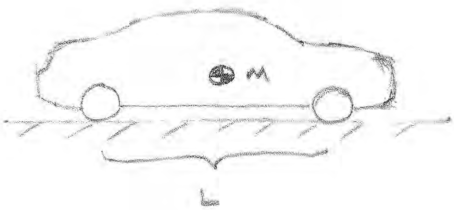
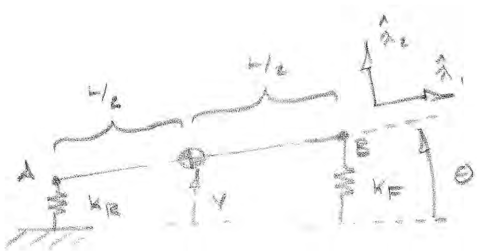
MATT BYRNE
MAE 5735
FINAL EXAM Q.



Problem Statement: Given a car traveling along a path, determine the equations of motion. Assume two dimensional analysis

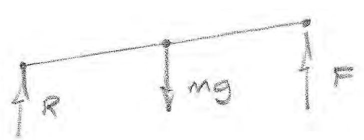
Solution: Assumptions – simple suspension (i.e. tires, springs and chassis modeled by single spring at either contact patch)
– suspension is undamped
– chassis has equal weight distribution



$\hat{j}$ $\hat{i}$

$\hat{\lambda}_1 = \hat{i}\cos\theta + \hat{j}\sin\theta$
$\hat{\lambda}_2 = -\hat{i}\sin\theta + \hat{j}\cos\theta$

$$\vec{r}_{A/G} = y\hat{j} - \tfrac{L}{2}\hat{\lambda}_1 = y\hat{j} - \tfrac{L}{2}\cos\theta\,\hat{i} - \tfrac{L}{2}\sin\theta\,\hat{j}$$
$$= (y - \tfrac{L}{2}\sin\theta)\hat{j} - \tfrac{L}{2}\cos\theta\,\hat{i}$$
$$\vec{r}_{B/G} = y\hat{j} + \tfrac{L}{2}\hat{\lambda}_1 = y\hat{j} + \tfrac{L}{2}\cos\theta\,\hat{i} + \tfrac{L}{2}\sin\theta\,\hat{j}$$
$$= (y + \tfrac{L}{2}\sin\theta)\hat{j} + \tfrac{L}{2}\cos\theta\,\hat{i}$$



$$\Delta\vec{r}_{A/G} = (y - \tfrac{L}{2}\sin\theta)\hat{j} - \tfrac{L}{2}(\cos\theta - 1)\hat{i}$$
$$\Delta\vec{r}_{B/G} = (y + \tfrac{L}{2}\sin\theta)\hat{j} + \tfrac{L}{2}(\cos\theta - 1)\hat{i}$$

Apply small angle approximation:

$$\Delta\vec{r}_{A/G} = (y - \tfrac{L}{2}\theta)\hat{j} \qquad \Delta\vec{r}_{B/G} = (y + \tfrac{L}{2}\theta)\hat{j}$$

I wrote on the back.
I wasn't thinking...

Calculate forces applied at tips:

$$\left.\begin{aligned} R &= k_R \, \Delta \bar{r}_{A/G} = k_R \left(\frac{mg}{2k_R} - y + \tfrac{L}{2}\theta \right) \\ F &= k_F \, \Delta \bar{r}_{B/G} = k_F \left(\frac{mg}{2k_F} - y - \tfrac{L}{2}\theta \right) \end{aligned}\right\} \text{extra term from static deflection}$$

Apply linear momentum balance in $\hat{j}$ direction:

$$\sum F = \left[k_R \left(\frac{mg}{2k_R} - y + \tfrac{L}{2}\theta \right) + k_F \left(\frac{mg}{2k_F} - y - \tfrac{L}{2}\theta \right) - mg \right] \hat{j} = m\ddot{y}\,\hat{j}$$

$$\longrightarrow m\ddot{y} + (k_R + k_F)y + (k_F - k_R)\tfrac{L}{2}\theta = 0$$

Apply angular momentum balance about com:

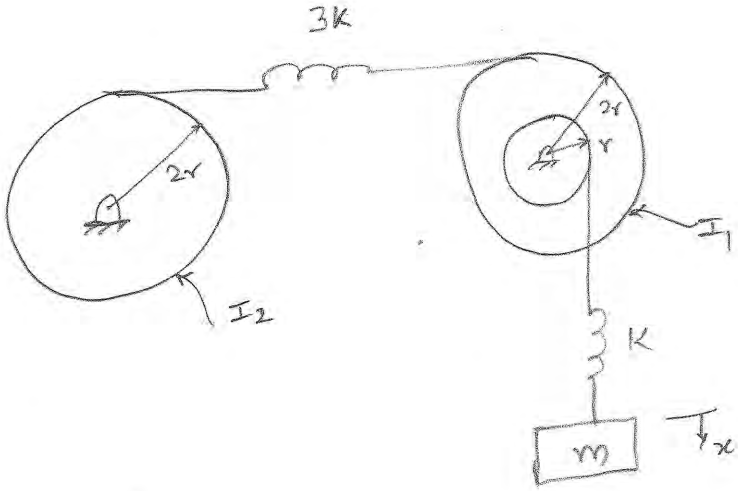
$$\bar{r}_{A/G} \times \bar{R} + \bar{r}_{B/G} \times \bar{F} = I\ddot{\theta}$$

$$\left(\frac{-Lk_R}{2} \right)\left(\frac{mg}{2k_R} - y + \frac{L\theta}{2} \right) + \frac{Lk_F}{2}\left(\frac{mg}{2k_F} - y - \frac{L\theta}{2} \right) = I\ddot{\theta}$$

$$\longrightarrow I\ddot{\theta} + \left[\frac{L^2(k_R + k_F)}{4} \right]\theta + \left[\frac{L(k_F - k_R)}{2} \right] y = 0$$

Arkun Chiplunkar

#36 Sample exam Problem



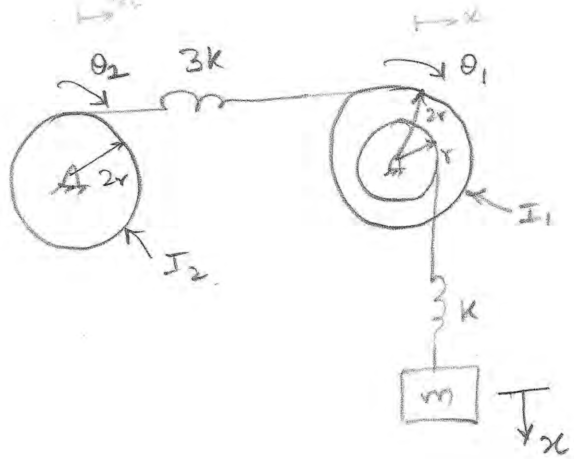
I_1, I_2 are moment of inertias of the big rollers.

(a) What is the DOF of the system? Suggest the generalized coordinates.

(b) Derive the equations of motion for the system using any method learned in class.

(c) Write the EOMs in matrix form.

Solution:



(a) DOF is $\underline{3}$. Generalized coordinates θ_1, θ_2, x

(b) Deriving using lagrange's method

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m \dot{x}^2$$

$$= \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} 3K (x_1 - x_2)^2 + \frac{1}{2} K (x - r\theta_1)^2$$

$$= \frac{1}{2} 3K (2r\theta_1 - 2r\theta_2)^2 + \frac{K}{2} (x - r\theta_1)^2$$

$$\therefore L = T - V = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m \dot{x}^2 - \frac{3K}{2} (2r\theta_1 - 2r\theta_2)^2 - \frac{K}{2} (x - r\theta_1)^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = I_1 \ddot{\theta}_1 + \frac{3K}{\cancel{2}} \cdot \cancel{2} (2r\theta_1 - 2r\theta_2) \cdot 2r - \frac{K\cancel{2}}{\cancel{2}} (x - r\theta_1)(r) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = I_2 \ddot{\theta}_2 - \frac{3K}{\cancel{2}} \cdot \cancel{2} (2r\theta_1 - 2r\theta_2) \cdot 2r = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + K(x - r\theta_1) = 0$$

(c)
$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{x} \end{bmatrix} + K \begin{bmatrix} 13r^2 & -12r^2 & -Kr \\ -12r^2 & 12r^2 & 0 \\ -Kr & 0 & K \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider our favorite 3 mass system: 3 equal masses in a line separated by 4 springs. However, these springs are special. Instead of obeying $V = \frac{1}{2}kx^2 \implies F_s = kx$, they obey $V = kg(x)$ $\implies F_s = k\frac{dg}{dx}$, where $g(x)$ is an arbitrary function. What are the approximate normal modes of the system?

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Problem # 38. A Final Exam Question

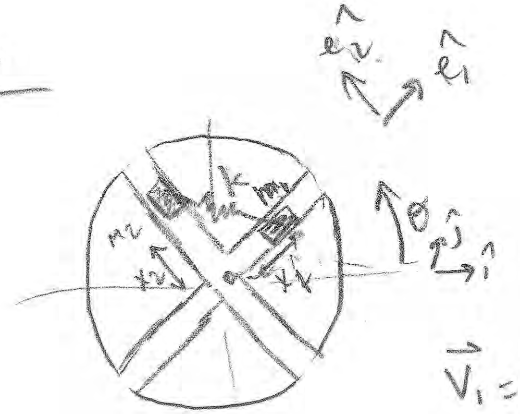
A rigid turntable (m_t, I_t) is free to rotate about a hinge at it's center. It has in it two straight frictionless slots that pass through its center. These two slots are perpendicular to each other. In each slot, there is a mass that slides (m_1, m_2). Use rotation of disc θ from the position in which the first slot is horizontal and the second slot is perpendicular, and the distances s_1 and s_2 the masses are from the hinge.
Finally, a spring (k) connects these two masses. It has zero rest length.

a) Find the accelerations of the masses in terms of $\theta, \dot{\theta}, \ddot{\theta}, x_1, \dot{x}_1, \ddot{x}_1, x_2, \dot{x}_2, \ddot{x}_2$. You can use $\hat{e}_1$ which aligns with the first slot and $\hat{e}_2$ which aligns with the second slot.

b) Find the equations of motion in any method you would like.

