m = 1\,\mathrm{kg}$.

   (b) Numerically calculate (by integrating $\dot{W} = P$ along with the state variables) the work done by the drag force. Compare this with the change of the total energy. Make a plot showing that the difference between the two goes to zero as the integration gets more and more accurate.

6. **What means "rate of change of angular momentum"?** Consider a moving particle P. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$)?

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$$

   In each case say whether the definition works i) in general, or ii) for some special cases concerning the motions of P and C that you name.

   (a) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C'}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}}m$,
   where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.

   (b) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/0}}m$.

   (c) $\vec{\boldsymbol{H}}_{/C} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{v}}_{\mathrm{P/C}}m$.

   That is, for each possible definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\vec{\boldsymbol{r}}_{\mathrm{P/C}} \times m\vec{\boldsymbol{a}}_{\mathrm{P}/\mathcal{F}}$.

7. **Mechanics of two or more particles**

   (a) For two particles with mass m_1 and m_2 what is the period of circular motion if the distance between the particles is d and the only force is the force between them, $F = Gm_1m_2/r^2$?

   (b) Pick numbers for G, m_1, m_2 and r and, using appropriate initial conditions, test your analytical result with a numerical simulation. Make any plots needed to make your result convincing.

   (c) For three equal particles, $m_1 = m_2 = m_3 = 1$ and $G = 1$ what is the angular speed for circular motion on a circle with diameter of $d = 1$?

   (d) Check your result with a numerical simulation.

8. **Montgomery's eight.** (From Ruina/Pratap). Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

   and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

   For each of the problems below show accurate computer plots and explain any curiosities.

   (a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units.

   (b) Same as above, but run for 10 time units.

   (c) Same as above, but change the initial conditions slightly.

   (d) Same as above, but change the initial conditions more and run for a much longer time.

Wed. Sept 17

9. **Konig's Theorem** The total kinetic energy of a system of particles is

$$E_{\mathrm{K}} = \frac{1}{2}\sum m_i v_i^2.$$

   (a) Derive an expression of this form

$$E_{\mathrm{K}} = \frac{1}{2}m_{\mathrm{tot}}v_G^2 + \text{...you fill in the rest....}$$

(b) Is it always true that

$$\left(\sum \vec{F}^{ext}\right) \cdot \vec{v}_{\mathrm{G}} = \frac{d}{dt}\left(\frac{1}{2} m_{\mathrm{tot}} v_G^2\right)?$$

Defend your answer with unassailable clear reasoning (that is, a proof or a counter-example).

(c) Is it always true that the power of internal forces is equal to the rate of change of the quantity you filled in part (a) above (just the second half of the full expression)? Provide a proof or a counter-example. (A good solution is expected from those in 5730).

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10. **What means "rate of change of angular momentum" for a SYSTEM of particles?** Consider a system of moving particles with moving center of mass at G. Consider also a moving point C (moving relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions of $\vec{H}_{/\mathrm{C}}$ Is the following equation of motion true (that is, consistent with $\vec{F} = m\vec{a}$)?

$$\vec{M}_{\mathrm{C}} = \dot{\vec{H}}_{/\mathrm{C}}$$

In each case say whether the definition works i) in general, or ii) for some special cases concerning the motions of P and C that you name.

(a) $\vec{H}_{/C} = \sum \vec{r}_{\mathrm{i/C'}} \times \vec{v}_{\mathrm{i/C'}} m_i$,
where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.
(Hint: this definition is good one, always!)

(b) $\vec{H}_{/C} = \sum \vec{r}_{\mathrm{i/C}} \times \vec{v}_{\mathrm{i/0}} m_i$.
(This strange definition is used in the classic book by Housner and Hudson)

(c) $\vec{H}_{/C} = \sum \vec{r}_{i/C} \times \vec{v}_{i/C} m_i$.
(Hint: this is the most important candidate definition, but it's only good for special kinds of C, namely: C = COM, C is fixed and ...)

That is, for each possible definition of $\vec{H}_{/\mathrm{C}}$ you need to calculate $\dot{\vec{H}}_{/\mathrm{C}}$ by differentiation and see if and when you get $\sum \vec{r}_{\mathrm{i/C}} \times \vec{a}_{\mathrm{i/0}}$. If you are short for time just consider cases (a) and (c) and note their agreement if C is stationary or if C=G.

11. **Rotation with zero angular momentum.** This is a concrete example showing that a system can rotate while having zero angular momentum at all times. If you don't like this example, or want to do a different one for any reason, any substitution is fine. But it needs to be backed by a clear (numerical is fine) calculation.

Consider a massless rigid plate in space. On this plate are marked x and y axes. There are 4 equal masses attached to the plate. Two of them are welded (glued, fixed) to the plate on the y axis at $y = \pm 2$. Two of them are forced, by massless machinery that rides on the plate, to move in a circle with radius $r = 1$ centered at $x = \pm 2$, and starting at $x = \pm 3$, respectively. They move such that the line connecting them goes through the origin at all times. The right mass moves according to this equation, using polar coordinates centered at $y = 0, x = 2$:

$$r = 1, \qquad \theta = (1 - \cos(t))\,\pi \quad \text{for} \quad 0 \le t \le \pi.$$

That is, the mass goes around the circle once in a nice smooth motion. The other mass moves accordingly (same motion, reflected through the origin).