Solution

Part a)



Mass 1

$$\vec{r}_1 = x_1 \hat{e}_1$$

$$\vec{V}_1 = \dot{\vec{r}}_1 = \dot{x}_1 \hat{e}_1 + x_1 \dot{\theta} \hat{e}_2$$

$$\vec{a}_1 = \ddot{\vec{r}}_1 = \ddot{x}_1 \hat{e}_1 + \dot{x}_1 \dot{\theta} \hat{e}_2 + \dot{x}_1 \dot{\theta} \hat{e}_2 + x_1 \ddot{\theta} \hat{e}_2 - x_1 \dot{\theta}^2 \hat{e}_1$$

$$\boxed{\vec{a}_1 = \left(\ddot{x}_1 - x_1 \dot{\theta}^2\right)\hat{e}_1 + \left(2\dot{x}_1 \dot{\theta} + x_1 \ddot{\theta}\right)\hat{e}_2}$$

Mass 2

$$\vec{r}_2 = x_2 \hat{e}_2$$

$$\vec{V}_2 = \dot{\vec{r}}_2 = \dot{x}_2 \hat{e}_2 - x_2 \dot{\theta} \hat{e}_1$$

$$\vec{a}_2 = \ddot{\vec{r}}_2 = \ddot{x}_2 \hat{e}_2 - \dot{x}_2 \dot{\theta} \hat{e}_1 - \dot{x}_2 \dot{\theta} \hat{e}_1 - x_2 \ddot{\theta} \hat{e}_1 + x_2 \dot{\theta}^2 \hat{e}_2$$

$$\boxed{\vec{a}_2 = \left(-2\dot{x}_2 \dot{\theta} - x_2 \ddot{\theta}\right)\hat{e}_1 + \left(\ddot{x}_2 - x_2 \dot{\theta}^2\right)\hat{e}_2}$$

Part b) Use Lagrange Equations

$$T = \frac{1}{2} I_t \dot{\theta}^2 + \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \Rightarrow \text{Kinetic Energy}$$

$$V_1^2 = \vec{V}_1 \cdot \vec{V}_1 = \left\|\left(\dot{x}_1 \hat{e}_1 + x_1 \dot{\theta} \hat{e}_2\right)\right\|^2 = \dot{x}_1^2 + x_1^2 \dot{\theta}^2$$

$$V_2^2 = \vec{V}_2 \cdot \vec{V}_2 = \left\|\left(\dot{x}_2 \hat{e}_2 - x_2 \dot{\theta} \hat{e}_1\right)\right\|^2 = \dot{x}_2^2 + x_2^2 \dot{\theta}^2$$

$$\therefore T = \frac{1}{2} I_t \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 x_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 x_2^2 \dot{\theta}^2$$

$$= \frac{1}{2}\left(I_t + m_1 x_1^2 + m_2 x_2^2\right)\dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k \| \vec{r}_1 - \vec{r}_2 \|^2 = \frac{1}{2} k \| x_1 \hat{e}_1 - x_2 \hat{e}_2 \|^2 = \frac{1}{2} k (x_1^2 + x_2^2)$$

$$\mathcal{L} = T - V = \frac{1}{2}(I_t + m_1 x_1^2 + m_2 x_2^2)\dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_1^2 + x_2^2)$$

- $$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (I_t + m_1 x_1^2 + m_2 x_2^2)\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (I_t + m_1 x_1^2 + m_2 x_2^2)\ddot{\theta} + (2m_1 x_1 \dot{x}_1 + 2m_2 x_2 \dot{x}_2)\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \boxed{(I_t + m_1 x_1^2 + m_2 x_2^2)\ddot{\theta} + (2m_1 x_1 \dot{x}_1 + 2m_2 x_2 \dot{x}_2)\dot{\theta} = 0}$$

Eq 1

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{\partial \mathcal{L}}{\partial x_1} = m_1 x_1 \dot{\theta}^2 - k x_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \ddot{x}_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow \boxed{m_1 \ddot{x}_1 - m_1 x_1 \dot{\theta}^2 + k x_1 = 0}$$

Eq 2

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = m_2 x_2 \dot{\theta}^2 - k x_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \ddot{x}_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} - \frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow \boxed{m_2 \ddot{x}_2 - m_2 x_2 \dot{\theta}^2 + k x_2 = 0}$$

Eq 3

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HW13

Problem 38 - Write your exam question.

Consider the following perpetual motion toy fixed about point O.

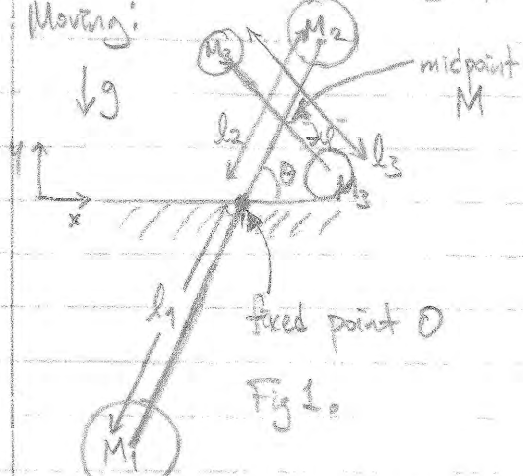


Fig 1.

There are two main components to the toy:

1. The main arm of length (l_1+l_2), with point masses m_1 & m_2 at the either end
2. A side arm of length l_3 with point masses m_3 & m_3 at the either end.

The two arms intersect at the midpoint of l_2 & l_3.

a) Find the equation of motion for θ & φ (θ & φ are measured to be positive for counterclockwise rotations. Thus, in the picture shown, θ is positive, while φ is negative, and $-\varphi$ is just shown in the diagram to explicitly denote this fact)

Angle def:

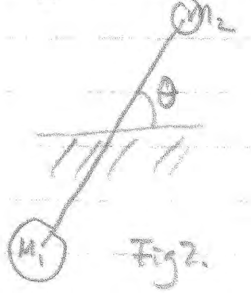
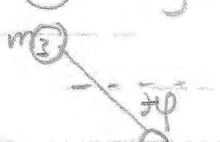


Fig 2.



b) In this problem, the rotation of the main bar was about some fixed point O. However, in commercial designs, perpetual motion toys are never made this way. Rather, the main bar cradles in a half-circle guide as shown in Fig 3. From the result in part (a), briefly explain why this may be the case.
(Trick question... no short answer is possible).

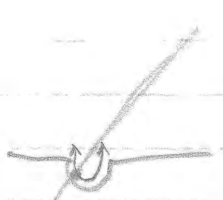


Fig 3

Sungkyu

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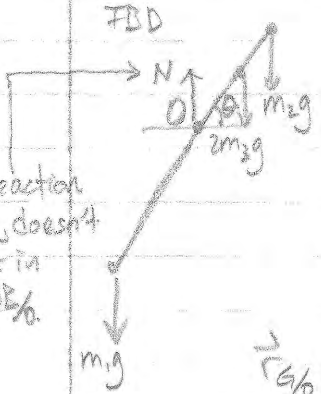
HW13

a) AMB of the whole system about point O



$$\sum \vec{M}_{/O} = (l_1 m_1 g - l_2 m_3 g - l_2 m_2 g)\cos\theta \, \hat{k}$$

$$\sum \dot{\vec{H}}_{/O} = \vec{r}_{G/O} \times m\vec{a}_{G/O} + I^G \ddot{\theta} \hat{k}$$

Center of mass will always be along the main bar. Taking the weighted average:

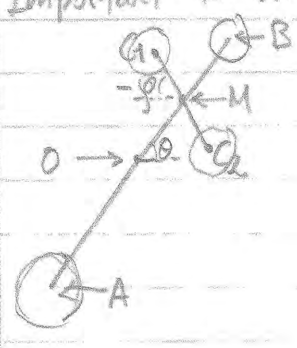
$$\vec{r}_{G/O} = \left(\frac{l_2 m_3 + l_2 m_2 - l_1 m_1}{l_1 + l_2}\right)(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath}) \Big/ (m_1 + m_2 + 2m_3)$$

$$\vec{v}_{G/O} = \left(\frac{l_2 m_3 + l_2 m_2 - l_1 m_1}{l_1 + l_2}\right)(-\sin\theta \hat{\imath} + \cos\theta \hat{\jmath})\dot{\theta} \Big/ (m_1 + m_2 + 2m_3)$$

$$\vec{a}_{G/O} = \left[-\vec{r}_{G/O} \cdot \dot{\theta}^2 + \left(\frac{l_2 m_3 + l_2 m_2 - l_1 m_1}{l_1 + l_2}\right)(-\sin\theta \hat{\imath} + \cos\theta \hat{\jmath})\ddot{\theta}\right] \Big/ (m_1 + m_2 + 2m_3)$$

$$\Rightarrow \sum \dot{\vec{H}}_{/O} = \left[(m_1 + m_2 + 2m_3)^{-1}\left(\frac{l_2 m_3 + l_2 m_2 - l_1 m_1}{l_1 + l_2}\right)^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + I^G \ddot{\theta}\right]\hat{k}$$

Important to note here that I^G changes depending on the configuration



$$I^G = m_1(\vec{r}_{A/G})\cdot(\vec{r}_{A/G}) + m_2(\vec{r}_{B/G})\cdot(\vec{r}_{B/G}) + m_3(\vec{r}_{C_1/G})\cdot(\vec{r}_{C_1/G}) + m_3(\vec{r}_{C_2/G})\cdot(\vec{r}_{C_2/G})$$

$$\vec{r}_{A/G} = \vec{r}_{A/O} + \vec{r}_{O/G} \quad \text{where } \vec{r}_{O/G} = -\vec{r}_{G/O}$$
$$= l_1(-\cos\theta \hat{\imath} - \sin\theta \hat{\jmath}) - \vec{r}_{G/O}$$

$$\vec{r}_{B/G} = \vec{r}_{B/O} - \vec{r}_{G/O} = l_2(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath}) - \vec{r}_{G/O}$$

$$\vec{r}_{C_1/G} = \vec{r}_{C_1/O} - \vec{r}_{G/O} = \vec{r}_{C_1/M} + \vec{r}_{M/O} - \vec{r}_{G/O}$$

note current value of φ in the configuration is negative. → where $\vec{r}_{C_1/M} = (\cos(-\varphi)\hat{\imath} + \sin(-\varphi)\hat{\jmath})(l_3/2)$

$$\vec{r}_{M/O} = \left(\tfrac{1}{2} l_2\right)(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath})$$

$$\vec{r}_{C_2/G} = \vec{r}_{C_2/M} + \vec{r}_{M/O} - \vec{r}_{G/O}, \quad \vec{r}_{C_2/M} = -\vec{r}_{C_1/M}$$

Putting everything together gives (and cancelling $\hat{k}$)

$$\cos\theta = f(\theta, \varphi, \ddot{\theta}) \quad \text{or} \quad \ddot{\theta} = f(\theta, \varphi)$$

where we don't bother with all the algebra.

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HW13

<u>Angular momentum balance about point M' that instantaneously matches with point M (for the second bar)</u>

$\sum M_{/M'} = 0$ (symmetry).

$$\sum \dot{H}_{/M'} = \cancel{\vec{r}_{G/M'}}^{\,0} \times m\vec{a}_{G/M'} + I^G \ddot{\varphi}$$

Here, $I^G = m_3 (l_3/2)^2 \times 2$, but more importantly: $\ddot{\varphi} = 0. \rightarrow \varphi = \varphi_0 + \dot{\varphi}_0 t$.

And thus, we have the equation of motion on θ & φ.

b) It is because the φ movement is too boring without this additional complication to the geometry of the problem, and commercial perpetual motion toys have to look cooler...
Except NOT!
Changing that geometry will have no effect on the AMB for φ.

Naturally, question arises... How does φ change in commercial toys? A quick search on youtube for Mars-Perpetual Motion will show a VERY convincing evidence (at least initially) that φ is unaffected by θ. The secret lies in the fact that commercial toys use batteries & magnetic masses, which the AMB formulation does not grasp.

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HW 38

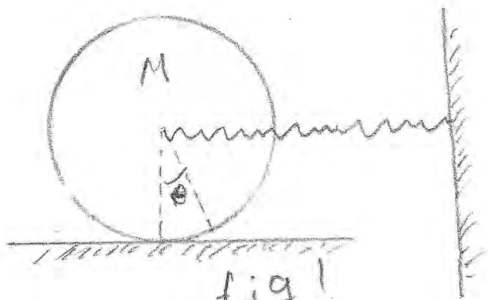
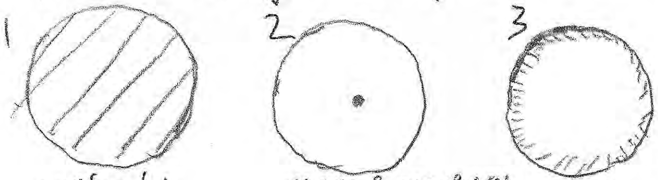


fig 1

Final Exam problem:

There are 3 type of sphere all with mass M



uniformly distributed mass. | mass concentrated at the center | hollow sphere.

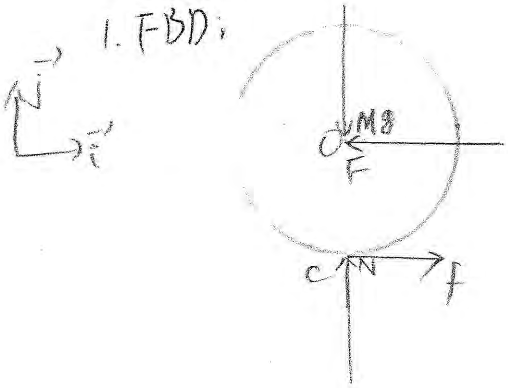
As shown in Fig. 1, sphere is initially at rest, and connected to a spring. The spring is now compressed with length of Δx. Assume no slip.

Question:

1. without calculations try to explain which kind of sphere can acquire fastest velocity and which can get largest magnitude?
2. Try to get the equation of motion with at least 3 different methods, solve the equation to ~~show~~ prove your conclusions in Question 1.

Solution:

1. FBD:



Consider the whole system, because there is no displacement corresponding to the friction, the energy is conserved. for ~~both~~ all 3 type of sphere, they have same energy, so the magnitude should be equal. When the sphere start to roll, some of the energy will be distributed for rolling, which is $\frac{1}{2}I\dot{\theta}^2$, so with smaller I, the velocity will be greater.

So, all magnitude are equal.

Concentrated mass (type 2) sphere will get the fastest velocity in the vibration.

2. AMB/C: $\vec{M} = \dot{\vec{H}}$

$\therefore k(\Delta x - \theta R)R\vec{k} = R\vec{j} \times MR\ddot{\theta}(-\vec{i}) + I\ddot{\theta}\vec{k} = (I + MR^2)\ddot{\theta}\vec{k}$

$\therefore (I + MR^2)\ddot{\theta} + kR^2\theta = k\Delta x.$

Conservation of Energy: $\frac{d}{dt}E_{tot} = 0$

$E_{tot} = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M(R\dot{\theta})^2 + \frac{1}{2}k(\Delta x - R\theta)^2$

$\therefore \frac{d}{dt}E_{tot} = I\dot{\theta}\ddot{\theta} + MR^2\dot{\theta}\ddot{\theta} + k(\Delta x - R\theta)(-R)\dot{\theta} = 0$

$\therefore (I + MR^2)\ddot{\theta} + kR^2\theta = k\Delta x$

Lagrange Eq: $\mathcal{L} = E_k - E_p$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$

$E_k = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(R\dot{\theta})^2$. $\therefore \frac{d}{dt}(I\dot{\theta} + mR^2\dot{\theta}) = -k(\Delta x - R\theta)(-R)$

$= k(\Delta x - R\theta)R$

$E_p = \frac{1}{2}k(\Delta x - R\theta)^2$

$\therefore (I + MR^2)\ddot{\theta} = k(\Delta x - R\theta)R \quad \therefore (I + MR^2)\ddot{\theta} + kR^2\theta = k\Delta x R$

~~guess θ = A sin~~

$\theta = \frac{k\Delta x R}{kR^2} + A\sin\sqrt{\frac{kR^2}{I+MR^2}}\,t + B\cos\sqrt{\frac{kR^2}{I+MR^2}}\,t$

$= \frac{\Delta x}{R} + A\sin\sqrt{\frac{kR^2}{I+MR^2}}\,t + B\cos\sqrt{\frac{kR^2}{I+MR^2}}\,t$

$\therefore \theta|_{t=0} = \frac{\Delta x}{R} + B = 0 \qquad \therefore B = -\frac{\Delta x}{R}$

$\dot{\theta}|_{t=0} = A\omega\cos\omega t = 0 \qquad A = 0$

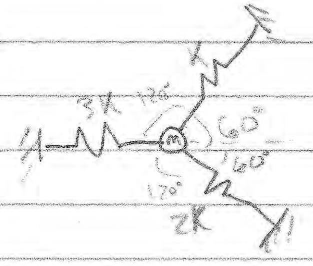
$\boxed{\therefore \theta = \frac{\Delta x}{R} - \frac{\Delta x}{R}\cos\sqrt{\frac{kR^2}{I+MR^2}}\,t}$

∴ No matter what I is, magnitude is the same, but with smaller I, $\dot{\theta} = \frac{\Delta x}{R}\cdot\sqrt{\frac{kR^2}{I+MR^2}}\sin\sqrt{\frac{kR^2}{I+MR^2}}\,t$ is higher.

Francis Lora
FML37
MAE 5735
Due 12/3/12

36 + 37 Busy work

38



a) find equation of motion for the mass supported by 3 springs spaced equilaterally.
b) put in M and K matrices
c) find the natural frequencies
d) find Modes

$\hat{j}$, $\hat{i}$

a) $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$V = \frac{1}{2}(3K)x^2 + \frac{1}{2}(2K)(\Delta l_{2K})^2 + \frac{1}{2}K(\Delta l_K)^2$

$\Delta l_{2K} = x\cos(60) - y\sin(60) = \frac{1}{2}x - \frac{\sqrt{3}}{2}y$

$\Delta l_{2K}^2 = \frac{1}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$

$\Delta l_K = x\cos(60) + y\sin(60) = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$

$\Delta l_K^2 = \frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$

$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{3}{2}Kx^2 - K(\frac{1}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2) - \frac{K}{2}(\frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2)$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - K(\frac{3}{2} + \frac{1}{4} + \frac{1}{8})x^2 + K(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4})xy + (\frac{3}{8} + \frac{3}{4})Ky^2$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{15}{8}Kx^2 + \frac{\sqrt{3}}{4}Kxy + \frac{9}{8}Ky^2$

$\frac{d\mathcal{L}}{dx} = -\frac{15}{4}Kx + \frac{\sqrt{3}}{4}Ky \qquad \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{x}} = m\ddot{x}$

$\frac{d}{dt}\frac{d\mathcal{L}}{d\dot{x}} - \frac{d\mathcal{L}}{dx} = \boxed{m\ddot{x} + (\frac{15}{4}x - \frac{\sqrt{3}}{4}y)K = 0}$

$\frac{d\mathcal{L}}{dy} = \frac{\sqrt{3}}{4}Kx - \frac{3}{4}Ky \qquad \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{y}} = m\ddot{y}$

$\Rightarrow \boxed{m\ddot{y} + (\frac{9}{4}y - \frac{\sqrt{3}}{4}x)K = 0}$

b) $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad K = k\begin{bmatrix} \frac{15}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{bmatrix}$ for $z = \begin{bmatrix} x \\ y \end{bmatrix}$

$\ddot{z} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$

frequencies?

c) assume sinusoidal answer

$$\text{Det}\begin{vmatrix} -\omega^2 m + \frac{15}{4}K & -\frac{\sqrt{3}}{4}K \\ -\frac{\sqrt{3}}{4}K & -\omega^2 m + \frac{9}{4}K \end{vmatrix} = 0$$

$$\left(-\omega^2 m + \frac{15}{4}K\right)\left(-\omega^2 m + \frac{9}{4}K\right) - \frac{3}{16}K^2 = 0$$

$$\omega^4 m^2 - \frac{24}{4} mK\omega^2 + \frac{132}{16}K^2 = 0$$

$$\omega^2 = \frac{\frac{24}{4}mK \pm \sqrt{\left(\frac{24}{4}mK\right)^2 - \frac{132}{4}m^2K^2}}{2m^2} = \frac{24}{8}\frac{K}{m} \pm \frac{K}{2m}\sqrt{36-33}$$

$$\omega^2 = \frac{K}{2m}\left(6 \pm \sqrt{3}\right)$$

$$\boxed{\begin{aligned} \omega_1 &= 1.461\sqrt{\tfrac{K}{m}} \\ \omega_2 &= 1.966\sqrt{\tfrac{K}{m}} \end{aligned}}$$

d) Modes?