Use conservation of angular momentum for the system to find the net rotation ϕ of the masses on the y axis at the end of one rotation of the $x-$ axis masses. Numerical integration is fine. [Hint: The equation $\vec{\boldsymbol{H}}_{/0} = \vec{\boldsymbol{0}}$ tells you $\dot{\phi}(t)$.]

12. **Two masses** This problem has 2 independent educational goals:

    (a) Motivate the use of kinematic constraints.

    (b) Introduce the simplest of a class of vibrations problems you should master. At this point it is mastery of derivation of the equations. You should check that you can reproduce the lecture example with *no* sign errors without looking up anything.

    Two masses m_1 and m_2 are constrained to move frictionlessly on the x axis. Initially they are stationary at positions $x_1(0) = 0$ and $x_2(0) = \ell_0$. They are connected with a linear spring with constant k and rest length ℓ_0. A force is applied to the second mass. It is a step, or '**Heaviside**' function

$$F(t) = F_0 H(t) = \begin{cases} 0 & \text{if } t < 0 \\ F_0 & \text{if } t \geq 0 \end{cases}$$

    (a) Write code to calculate, plot and (optionally) animate the motions for arbitrary values of the given constants.

    (b) Within numerical precision, should your numerical solution always have the property that $F = (m_1 + m_2)a_G$ where $x_G = (x_1 m_1 + x_2 m_2)/(m_1 + m_2)$? (As always in this course, yes or no questions are not multiple choice, but need justification that another student, one who got the opposite answer, would find convincing.)

    (c) Use your numerics to demonstrate that if k is large the motion of each mass is, for time scales large compared to the oscillations, close to the center of mass motion.

    (d) 5730 only: Using analytic arguments, perhaps inspired by and buttressed with numerical examples, make the following statement as precise as possible:

    For high values of k the system nearly behaves like a single mass.

    Of course, in detail, the system has 2 degrees of freedom (DOF). So you are looking for a way to measure the extent to which the system is 1 DOF, and in which conditions (for which extreme values of parameters and times) the system is close to 1 DOF by that measure. There is not a simple single unique answer to this question.

Wed. Sept 24

13. **Two masses constrained** This is an elaboration of the problem above, replacing the two masses with a rigid rod. As per lecture, set up the DAEs and solve them using Matlab using numbers of your choice. Note the increasing error (as time progresses) in the satisfaction of the constraint. Compare this solution with the the method from

the problem above (where you use some very large value of k). Which one is faster? more accurate in predicting COM motion?

14. **Simple pendulum.** Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ as many ways as you can without looking anything up in books. For example, in all cases using polar coordinates,

    (a) linear momentum and manipulate the equations to eliminate constraint force
    (b) linear momentum, dot with $\hat{\boldsymbol{e}}_\theta$
    (c) linear momentum, cross with $\vec{\boldsymbol{r}}$
    (d) angular momentum
    (e) conservation of energy
    (f) power balance
    (g) Lagrange equations (if you know them already, not if you don't).

15. **Pendulum numerics.** Set up the pendulum in cartesian coordinates. Express the constant length constraint as a set of linear equations restricting the acceleration. Solve these (3 2nd order) DAE equations with numerical integration and initial conditions and parameters of your choosing. No polar coordinates allowed. Quantitatively compare your solution with a solution of the simple pendulum equations (For the comparison you need to either compute x from θ or *vice versa*. Integrate for a long enough time so you can detect drift away from satisfying the kinematic constraint.

16. **Pendulum with an awkward parameterization** By any means you like, for a simple pendulum find the equations of motion using y (horizontal position) as your parameterization of the configuration. That is, find a 2nd order differential equation determining $\ddot{y}$ in terms of y, $\dot{y}$ and physical parameters (g, m, ℓ). Using numerics, quantitatively compare the solution of this ODE with a solution of the simple pendulum equations (Note, you can assume the pendulum is hanging down, hence $x > 0$. This problem is different from the DAE problem above in that you should obtain a single 2nd order ODE, not a set of 3 equations.

17. **2D Dumbell.** Two equal masses $m = 1$ are constrained by a rod to be a distance $\ell = 1$ apart. At $t = 0$ they have equal and opposite velocities ($v = 1$) perpendicular to the rod. Use a set of 5 DAEs ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ & the constraint equation $(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell^2$ with numerical integration to find the subsequent motion. Use plots and/or animation to help debug your code. Using what you know about systems of particles (*e.g.,* momentum, angular momentum, constraint equation, energy) quantify as many different numerical errors as you can.

18. **Canonball.** A cannon fires a projectile that experiences gravity and quadratic drag $F_D = c_D v^2$). Cannonball mass $m = 1\,\mathrm{kg}$, acceleration due to gravity $g = 10\,\mathrm{m/s^2}$, and drag coefficient is $c_D = 0.5\,\mathrm{kg/\,m}$ (same example used in class).