for $\omega_1 = \sqrt{3 - \frac{\sqrt{3}}{2}}\left(\sqrt{\frac{K}{m}}\right)$

$$\left(-\left(3 - \frac{\sqrt{3}}{2}\right) + \frac{15}{4}\right)X_1 = \frac{\sqrt{3}}{4}X_2$$

$$\left(3 - 2\sqrt{3}\right)X_1 = \sqrt{3}\,X_2$$

$$\boxed{\text{mode shape 1} = \begin{bmatrix} 1 \\ \frac{3}{\sqrt{3}} - 2 \end{bmatrix}}$$

for $\omega_2 = \sqrt{3 + \frac{\sqrt{3}}{2}}\sqrt{\frac{K}{m}}$

$$-\frac{\sqrt{3}}{4}K X_1 + \left(-\left(3 + \frac{\sqrt{3}}{2}\right)K + \frac{9}{4}K\right)X_2 = 0$$

$$-\sqrt{3}\,X_1 = \left(3 - 2\sqrt{3}\right)X_2$$

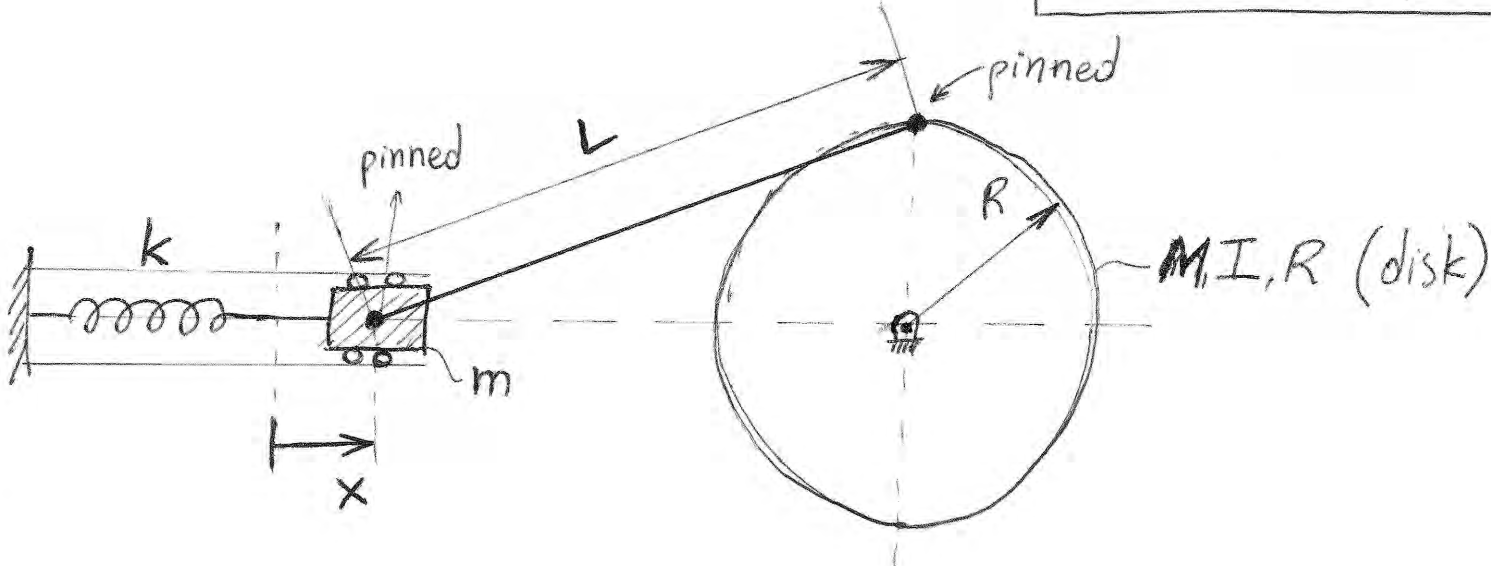
$$\boxed{\text{mode shape 2} = \begin{bmatrix} 1 \\ \frac{\sqrt{3}}{2\sqrt{3} - 3} \end{bmatrix}}$$

Problem #38 : Final exam question

Spyros Manolatopoulos
sm2296
1st Year PhD in ME

MAE 5735

Due: 12/3/2012



A mass m sliding, without friction, is connected to a rotating disk by a rigid, but massless, rod of length L, as shown in the picture. The mass is also connected to the wall by a linear spring k.

Questions:

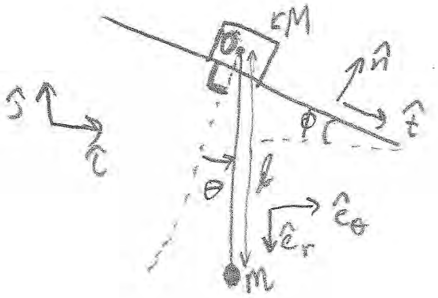
* Will the disk always rotate in the same direction? (CW/CCW) Elaborate on the possible motions of the system.
* Derive the nonlinear equations of motion.
* Find, and optionally characterize, the system's equilibria.
* Find the frequency of small vibrations. (Hint: Take the case $L \gg R$)

(This may be more of a kinematics problem!)

Ajay Patel
MAE 4735
Final Exam Question

Find equations of motion for the system

A simple pendulum attached to a sliding block on a frictionless incline,
Pendulum mass m, length l
Block mass M



AMB @ O for system

$$\vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$l\hat{e}_r \times -mg\hat{\jmath} = l\hat{e}_r \times m(\ddot{s}\hat{t} + l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r)$$

$$\hat{k}\cdot\{\} \Rightarrow -mgl\sin(\theta-\phi) = m\ddot{s}\,l\cos\theta + ml^2\ddot{\theta}$$

LMB $\cdot\,\hat{t}$ for system

$$\hat{t}\cdot(-Mg - mg)\hat{\jmath} = \hat{t}\cdot(M\ddot{s}\hat{t} + m(\ddot{s}\hat{t} + l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r))$$

$$(M+m)\sin\phi = (M+m)\ddot{s} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\begin{bmatrix} l & \cos\theta \\ ml\cos\theta & M+m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{s} \end{bmatrix} + \begin{bmatrix} g/l\,\sin(\theta-\phi) \\ ml\dot{\theta}^2\sin\theta - (M+m)g\sin\phi \end{bmatrix} = 0$$

Assume small amplitude motion and a small ϕ.
Find a normal mode.

$$\begin{bmatrix} l & 1 \\ ml & M+m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{s} \end{bmatrix} + \begin{bmatrix} g/l & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ s \end{bmatrix} = \begin{bmatrix} \phi \\ (M+m)g\,\phi \end{bmatrix}$$

can be $\sin\phi$ since they are constant

$$\omega_1 = 0 \qquad v^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

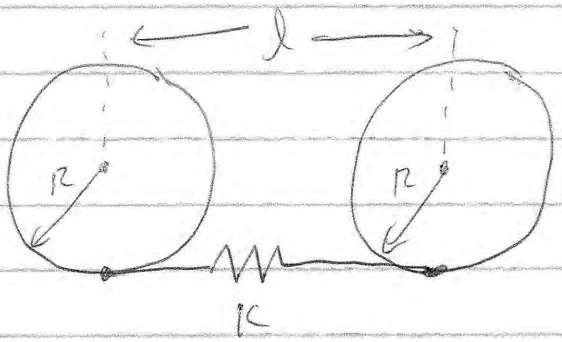
Cart simply slides down pulling pendulum behind it.

Ethan Rit

Final Question

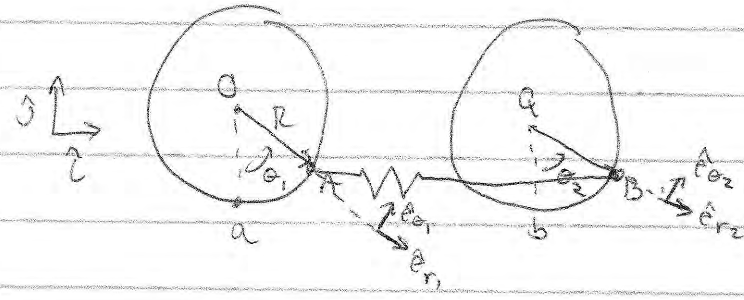
& I'm handing in the other problems Late (finishing project) p①

Two identical, uniform disks of radius R and mass M are pinned at their centers, free to rotate. They are separated by a distance $l > 2R$, and connected by a spring of stiffness K, rest length l at a point on the rim of each disk:



- Find the equations of motion
- Find the mode shapes and natural frequencies for small motions.

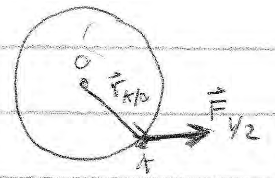
First, parameterize the problem:



$$\hat{e}_{r_1} = -\cos(\theta_1)\hat{j} + \sin(\theta_1)\hat{i}$$
$$\hat{e}_{r_2} = -\cos(\theta_2)\hat{j} + \sin(\theta_2)\hat{i}$$

First, AMB around the left-hand disk, point O:

FBD:



$$\Rightarrow \vec{M}_{/O} = \vec{r}_{A/O} \times \vec{F}_{1/2} = \dot{\vec{H}}_{/O} = I\ddot{\theta}_1 = \frac{MR^2}{2}\ddot{\theta}_1$$

where $\vec{F}_{1/2} = F_{1/2} \dfrac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}$

We know that $\vec{r}_{B/A} = \vec{r}_{B/O} - \vec{r}_{A/O} = (R_2 \hat{e}_{r_2} + l\hat{i}) - R_1 \hat{e}_{r_1}$

$$\Rightarrow \vec{r}_{B/A} = (l + R(\sin(\theta_2) - \sin(\theta_1))\hat{i} + R(\cos(\theta_1) - \cos(\theta_2))\hat{j}$$

And $|\vec{r}_{B/A}| = \sqrt{\vec{r}_{B/A} \cdot \vec{r}_{B/A}} = \sqrt{(l + R(\sin(\theta_2) - \sin(\theta_1))^2 + (R(\cos(\theta_1) - \cos(\theta_2))}$

The magnitude of $F_{1/2}$ is proportional to spring displacement: $F_{1/2} = (|\vec{r}_{B/A}| - l)K$

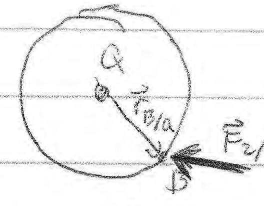
so, if $\vec{r}_{A/O} = R\hat{e}_{r_1} = -R\cos(\theta_1)\hat{j} + R\sin(\theta_1)\hat{i}$

$$\Rightarrow \vec{M}_{/O} = \vec{r}_{A/O} \times \vec{F}_{1/2} = \Big[R\cos(\theta_1)(l + R(\sin(\theta_2) - \sin(\theta_1))) \ldots$$
$$\ldots + R\sin(\theta_1)(R(\cos(\theta_1) - \cos(\theta_2)))\Big] \cdot \left[\frac{(|\vec{r}_{B/A}| - l)K}{|\vec{r}_{B/A}|}\right] = \frac{MR^2}{2}\ddot{\theta}_1$$