    (a) Find a trajectory (using whichever numerical tool you choose, but dont just guess!) so that the projectile hits the ground at $x = 2\,\mathrm{m}$ given using a launch speed of $v_0 = 10\,\mathrm{m/s}$. Find the launch angle and a plot of the trajectory.

(b) Imagine you want an efficient cannon that can hit a point at least 2 m away with the minimum possible launch speed. Find the angle and launch velocity that achieve this.

19. (5730 only). Consider the Montgomery 8 (or not?) problem again (3 point masses mutually attracted by gravity) using $m_i = G = 1$. Find a periodic solution by optimizing both the initial conditions and simulation time so that the error between initial and final state (positions and velocities) goes to 0. Afterwards simulate for 5+ periods to demonstrate that your solution is periodic. The following initial conditions are a reasonable guess:

    Mass 1: X1=-0.755; Y1=0.355; Vx1=0.9955; Vy1=0.07855;

    Mass 2: X2=1.155; Y2=-0.0755; Vx2=0.1055; Vy2=0.4755;

    Mass 3: X3=-0.4055; Y3=-0.3055; Vx3=-1.1055; Vy3=-0.5355;

    A good initial guess for the period is 8 . Hint: Youll be most successful if you bound your search near to these values. If you dont use variable bounds (e.g. in FMINCON), youll probably find the trivial solution where everything equals 0. If you want to go on with this problem, you can try it again using Matlabs BVP4C command (which does this particular problem more elegantly).

Problems below not yet assigned

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20. **Braking stability** 2D, looking down. Consider the steering stability of a car going straight ahead with either the front brakes locked or the rear brakes locked. The steering is locked and straight ahead. For simplicity assume that the center of mass is at ground height between the front and back wheels. Assume that the locked wheels act the same as a single dragging point on the centerline of the car midway between the wheels.

    (a) Develop the equations of motion.

    (b) Set them up for computer solution.

    (c) For some reasonable parameters and initial conditions find the motion and make informative plots that answer the question about steering stability. Note, in this problem where there is no steady state solution you have to make up a reasonable definition of steering stability.

    (d) See what analytical results you can get about the steering stability (as dependent on the car geometry, mass distribution, the coefficient of friction and the car speed). As much as you have time and interest, illustrate your results with graphs and animations of numerical integrations.

    (e) **Hints.** Check that your governing equations reduce to the lecture equations when the friction is zero. Check special cases of the numerical solutions with solutions you know other ways. A challenge is to think of as many of these as you can (even if you don't check all of them). That is, for some special parameter values and/or initial conditions you know features of the solution (examples: 1) no friction means energy is conserved 2) with friction and no initial rotation rate slowing is with constant acceleration, etc).

21. **Review.** Spend 3 hours doing problems of your choice from Ruina/Pratap. Pick problems at the edge of your confidence level. Make sure to draw clear free body diagrams and to use clear problem setups and clear vector notation. Hand in your work.

22. **Review again.** Again spend 3 hours doing problems of your choice from Ruina/Pratap. Pick problems at the edge of your confidence level. Make sure to draw clear free body diagrams and to use clear problem setups and clear vector notation. Hand in your work.

23. **Double pendulum** Consider the double pendulum made with two bars. Hinges are at the origin 0 and the elbow E. For definiteness (and so we can check solutions against each other) both bars are uniform with the same length $\ell = 1$ (in some consistent unit system). $g = 10$. Neglect all friction and assume there are no joint motors.

    (a) Set up and numerically solve (there is no analytic solution) to the governing equations that you find using AMB. You may refer to lecture notes, but you should be able to do it on your own by the time you hand in the work. Assume that at $t = 0$ the upper arm is horizontal, sticking to the right, and the fore-arm is vertical up (like looking from the front at a driver using hand signals to signal a right turn). Integrate until $t = 10$. Draw the (crazy) trajectory of the end of the forearm.

    (b) Use Lagrange equations to find the governing equations and, either by comparing equations or by comparing numerical solutions, show that the governing equations are the same as those obtained using AMB.

24. **Mass in slot on turntable.** A rigid turntable (m_t, I_t) is free to rotate about a hinge at it's center. It has in it a straight frictionless slot that passes a distance d from it's center. A mass m_s slides in the slot. For minimal coordinates use rotation of the disk θ from the position in which the slot is horizontal and below the disk center, and the distance s the mass is from the point where the slot is closest to the center of the disk.

    (a) find the acceleration of the mass in terms of $d, \theta\dot{\theta}, \ddot{\theta}, s, \dot{s}$ and $\ddot{s}$. Do this three different ways and check that all give the same answer when reduced to x and y coordinates.

    i. Write the position of the mass in terms of d, θ and s using base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. Differentiate twice.

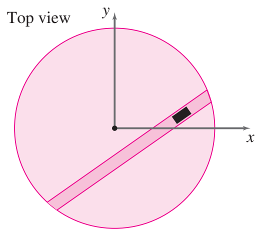
    ii. Write the position using $\hat{\boldsymbol{\imath}}'$, which aligns with the slot, and $\hat{\boldsymbol{\jmath}}'$. Differentiate twice using that $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}' = \omega \hat{\boldsymbol{\jmath}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}' = -\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}' = -\omega \hat{\boldsymbol{\imath}}'$

    iii. Use the five-term acceleration formula (using $\vec{\boldsymbol{v}}_{\text{rel}} = \dot{s}\hat{\boldsymbol{\imath}}'$ and $\vec{\boldsymbol{a}}_{\text{rel}} = \ddot{s}\hat{\boldsymbol{\imath}}'$).

    (b) Using the most convenient expression above, find the equations of motion (That is, find $\ddot{\theta}$ and $\ddot{s}$ in terms of fixed parameters and position and velocity variables.).

One way to do this would be to use AMB for the whole system about the center and to use {LMB for the mass}$\cdot\hat{\boldsymbol{\imath}}'$.

(c) Assume ICs that $s(0) = 0, \dot{s}(0) = v_0, \theta(0) = 0$ and $\dot{\theta}(0) = \omega_0 > 0$. As $t \to \infty$ does $\theta \to \infty$? (As for all questions, please explain in a way that would convince a non-believer.)



25. **Mass and spring vibration.** The harmonically forced vibration of a damped oscillator is given by this equation:

$$m\ddot{x} + c\dot{x} + kx = A\cos\omega_0 t + B\sin\omega_0 t$$

(a) Assume that the mass is connected to the spring and to the dashpot, the other ends of which are at C and D, respectively. Define x as the displacement of the mass in inertial space. For each of the cases below, find A and B. The latter three cases are from excitation by a moving base.

   i. C and D are fixed and a force $F = F_0 \sin\omega_0 t$ acts on the mass.

   ii. C is fixed and D oscillates with $\delta = \delta_0 \sin\omega_0 t$.

   iii. D is fixed and C oscillates with $\delta = \delta_0 \sin\omega_0 t$.

   iv. C and D oscillate together with $\delta = \delta_0 \sin\omega_0 t$.

(b) For the following problems, solve the governing equation above numerically using, say ODE45 using various appropriate forcings and initial conditions. For definiteness use the underdamped case $m = 1, c = 1, k = 1$.

   i. Set $A = 0, B = 0$ and $c = 0$. Using numerics find the natural frequency ω_n. Do this using 'events' (for the mass released from rest, find the time until the velocity gets to zero from above). Then calculate the frequency from this measure period. Compare the result with the analytic result $\omega_n = \sqrt{k/m}$.

   ii. Now set $A = 0, B = 0$ and $c = 1$ and find the damped natural frequency ω_d. Compare this with the analytic result.

   iii. Using 'events' that do not terminate the integration, and using the logarithmic decrement method, find the damping ratio ζ ('zeta'). Compare with the analytic result.

   iv. Draw a frequency response curve (each point on this curve requires a full simulation). For example, use A or $B = 0$ and the other equal to 1 and look at the amplitude of steady state response. The hard part here is running the simulation long enough so that the response is "steady state". Compare this curve with an analytically derived curve. Using the numerics, find, as accurately as you can, the frequency at which the amplitude of the steady state response is maximum.

v. Compare the three frequencies: 1) natural frequency, 2) damped natural frequency and 3) 'resonance' frequency (frequency which gives maximum amplitude response). Note their order and note how close, or not, they are to each other.

26. **Bead on parabolic wire** For a frictionless point-mass bead sliding on a rigid wire on the curve $y = cx^2$ with gravity in the $-y$ direction, find the equation of motion.

    (a) Derive the equations of motion using Lagrange equations. Use, say the projection of the position on the x axis as the generalized coordinate.

    (b) Derive the equations of motion using Newton's laws. First write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ with an unknown constraint force orthogonal to the wire. Then dot both sides with a vector tangent to the wire. You should get the same answer as for part (a) with a very similar amount of algebra.

    (c) Find the frequency of small vibrations (find a formula for this in terms of some or all of m,g and c.

27. **Three masses normal modes.** Three equal masses are in a line between two rigid walls. They are separated from each other and the walls by four equal springs.

    (a) Write the equations of motion in matrix form.

    (b) By guessing/intuition find one of the normal modes.

    (c) Using the MATLAB eig function find all three normal modes.

    (d) Using numerical integration, with masses released from rest with a normal mode shape ($\vec{\boldsymbol{x}}_0 = \vec{\mathbf{u}}_i$), show that you get normal mode (synchronous) oscillations.

28. **Double Pendulum normal modes.** Use your double pendulum solutions with the following simplifications:

    (a) both links have the same length ℓ;

    (b) all mass is in two equal point masses (one at the elbow, one at the hand), so $I_1 = I_2 = 0$;

    (c) linearize: drop all terms that involve products like $\dot{\theta}_1^2, \dot{\theta}_2^2$ or $\dot{\theta}_1^2$. For both θs replace $\sin\theta$ with θ and $\cos\theta$ with 1.