(Equation 1)

p ③

Next, AMB around the right disk, point Q:



$$\vec{M}_{/Q} = \vec{r}_{B/Q} \times \vec{F}_{2/1} = \dot{\vec{H}}_{/Q} = I\ddot{\theta}_2 = \frac{MR^2}{2}\ddot{\theta}_2$$

we know that $\vec{F}_{2/1} = -\vec{F}_{1/2}$

And if $-\vec{F}_{1/2} = F_{1/2}\dfrac{-\vec{r}_{B/A}}{|\vec{r}_{B/A}|} = \dfrac{F_{1/2}}{|\vec{r}_{B/A}|}\vec{r}_{A/B}$

And $\vec{r}_{A/B} = \vec{r}_{A/O} - \vec{r}_{B/O} = (-R\hat{e}_{r_2} - l\hat{i}) + R\hat{e}_{r_1}$

$$= \big(R(\sin(\theta_1) - \sin(\theta_2)) - l\big)\hat{i} + R(\cos(\theta_2) - \cos(\theta_1))\hat{j}$$

since $\vec{r}_{B/Q} = R\hat{e}_{r_2} = -R\cos(\theta_2)\hat{j} + R\sin(\theta_2)\hat{i}$

(Equation 2)

$$\Rightarrow \vec{M}_{/Q} = \Big[R\cos(\theta_2)\big(R(\sin(\theta_1) - \sin(\theta_2)) - l\big) + R_2\sin(\theta_2)\big(R(\cos(\theta_2) - \cos(\theta_1))\big)\Big] \cdot \ldots$$

$$\frac{(|\vec{r}_{B/A}| - l)K}{|\vec{r}_{B/A}|} = \frac{MR^2}{2}\ddot{\theta}_2$$

For small angles,

$$① \Rightarrow R(l + R(\theta_2 - \theta_1))\frac{\left(\sqrt{(l + R(\theta_2 - \theta_1))^2} - l\right)K}{\sqrt{(l + R(\theta_2 - \theta_1))^2}} = \frac{MR^2}{2}\ddot{\theta}_1$$

$$\Rightarrow KR^2(\theta_2 - \theta_1) = \frac{MR^2}{2}\ddot{\theta}_1 \Rightarrow \boxed{\ddot{\theta}_1 + \frac{2k}{M}\theta_1 - \frac{2k}{M}\theta_2 = 0}$$

By same process, $② \Rightarrow \boxed{\ddot{\theta}_2 - \frac{2k}{M}\theta_1 + \frac{2k}{M}\theta_2 = 0}$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\\ddot{\theta}_2\end{bmatrix} + \frac{2k}{m}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix} = \underset{\sim}{0}$$

M is trivial to invert, so we have $M^{-1}K = K = \frac{2k}{m}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

Evecs/Evals are also easy to find,

$$\omega_1 = 0, \quad v_1 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \omega_2 = 2\sqrt{k/m}, \quad v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

Anthony Savas
MAE 5735
HW 13, Due Dec. 3, 2012

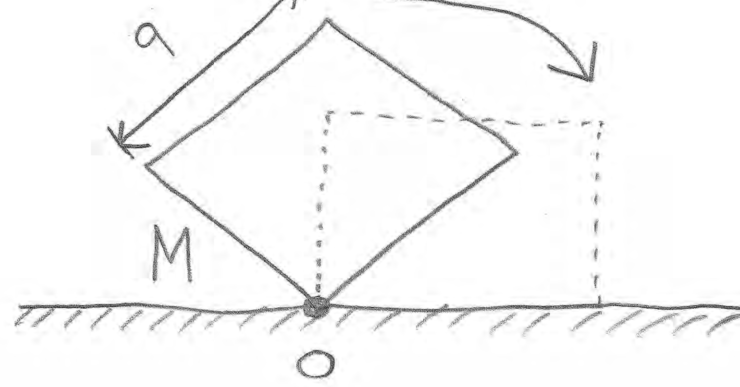
38) Final Exam Question

This problem concerns the motion of a flat plate which is fixed to the ground at one corner. The plate is initially standing upright on its fixed corner and is then given a slight perturbation, causing it to rotate and fall onto its side.

a) Find an equation of motion for this system any way you like.

b) Determine the angular velocity of the plate when it impacts the ground.

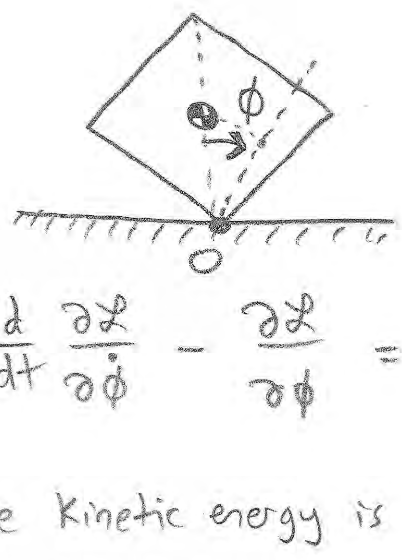
Take the moment of inertia of the plate about its fixed corner to be $I = \frac{2}{3} M a^2$, where M is the plate's mass and a is the length of a side of the plate. The scenario is shown below:



38) (continued)

Final Exam Question: Solution

a) Let's use Lagrange equations to obtain an equation of motion for this system. A generalized coordinate is the angle through which the center of mass of the plate moves as it falls, which we will call ϕ.

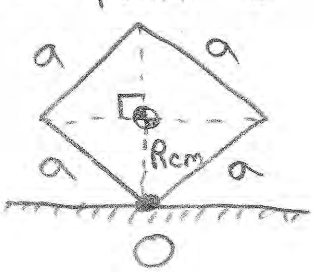


Thus, $\phi = 0°$ when the plate is standing on its corner.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{where } \mathcal{L} = T - V$$

The kinetic energy is all due to rotation $\Rightarrow T = \frac{1}{2} I \dot{\phi}^2$

The potential energy can be found by defining a distance R_{CM} from point O to the center of mass:



From basic geometry,

$$R_{cm}^2 + R_{cm}^2 = a^2$$

$$\Rightarrow \quad R_{cm} = \frac{a}{\sqrt{2}}$$

38) (continued)

Final Exam Question : Solution (cont'd)

If we take the zero potential energy line to be ground level, we have

$$V = M g R_{cm} \cos\phi = M g \left(\frac{a}{\sqrt{2}}\right) \cos\phi$$

$$\Rightarrow \mathcal{L} = T - V = \left(\tfrac{1}{2} I \dot{\phi}^2\right) - \left(M g \frac{a}{\sqrt{2}} \cos\phi\right)$$

with $I = \frac{2}{3} M a^2$ we get $\mathcal{L} = \frac{1}{3} M a^2 \dot{\phi}^2 - M g \frac{a}{\sqrt{2}} \cos\phi$

Taking appropriate derivatives : $\frac{\partial \mathcal{L}}{\partial \phi} = M g \frac{a}{\sqrt{2}} \sin\phi$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{2}{3} M a^2 \dot{\phi}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{2}{3} M a^2 \ddot{\phi}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{2}{3} M a^2 \ddot{\phi} - M g \frac{a}{\sqrt{2}} \sin\phi = 0$$

$$\boxed{\ddot{\phi} - \frac{3g}{2\sqrt{2}\, a} \sin\phi = 0}$$

b) Determine $\dot{\phi}_{final}$. The above E.O.M. is transcendental $\rightarrow$ doesn't help us

$\Rightarrow$ Use energy conservation

$$T + V = \text{constant}$$

$$\Rightarrow \cancel{T_o}^{\,0} + V_o = T_{final} + V_{final}$$

$$\left(M g \frac{a}{\sqrt{2}} \cos(\phi_o)\right) = \left(\tfrac{1}{2} I \dot{\phi}_f^2\right) + \left(M g \frac{a}{\sqrt{2}} \cos(\phi_f)\right)$$

with $\phi_o = 0°$ and $\phi_f = 45°$ you get

$$\tfrac{1}{2} I \dot{\phi}_f^2 = M g \frac{a}{\sqrt{2}} \left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\dot{\phi}_f^2 = \frac{2 M g a \left(\frac{\sqrt{2}-1}{2}\right)}{I}$$

$$\dot{\phi}_f^2 = \frac{2 M g a \left(\frac{\sqrt{2}-1}{2}\right)}{\left(\frac{2}{3} M a^2\right)}$$

$$\Rightarrow \boxed{\dot{\phi}_f = \sqrt{\frac{3(\sqrt{2}-1) g}{2a}}}$$

Tom Schmidt
MAE 5180

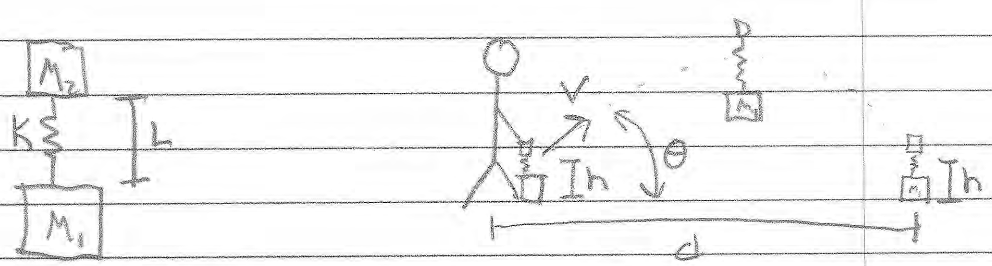
Final exam question

You have developed a new game in which the goal is to throw an object and hit a target a distance d away (think bean bags). However, instead of throwing a bean bag we are now throwing two masses attached by a spring. The spring mass system will be tossed underhand by holding mass 2. Let $M_1 > M_2$. Assume M_1 is large enough that it is always down with M_2 directly above it (i.e. ignore rotation about the center of mass), meaning that M_1 always hits the ground first.

A) Find the equation of motion of the center of mass of the system assuming it is released from a height h with a velocity V at an angle θ.

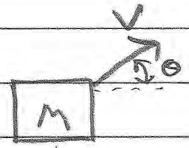
B) Find the initial stretch in the spring when it is released (Assume rest length L)

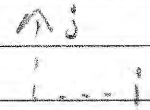
C) Find the natural frequency of the system



Note: Simplify the throwing motion to pivoting about M_2.