    Thus write the small amplitude double pendulum equations in this form:

$$M\ddot{\vec{\theta}} + K\vec{\theta} = \vec{\mathbf{0}}.$$

    Here, $\vec{\theta} = [\theta_1 \quad \theta_2]'$, and M and K are 2×2 symmetric matrices whose entries are expressions involving, m,g and ℓ. Use the normal mode approach (e.g., the problem above) to find the normal modes. Use one of these, with small amplitude, as as initial conditions for your full non-linear simulator and show that you get (nearly) synchronous motion.

29. **Normal modes, solving an IVP.** (IVP = Initial Value Problem). For your three mass system (above), find the motion if you are given initial positions and velocities. In some special cases (including at least one normal mode shape) check your motion against direct integration of the ODEs.

30. **Rolling cylinder** A uniform cylinder with mass m and radius r rolls without slip inside a cylinder with radius R. Gravity g pulls it down.

    (a) Are the full non-linear differential equations the same as those of a pendulum? If so, or not, explain why this is expected.

    (b) In terms of some or all of m, g, r and R find the frequency of small oscillation near the bottom.

31. **Cart and pendulum** A cart m_1 slides frictionlessly on a level surface. A massless stick with length ℓ is hinged to it with a mass m_2 at the end. Take $\theta = 0$ to be the configuration when the pendulum is straight down. Use gravity g.

    (a) Find the full non-linear governing equations at least two different ways and show that they agree.

    (b) Linearize the equations for small deviations from the configuration where the pendulum hangs straight down.

    (c) Write the equations in standard vibration form: $M\ddot{\vec{x}} + K\vec{x} = \vec{\mathbf{0}}$.

32. **Normal Mode Numerics** Much of this problem solution can be done by recycling previous solutions. Given M, K, x_0, v_0 one can find $x(t)$ and $v(t)$ three ways.

    (a) Write a matlab function `SOLVENUM`

    ```
    [xmatrix] = solvenum(M,K,x0,v0,tspan);
    ```

    The output is an array, each row of which are the values of x at the corresponding time in span. Use `ODE45` and any other functions you write.

    (b) Write a matlab function `SOLVEMODE`

    ```
    [xmatrix] = solvemode(M,K,x0,v0,tspan);
    ```

    that solves the same problem by using eigenvectors of $M^{-1}K$. Make sure your function works even when K is indefinite (even when there are motions that have no potential energy).

    (c) Write a matlab function `SOLVENORM`

    ```
    [xmatrix] = solvenorm(M,K,x0,v0,tspan);
    ```

    that solves the same problem by using eigenvectors of $M^{-1/2}KM^{-1/2}$. Make sure your function works even when K is indefinite (even when there are motions that have no potential energy).

    (d) Show that all three functions above give the same solution for the linearized cart and pendulum.

33. **Cylinder in a pipe.** . A thin-walled hollow cylinder with radius R and mass M rolls without slip on level ground. Inside it rolls, without slip, a disk with radius $r < R$ and mass m. Gravity g points down. Find the equations of motion (two ways if you have time and energy and would find it educational). Find the modes of small oscillation and their frequencies. One of these should make clear intuitive sense. Can you find at least one special case in which you can check the other? [For those interested in such: can you find a conservation law associated with the translation invariance of the governing equations? (There is one, but I don't know what it is)]

34. **Mass in slot on a turntable.** 2D. A turntable with mass M and moment of inertia I is held in place at its center with a bearing and a torsional spring k_t. Along one diameter of the disk is a slot in which a mass m slides with no friction. A zero-rest-length spring pulls it to the center with spring constant k. Find the equations of motion at least two different ways. Find the normal modes and frequencies.

35. **Normal modes by inspection.** For each of the systems below find as many normal modes, and their frequencies, as you can **without** doing matrix calculations. Then, if you like and can, check your work with matrix calculations.

    (a) 1D. Three equal masses in a line connected by two springs. No springs are connected to ground.

    (b) 3D. Two unequal masses, m_1 and m_2, are at points $\vec{r}_1$ and $\vec{r}_2$ in 3D space and are connected by one spring k. No springs are connected to ground.

    (c) 2D. 4 point masses are arranged in a square. The 4 edges are equal massless springs.

    (d) A regular hexagon has equal point masses at the vertices and equal springs on the edges. No springs are connected to ground. Just find one mode of vibration that does *not* have zero frequency.

    (e) 1D. An infinite line of equal point masses m is connected by an infinite line of equal springs k. One normal mode oscillation is given by

    $$\vec{v} = [\cdots -1 \quad 1 \quad -1 \quad 1 \quad -1 \ldots]'$$

    with $\omega = 2\sqrt{k/m}$. Find another mode and frequency. Challenge: find more. [Hint 1: this problem has impenetrably beautiful simple solutions which you might, with some luck, guess and, with some skill, check. Hint 2: If you assume the solution is periodic with period n then the stiffness matrix can be written as an $n \times n$ matrix. You can use this to numerically find normal modes which, if you look at them (say, plot the vector components vs their indecies) should reveal a pattern. Then you can use that same matrix to check if you detected the pattern correctly.]

36. **Intro to damped modes.** Consider the set of ODEs

$$\dot{z} = Az$$

where z is a list of scalar functions of time and A is a constant real matrix. Here you are to test, in Matlab, the basic theory of the solutions of equations like this.

(a) Generate a fairly random $n \times n$ matrix A using `RAND` or any other way. You can use any positive integer $3 \leq n \leq 100$ that pleases you.

(b) Find any eigenvalue λ and associated eigenvector v of A (these will undoubtedly be complex).

(c) For a sequence of, say, 100 or 1000 times, starting at $t = 0$, plot the real part of $e^{\lambda t} v_1$ versus t, where v_1 is the first component of the eigenvector. Pick a length of time where the curve is variable enough to be interesting, but not so variable that no details can be detected.

(d) Find the vector w which is the real part of v.

(e) Solve

$$\dot{z} = Az$$

using `ODE45` with the initial condition $z_0 = w$. Plot z(1) vs t from this solution and compare it with the plot above. If nothing pops out, someone made a mistake. Explain the interesting relation as best you can.

37. **Old Qualifying exams.** Four documents on the course WWW page have old qualifying exams. Pick a 2D dynamics problem from that set. This should be a problem that you are challenged by. Take the poorly worded question and sloppy figure and make a clear figure and clear question. Write up a clear solution. If this takes less than 4 hours, do another and another until 4 hours are used up.

38. **Damped normal modes.** Consider 3 equal masses in a line held between 2 walls by 4 equal springs. A single small dashpot ($c = .1\sqrt{mk}$) connects the leftmost mass from the wall at its left. Assume the initial velocity is zero. Assume the initial position is a mode shape with mass one having a displacement of 1. Plot position vs time for the first mass for all three mode shapes. Comment on the similarity and differences between the results for the three methods below. Make any other revealing plot(s) you can think of.

    (a) Numerical ODE solution (an arbitrarily exact numerical method).

    (b) Solution using first order odes and the matrix exponential (an exact method, numerically evaluated).

    (c) Solution using modal damping:

        i. Pick α and β so that $C = \alpha M + \beta K$ gives about the right decay rate for the fastest and slowest modes (a numerically evaluated analytic expression for the slightly wrong problem).

        ii. Use the change of coordinates that reduces undamped problem to diagonal form in terms of modal coordinates:

        $$\ddot{\vec{r}} + \underbrace{P' M^{-1/2} C M^{-1/2} P}_{\hat{C}} \dot{\vec{r}} + \Lambda \vec{r} = \vec{\mathbf{0}}.$$

        Only if you are lucky is this $\hat{C}$ diagonal (for example if $C = \alpha M + \beta K$). If it isn't, which it isn't for this HW problem, replace the $\hat{C}$ matrix with the diagonal part of $\hat{C}$. Then find the solution for each mode (A numerically evaluated analytic solution to a more nearby problem).

39. **a) Old Qualifying exam, 2nd try.**
    Observation: No honest student could say "Could do, but I won't learn from this."

    i Pick a 2D dynamics Q-exam problem, hopefully one that is not trivial for you.

    ii Don't write on page backs. Staple this problem separate from your other HW. Page 1: problem statement, Page 2: start of solution. No more than 3 pages total.

    iii This should not be a record of your brainstorming, it should be a clear write-up of a solution. (Practice brainstorming. Just don't show that here.)

    iv State assumptions. Describe methods if the calculation is too long for details to show. Name generalizations and how you could/would deal with them.

    v Write in such a way that you will make a competent reader, say a TA or professor in a course like this, think you are a clear thinker with mastery of the topic and good communication skills. Meanwhile convince another imagined skeptical member of this class that your solution is correct.

    vi MAIN INITIAL GOAL: Reduce the problem to one or more precisely defined mechanics problems. State reasonable assumptions. Don't need complete sentences. Clear sentence fragments & shorthand ok.

    vii SECONDARY GOAL: Use precise mechanics reasoning to solve, or set up solutions.

    viii THIRD GOAL: Describe why your solution does or does not make sense.

35. **b,c,d,...) Same as (a).**

40. **Rolling eccentric cylinder** A cylinder with radius R has center of mass G offset from the cylinder center C by a distance $d < R$. It has total mass M, radius R and moment of inertia I^G about it's center of mass. It rolls without slip down a ramp with slope γ, propelled by gravity g.

    (a) Find the equations of motion.
    (b) Find the needed coefficient of friction to enforce the rolling constraint.
    (c) After release from rest how far does it roll before it skips into the air.

    It's ok to use numerical solutions based on any non-trivial parameter choices. No need for parameter sweeps.

    Interesting extension if you have *lots* of time: With appropriate initial conditions, this can roll on a level ramp then skip, then do about a full revolution in the air, then land with no no relative velocity at impact (thus conserving energy) and continue rolling and skipping. This is explained, somewhat, in a paper on Ruina's www page: "A collisional model ..." (figure 4). The calculation details are like those in this paper "Persistent Passive Hopping ..."