A) Treat the system as a single mass of $M = M_1 + M_2$





Find time to top of arc

$\boxed{V_F = V_i + a\Delta T} \cdot j$

$a = \frac{F}{M} = \frac{g}{M}$

$V_i = V\sin(\theta)$

$V_F = 0$

$\Delta T = -\frac{1}{a} V\sin(\theta)$

Since the arc is conservative we can say that the time back to a height h is $-2\frac{1}{a}V\sin(\theta)$

$Y_F = Y_i + V\sin(\theta)\Delta t + \frac{1}{2}a\Delta t^2$

$Y_F = h - \frac{1}{a}V^2\sin^2(\theta) + \frac{1}{2}\cdot\frac{1}{a}V^2\sin^2(\theta)$

$\boxed{Y_F = h - \frac{1}{2}\frac{1}{a}V^2\sin^2(\theta)}$

$X_F = X_i + V\cos(\theta)\Delta t + \frac{1}{2}a\Delta t^2$

$\boxed{X_F = -\frac{1}{2}\cdot\frac{1}{a}V^2\sin(\theta)\cos(\theta)}$

B) The velocity V is the tangential velocity at the center of mass. First, find the center of mass

$(x)M_1 = (1-x)M_2$

$x(M_1 + M_2) = M_2$

$x = \frac{M_2}{M_1 + M_2}$ (distance from M_1 as a percent of L)

Find angular velocity

$w = \frac{V}{R} = \frac{V}{(1-x)L}$

Find Centripet Force on mass 1

$F = mRw^2$

$F = \frac{mV^2}{(1-x)^2 L}$

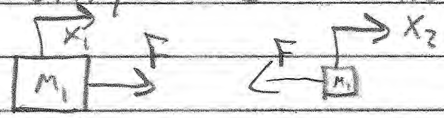
(Finished on next page)

define δ as the stretch in the spring

$F = K\delta$

$$\boxed{\delta = \frac{mv^2}{(1-x)^2 L}}$$

C) The full equation of motion will be a super position of the translation and oscillation. Consider only the oscillation for now



$F = K(x_2 - x_1)$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} w^2M_1 - K & K \\ K & w^2M_2 - K \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

Find the determinant

$(w^2M_1 - K)(w^2M_2 - K) - K^2 = 0$

$w^4M_1M_2 - Kw^2M_2 - Kw^2M_1 + K^2 - K^2 = 0$

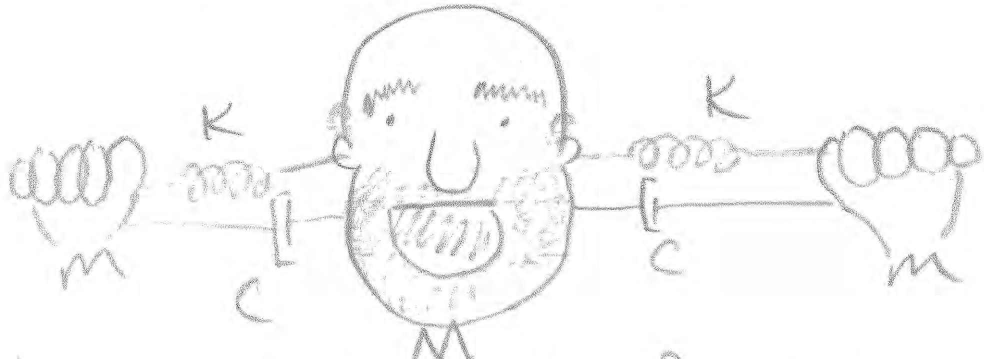
$w^2M_1M_2 - K(M_1 + M_2) = 0$

$w^2 = \dfrac{K(M_1 + M_2)}{M_1M_2}$

$$\boxed{w = \sqrt{\frac{K(M_1 + M_2)}{M_1M_2}}}$$

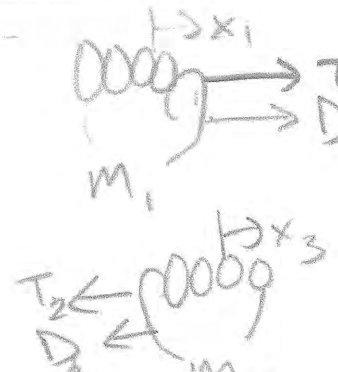
Neil Sen
MAE 4735

FINAL EXAM PROBLEM



In a mishap in the Biorobotics and Locomotion Lab, Professor Andy Ruina got his hands stuck to two equal springs, which are in turn stuck to his ears. His neck functions like a damper to keep his head in control. Write code to plot the motion.

FBD:



$$m_1\ddot{x}_1 = -K\,(x_1 - x_2) - c\,(\dot{x}_1 - \dot{x}_2)$$

$$m_3\ddot{x}_3 = K_1\,(x_2 - x_3) + c\,(\dot{x}_2 - \dot{x}_3)$$

$$M_2\ddot{x}_2 = K\,(x_1 - x_2) + c\,(\dot{x}_1 - \dot{x}_2) + K\,(x_3 - x_2) + c\,(\dot{x}_3 - \dot{x}_2)$$

```
function examProblem()
%Solves the equation exactly instead of numerically.

%Masses.
m1 = 1;
M2 = 5;

M= [m 0 0; 0 M 0; 0 0 m];

%Spring stiffnesses. All springs are equal, so only one variable is
%declared:
k = 1;

%Stiffness matrix:
K= [k -k 0; -k 2*k -k; 0 -k k];

%Damping matrix:
C= [c -c 0; -c 2*c -c; 0 -c c];

%Identiy matrix:
I= eye(3);

%Zeros:
Z= zeros(3);

%Super matrix:
A= [Z I; -inv(M)*K -inv(M)*C];

%ICs:
x0= [1 0 0]';
v0= [0 0 0]';
z0= [x0; v0];

tspan= [0:.01:100];

%Running it through the timesteps:
for i= 1:length(tspan)
    z= expm(A*tspan(i))*z0;
    x1(i)= z(1);
    x2(i)= z(2);
    x3(i)= z(3);
end

plot(tspan, x1, tspan, x2, tspan, x3);
xlabel('time')
ylabel('Position')
title('Position vs. Time: Exponential Solution')
legend('Mass 1', 'Mass 2', 'Mass 3')


end
```

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Problem 38

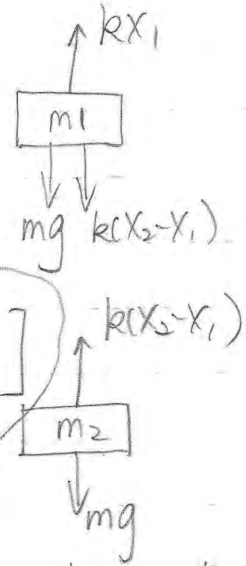
Consider a 2-mass-spring system hanging down a ceiling.
The 2 masses have equal mass m, and the springs are massless.
Spring constant is k for both springs.

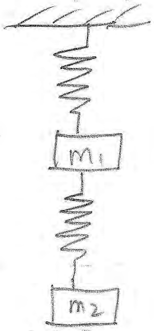
i) derive EOM

ii) find one normal mode and corresponding frequency

iii) provide Matlab code to plot the normal mode you found in ii)
(displacement vs. time for both masses)

Solution:





i) $\begin{cases} mg + k(x_2-x_1) - kx_1 = m\ddot{x}_1 \\ mg - k(x_2-x_1) = m\ddot{x}_2 \end{cases}$

$$\Rightarrow \underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M} \ddot{\vec{x}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}_{K} \vec{x} = mg\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ii) $M^{-1} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$ $\quad M^{-1}\cdot K = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix}$

$$\det(\lambda I - M^{-1}K) = \begin{vmatrix} \lambda - \frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & \lambda - \frac{k}{m} \end{vmatrix} = \left(\lambda - \frac{2k}{m}\right)\left(\lambda - \frac{k}{m}\right) - \frac{k^2}{m^2} = 0$$

$$\Rightarrow \lambda^2 - \frac{3k}{m}\lambda + \frac{k^2}{m^2} = 0$$

$$\Rightarrow \lambda = \frac{\frac{3k}{m} \pm \sqrt{\frac{9k^2}{m^2} - \frac{4k^2}{m^2}}}{2} = \frac{3k}{2m} \pm \frac{\sqrt{5}k}{2m}$$

use $\lambda = (3+\sqrt{5})\frac{k}{2m}$.

then $\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (3+\sqrt{5})\frac{k}{2m}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \begin{cases} \frac{2k}{m}x - \frac{k}{m}y = (3+\sqrt{5})\frac{k}{2m}x \\ -\frac{k}{m}x + \frac{k}{m}y = (3+\sqrt{5})\frac{k}{2m}y \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \frac{-\sqrt{5}-1}{3+\sqrt{5}} = -0.618 \end{cases}$$

SO, $\boxed{\text{normal mode: } \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}, \text{ frequency} = \sqrt{(3+\sqrt{5})\frac{k}{2m}}}$

iii) Matlab code:

```
function  exam
 tspan = 0:0.01:10 ;
 xo = [1; -0.618; 0; 0];
[t, x] = ode45(@f1, tspan, xo);
 plot (t, x)

function  xdot = f1(t, x)
 M = [m  0
      0  m];
 K = [2k  -k
      -k   k];
 F = mg[1
        1];
 xdd = M\(F - K·x(1:2));
 xdot = [x(3:4); xdd];
```

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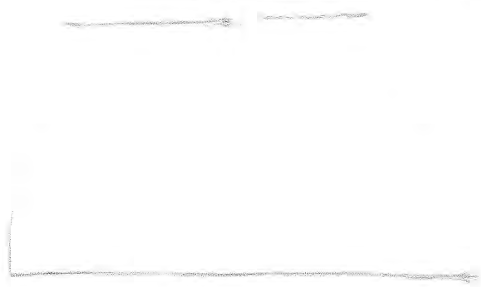
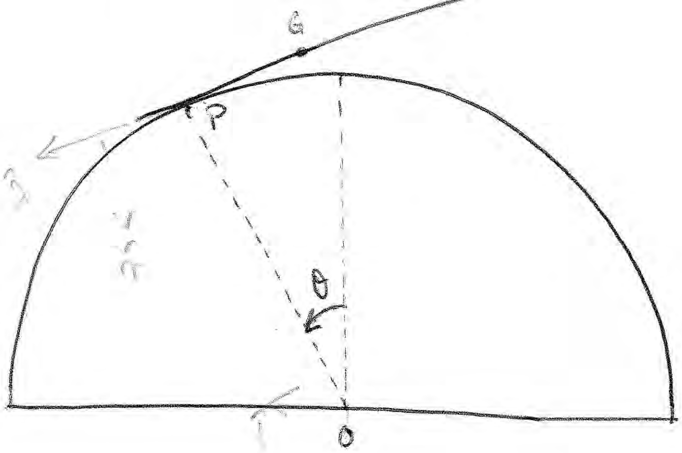
Problem 38

• describe the motion of a rod lying on the top of cylinder.