41. **Modal forcing** This problem is interesting. Consider our favorite 3 mass system, from problem 38 above: 3 equal masses in a line separated by 4 springs. Assume that all springs are parallel to dampers with $c = .1\sqrt{mk}$. If you are short of time, leave off the damping (use $c = 0$). Now consider this problem.

    (a) The system starts from rest at $t = 0$.
    (b) A force $F = F_0 \sin(\sqrt{\frac{2k}{m}}t)$ is applied to the left mass. You can think of this as a small force if you like (although the words "small" and "large" have no meaning in the solution of linear problems).

    Now solve this problem various ways and notice some interesting features by making relevant plots.

    (a) Solve using your favorite Matlab ODE solver.
    (b) Plot the positions of all three masses as a function of time. Make two plots: 1) Use a long enough time scale so that you get a sense of steady state response; 2) Use a short enough time so you can see the amplitude growing.
    (c) Use modal forcing for the mode $\vec{v} = [1 \quad 0 \quad -1]'$. Note that for this diagonalizeable problem, the modal solution is exact.
    (d) Make similar plots. Note that in this solution the middle mass does not move at all.

    Main question: Given that the modal solution is exact. And given that in that modal solution the middle mass does not move. What makes the right mass move?

42. **Write a final exam question.** You should write a clear candidate final exam question on one page. You should write a clear solution on the following page(s), 2 page max. Single sided (don't write on backs). These 2 or 3 pages should be stapled together separate from your other homework problems. Your question should not be like that of anyone else you know. Hand printing and hand drawing are fine. Some

of these will be scanned and posted before the final exam. At least one from the scanned bunch, possibly slightly modified, will be on the final exam.

A good final exam question has these properties:

(a) The question is clear.
(b) Most of the people who could do it well in a relaxed 6 hours could do it decently in just 30 minutes.
(c) Most people who mastered all the prerequisite course material, all of the course homeworks, all the lecture material and all of the readings should be able to do it.
(d) Most people who had the pre-requisite courses but didn't take this course (or its equivalent) should not be able to do it.
(e) Getting the problem right should be indicative of having (hopefully useful) skills and knowledge related to this course. For example, asking "What did Professor Ruina dress up as on Halloween?" might fulfill all of the requirements above, but not this one.
(f) Not too many jokes in the problem statement or in the solution. If it's too cute people get annoyed.
(g) The solution should be maximally illuminating. With minimum reading effort, someone who doesn't know how to do the problem should learn and understand how.
(h) One good type of problem would be of a type that you couldn't do at the course start, can do well now, and wish was on the final exam.

43. **Final computation project.** Due at the end of the semester. This is an extension of the double pendulum homework. The minimal version is to simulate and animate both a triple pendulum and also a 4-bar linkage. For the triple-pendulum the equations of motion should be found two different ways. In both problems the numerical solutions should be checked as many ways as possible (Energy conservation, limiting cases where simple-pendulum motion is expected, etc). Optional extras are a) to simulate and animate more complicated mechanisms of your choice (e.g., 4,5, n link pendulum or closed kinematic loop) and b) to find periodic motions.

    Deliverables:

    (a) Send one zip file called YourName4730.zip or Yourame5730.zip. By midnight Dec 4.
        i. That should be a compressed version of a single folder.
        ii. In that folder should be a collection of Matlab files
        iii. In that folder should be a *README file explaining how to use the Matlab files. It should be VERY EASY to use the files for simple demonstrations
        iv. In that folder should be a file called REPORT.pdf. It could be made from WORD, LateX or scanned handwork, or any mixture of those. It should explain what you have done, how, and give sample output. This is the main demonstration of your effort. As appendices this should include your documented matlab files.
    (b) On Dec 4, you will have 5 minutes to demonstrate animations of your simulations on your own laptop. Sign up for time on the course www page.

Problems below not yet finalized nor assigned

44. **Damping of homogeneous Solution.** Consider 3 equal masses m and 4 equal springs k in line (wall-spring1-mass1-spring2-mass2-spring3-mass3-spring4-wall). There is also a small dashpot $c \ll 2\sqrt{mk}$ parallel to spring 1. A force $F = F_0 \sin \omega t$ is applied to mass 3. The system is released from rest at $\vec{x} = [1\ 1\ 1]'$. Pick numbers you like for m, k, c, F_0 and ω.

    (a) Using ODE45, find the positions of the three masses vs time. Plot for long enough time that the transient motion has died out.

    (b) Set $c = 0$ and find the steady state motion using matrix methods.

    (c) Make a plot that shows that the solutions to the two problems above are very close at long times.

    (d) Explain (That is, show that you understand the moral of this story. This was all explained in lecture in the context of vibration isolation, but was not written on the board.)

45. **Rolling cylinder moment of inertia.**

    (a) Consider a cylinder with center of mass at its center rolling down a ramp. Find the equations of motion using I^G (the moment of inertia about the center of mass) and I^C (the moment of inertia about the ground contact point).

    (b) Do the problem above, but with a cylinder whose center of mass is off-center. Why don't the two methods give the same answer for this problem, whereas they do for (a)?

46. **Relation between axis-angle and rotation matrix.** Given the formula $\vec{r} = [R]\vec{r} = \cos\theta\ \vec{r} + (1 - \cos\theta)(\mathbf{n} \cdot \vec{r})\mathbf{n} + \sin\theta\mathbf{n} \times \vec{r}$ derived in lecture, write the rotation matrix in terms of the components of $\mathbf{n}$ and θ.

47. **Rotation of a book.** Draw a book. Now imagine

    (a) Rotating it first 90° about the z axis and then 90° about the x axis.

    (b) Rotating it first 90° about the x axis and then 90° about the z axis.

    (c) Draw the book before and after for each of these two cases.

    (d) Find R and R^T for each of these cases.

    (e) Find $\mathbf{n}$ and θ for each of these cases, draw the vector $\mathbf{n}$ on the pictures of the book for both of these cases.

48. **Free motion of a rigid body** Given that $[I^G] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$, a perfectly general rigid body in principal co-ordinates

    (a) For the special case that $A = B$ and $C = 2A$ find the motion very near spinning about the symmetry axis. Animate a disk moving like that from your simulation. Draw a 2D projection of the path of two particles on the boundary of the disk.

Note that for small angles these are almost circular paths tilted slightly one with respect to the other. Can you think of a simple explanation for this?

(b) Now consider a more general case $C > B > A$. Set up for numerical solution and animation the motion of such a body. For pictoral purposes you can represent the body as an ellipsoid, a box, or a jumping jack, as you find convenient or fun. Check in simulation that for rotation nearly about the 1 and 3 axes that the rotation is stable. Check that rotation about the 2 axis is not stable. The instability should show in animation, and in a plot of the position of some point on the nominal axis of rotation. Look at the motion for initial conditions in ω of $[01\epsilon]$ and see if you can qualitatively predict the motion without looking at your computer animations [Hint: angular momentum and energy are both conserved but the motion $\omega = [010]$ is unstable.]

49. **Steady precession of a rolling disk.** A disk or radius r rolls in steady circles on a ground track with radius R. The key is the ground contact condition that the velocity of the point on the disk which touches the ground is zero.

    (a) For a given disk, find all solutions. There is a 2 parameter family of them.
    (b) Assume that, instead of rolling, the disk slides with no friction (and the ground contact point need not have zero velocity). Find all steady solutions where the ground contact traces a circle. There is a 2 parameter family of these.
    (c) Find all solutions common to the two cases above.
    (d) Take a coin or flat disk and roll it on a flat ground. Look at it as it shutters. Which of the solutions above do you think the disk actually tracks? Why?

50. **Rolling cylinder**

    (a) Consider a cylinder with center of mass at its center rolling down a ramp. Find the equations of motion using I^G (the moment of inertia about the center of mass) and I^C (the moment of inertia about the ground contact point.
    (b) Do the problem above, but with a cylinder whose center of mass is off-center. Why don't the two methods give the same answer for this problem, whereas they do for (a)?

51. **Rolling disk, general motion** You must give a final demonstration of derivations and functioning software (30 min per student). Do these parts in any order that pleases you. Acknowledge clearly the source of all help that you got.

    (a) Derive the equations of motion of a rolling disk at least 2 different ways (not just steady precession, but general rolling motion). Some options: Newton-Euler with Euler angles, N-E with rotation matrices, Lagrange with Lagrange multipliers, viscous contact with the limit of viscosity going to zero, etc.
    (b) Set up the equations you get by both means for simulation.
    (c) Check that for some fairly arbitrary initial condition that both solutions above give the same motion.
    (d) See that at least one set of your equations reduces to the steady precession equations that you previously derived by other means.
    (e) Animate one of your solutions (showing a disk, the plane, and the path made by the contact point.).

(f) Check that Energy is conserved in your simulation.
(g) Show, in your simulations, that fast rolling is stable and slow rolling is not stable.
(h) Show by analysis that fast rolling is stable and slow rolling is not stable.

Problems below will not be assigned in Fall 2014

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52. **Euler's method.** If needed, review Euler's method of numerical solution of ordinary differential equations (ODEs) in your ODE text or from the WWW. Consider the differential equation $\dot{x} = x$ with initial conditions $x_0 = 1$ solved over the interval $0 \leq t \leq 1$. In all cases write your own code and do not use any Matlab ODE solvers.

    (a) Use any time step you like to calculate $x(1)$ and check that the result is reasonably close to $e \approx 2.718281828...$.
    (b) Solve the equation many times using time steps of $h = 10^{-n}$ there n = 0, 1, 2, ... as high as your computer and patience will allow. [hint: for large values of n storing intermediate values in the calculation is time consuming. And if you do store intermediate values, initialize variables with code like:

    ```
    x = zeros(1,10^n).
    ```

    Note that large n will take much time in any case.
    (c) Plot the error vs n on a log-log plot.
    (d) For what value of n is the error smallest?
    (e) Can you rationalize that result?
    (f) If you did not know the analytic result, how could you determine the optimal value of n for the most accurate solution?