Consider the motion of rod to be no slip and no skid motion.

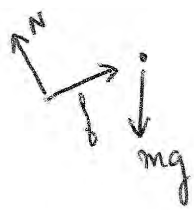


$$\hat{\lambda} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{n} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$\dot{\hat{n}} = -\dot{\theta}\,\hat{\lambda}$$
$$\dot{\hat{\lambda}} = \dot{\theta}\,\hat{n}$$

Let the center of mass of rod be at G

$$\vec{r}_{G/O} = -R\hat{n} - R\theta\hat{\lambda} \qquad \text{(using rolling condition)}$$
$$\vec{V}_{G/O} = -R\theta\dot{\theta}\,\hat{n}$$
$$\vec{a}_{G/O} = R\theta\dot{\theta}^2\hat{\lambda} - R\dot{\theta}^2\hat{n} - R\theta\ddot{\theta}\,\hat{n}$$

F.B.D.



Using AMB

$AMB_{/P}$

$$\Sigma\vec{M}_{/P} = \vec{r}_{G/P} \times mg\hat{j}$$
$$= -R\theta\hat{\lambda} \times mg\hat{j}$$
$$= -mgR\theta\cos\theta\,\hat{k}$$
$$\dot{\vec{H}}_{/P} = \vec{r}_{G/P} \times m\vec{a}_G + I\ddot{\theta}\hat{k}$$
$$= (-R\theta\hat{\lambda}) \times m\left[R\theta\dot{\theta}^2\hat{\lambda} - R\dot{\theta}^2\hat{n} - R\theta\ddot{\theta}\hat{n}\right] + I\ddot{\theta}\hat{k}$$
$$= \left[mR^2\theta\dot{\theta}^2 + mR^2\theta^2\ddot{\theta} + I\ddot{\theta}\right]\hat{k}$$

$$\Sigma\vec{M}_{/P} = \dot{\vec{H}}_{/P}$$

$$\Rightarrow \boxed{(I + mR^2\theta^2)\ddot{\theta} + mR^2\theta\dot{\theta}^2 + mgR\theta\cos\theta = 0}$$

Lagrange equations

$$T = \frac{1}{2} m R^2 \theta^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$V = mg\,[R\cos\theta + R\theta\sin\theta - R]$$

$$\mathcal{L} = T - V = \frac{1}{2} m R^2 \theta^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2 - mgR[\cos\theta + \theta\sin\theta - 1]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\theta^2\dot{\theta} + I\dot{\theta}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2mR^2\theta\dot{\theta}^2 + mR^2\theta^2\ddot{\theta} + I\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mR^2\theta\dot{\theta}^2 - mgR\theta\cos\theta$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \boxed{\left[I + mR^2\theta^2\right]\ddot{\theta} + mR^2\theta\dot{\theta}^2 + mgR\theta\cos\theta = 0}$$

~~For~~ For small oscillations

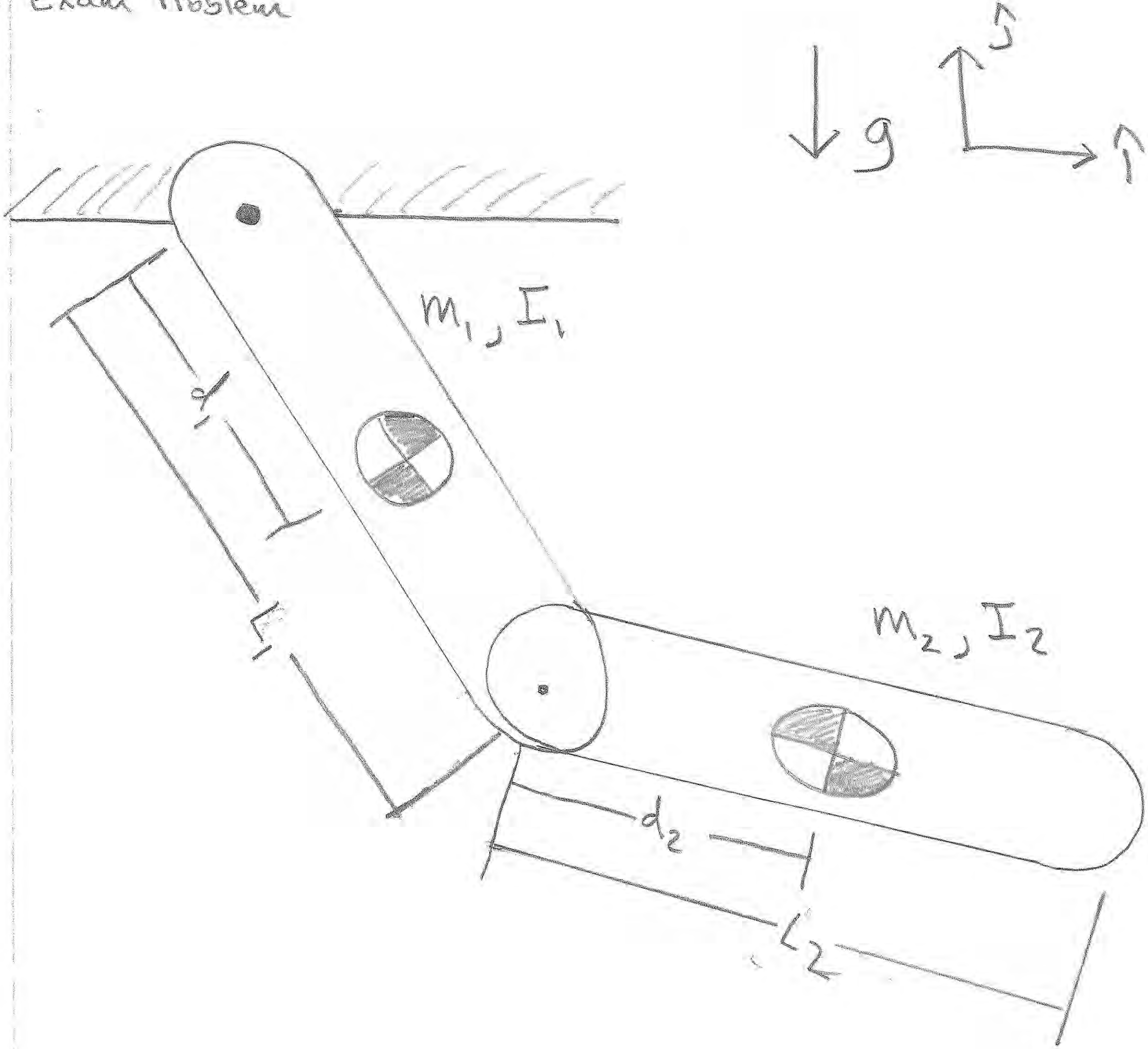
$$\dot{\theta}^2 \approx 0, \quad \theta^2 \approx 0, \quad \cos\theta \approx 1$$

$$\Rightarrow \quad I\ddot{\theta} + mgR\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \frac{mgR}{I}\theta = 0$$

$\therefore$ for small oscillations $\boxed{\omega = \sqrt{\frac{mgR}{I}}}$

Joshua Tashman
Final Exam Problem



Derive the differential algebraic equations for this double pendulum. These equations could be arranged in a matrix that could be used to solve them. Use $m_1, I_1, d_1, l_1, m_2, I_2, d_2, l_2$, and your choice of minimal coordinates.

Joshua Tashman

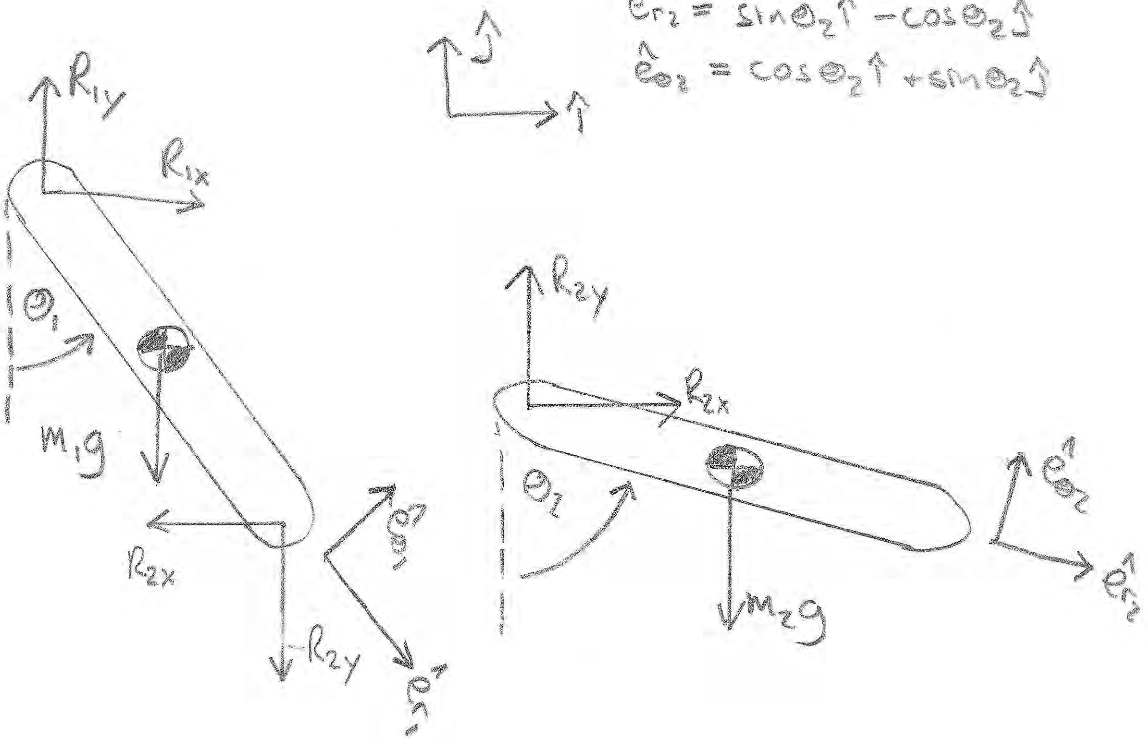
Final Exam Problem: Solution

$\hat{e}_{r_1} = \sin\theta_1 \hat{\imath} - \cos\theta_1 \hat{\jmath}$

$\hat{e}_{\theta_1} = \cos\theta_1 \hat{\imath} + \sin\theta_1 \hat{\jmath}$

$\hat{e}_{r_2} = \sin\theta_2 \hat{\imath} - \cos\theta_2 \hat{\jmath}$

$\hat{e}_{\theta_2} = \cos\theta_2 \hat{\imath} + \sin\theta_2 \hat{\jmath}$



The types of DAEs required are:

1 AMB 2 Constraints

2 LMB

LMB$_1$

$$\Sigma F = m_1 \vec{a}_1$$

$$\Sigma F = (R_{1x} - R_{2x})\hat{\imath} + (R_{1y} - R_{2y} - m_1 g)\hat{\jmath}$$

$$m_1 \vec{a}_1 = m_1 \ddot{x}_1 \hat{\imath} + m_1 \ddot{y}_1 \hat{\jmath}$$

$$\boxed{(R_{1x} - R_{2x})\hat{\imath} + (R_{1y} - R_{2y} - m_1 g)\hat{\jmath} = m_1(\ddot{x}_1 \hat{\imath} + \ddot{y}_1 \hat{\jmath})}$$

LMB$_2$

$$\Sigma F = m_2 \vec{a}_2$$

$$\Sigma F = (R_{2x})\hat{\imath} + (R_{2y} - m_2 g)\hat{\jmath}$$

$$m_2 \vec{a}_2 = m_2 \ddot{x}_2 \hat{\imath} + m_2 \ddot{y}_2 \hat{\jmath}$$

$$\boxed{(R_{2x})\hat{\imath} + (R_{2y} - m_2 g)\hat{\jmath} = m_2(\ddot{x}_2 \hat{\imath} + \ddot{y}_2 \hat{\jmath})}$$

Constraint 1

$$\vec{r}_{G1} = d_1 \sin\theta_1 \hat{\imath} - d_1 \cos\theta_1 \hat{\jmath}$$

$$\vec{V}_{G1} = d_1 \dot{\theta}_1 \cos\theta_1 \hat{\imath} + d_1 \dot{\theta}_1 \sin\theta_1 \hat{\jmath}$$

$$\boxed{(\ddot{X}_1 \hat{\imath} + \ddot{Y}_1 \hat{\jmath}) = (d_1 \ddot{\theta}_1 \cos\theta_1 - d_1 \dot{\theta}_1^2 \sin\theta_1)\hat{\imath} + (d_1 \ddot{\theta}_1 \sin\theta_1 + d_1 \dot{\theta}_1^2 \cos\theta_1)\hat{\jmath}}$$

Constraint 2

$$\vec{r}_{G2} = (L_1 \sin\theta_1 + d_2 \sin\theta_2)\hat{\imath} - (L_1 \cos\theta_1 + d_2 \cos\theta_2)\hat{\jmath}$$

$$\vec{V}_{G2} = (L_1 \dot{\theta}_1 \cos\theta_1 + d_2 \dot{\theta}_2 \cos\theta_2)\hat{\imath} + (L_1 \dot{\theta}_1 \sin\theta_1 + d_2 \dot{\theta}_2 \sin\theta_2)\hat{\jmath}$$

$$\boxed{\begin{aligned}(\ddot{X}_2 \hat{\imath} + \ddot{Y}_2 \hat{\jmath}) = {} & [L_1 \ddot{\theta}_1 \cos\theta_1 + L_1 \dot{\theta}_1^2 \sin\theta_1 + d_2 \ddot{\theta}_2 \cos\theta_2 - d_2 \dot{\theta}_2^2 \sin\theta_2]\hat{\imath} + \\ & [L_1 \ddot{\theta}_1 \sin\theta_1 + L_1 \dot{\theta}_1^2 \cos\theta_1 + d_2 \ddot{\theta}_2 \sin\theta_2 + d_2 \dot{\theta}_2^2 \cos\theta_2]\hat{\jmath}\end{aligned}}$$

AMB$_1$/end

$$\sum \vec{M}_1 = \dot{\vec{H}}_1$$

$$\sum \vec{M}_1 = d_1 \hat{e}_{r_1} \times -m_1 g \hat{\jmath} + L_1 \hat{e}_{r_1} \times (-R_{2x}\hat{\imath} - R_{2y}\hat{\jmath})$$

$$\sum \vec{M}_1 = -m_1 d_1 g \sin\theta_1 \hat{k} + (-R_{2x} L_1 \cos\theta_1 - R_{2y} L_1 \sin\theta_1)\hat{k}$$

$$\dot{\vec{H}}_1 = d_1 \hat{e}_{r_1} \times m_1(\ddot{x}_1 \hat{\imath} + \ddot{Y}_1 \hat{\jmath}) + I_1 \ddot{\theta}_1 \hat{k}$$

$$\dot{\vec{H}}_1 = [m_1 d_1 \cos\theta_1 \ddot{x}_1 + m_1 d_1 \sin\theta_1 \ddot{Y}_1 + I_1 \ddot{\theta}_1]\hat{k}$$

$$\boxed{\begin{aligned}&-[m_1 d_1 g \sin\theta_1 + R_{2x} L_1 \cos\theta_1 + R_{2y} L_1 \sin\theta_1] = \\ &[m_1 d_1 \cos\theta_1 \ddot{x}_1 + m_1 d_1 \sin\theta_1 \ddot{Y}_1 + I_1 \ddot{\theta}_1]\end{aligned}}$$

AMB$_2$/end

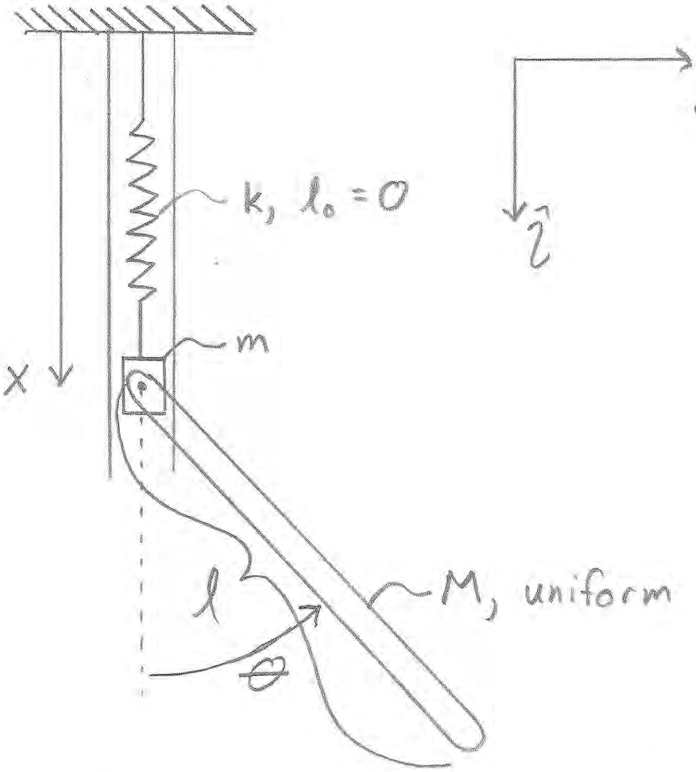
$$\sum \vec{M}_2 = \dot{\vec{H}}_2$$

$$\sum \vec{M}_2 = d_2 \hat{e}_{r_2} \times -m_2 g \hat{\jmath} = -m_2 d_2 g \sin\theta_2$$

$$\dot{\vec{H}}_2 = d_2 \hat{e}_{r_2} \times m_2(\ddot{x}_2 \hat{\imath} + \ddot{Y}_2 \hat{\jmath}) + I_2 \ddot{\theta}_2 \hat{k} = [m_2 d_2 \cos\theta_2 \ddot{x}_2 + m_2 d_2 \sin\theta_2 \ddot{Y}_2 + I_2 \ddot{\theta}_2]$$

$$\boxed{-m_2 d_2 g \sin\theta_2 = [m_2 d_2 \cos\theta_2 \ddot{x}_2 + m_2 d_2 \sin\theta_2 \ddot{Y}_2 + I_2 \ddot{\theta}_2]}$$

Adam Trolz
Final Question



A mass m is attatched to a spring $(k,\ l_0 = 0)$ which is connected on its other end to solid ground. A uniform pendulum of mass M and length l is hinged on the first mass.
Find the mode shapes and frequencies of small oscillations. Answer in terms of x, θ, m, M, l, k, or time derivatives of these quantities (where applicable). Use any method.

Solution: let $\hat{e}_R = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

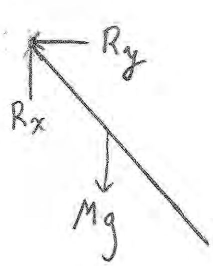
$\vec{r}_G = x\hat{i} + d\hat{e}_R$ (pendulum c.g.)

$\vec{v}_G = \dot{x}\hat{i} + d\dot{\theta}\hat{e}_\theta$

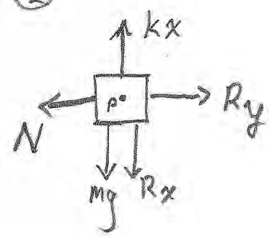
$\vec{a}_G = \ddot{x}\hat{i} + d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_R = (\ddot{x} - d\ddot{\theta}\sin\theta - d\dot{\theta}^2\cos\theta)\hat{i} + (d\ddot{\theta}\cos\theta - d\dot{\theta}^2\sin\theta)\hat{j}$

FBDs

①



②



① AMB$_{/P}$: $\sum \vec{M}_{/P} = \dot{\vec{H}}_{/P}$

$d\hat{e}_R \times Mg\hat{i} = d\hat{e}_R \times M(\ddot{x}\hat{i} + d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_R) + I\ddot{\theta}\hat{k}$

$\{-dMg\sin\theta\,\hat{k} = -dM\ddot{x}\sin\theta\,\hat{k} + d^2M\ddot{\theta}\hat{k} + I\ddot{\theta}\hat{k}\}\cdot\hat{k}$

$dM(\ddot{x} - g)\sin\theta = (d^2M + I)\ddot{\theta}$

① LMB: $\sum \vec{F} = M\vec{a}$

$\hat{i}\cdot\{-R_x\hat{i} - R_y\hat{j} + Mg\hat{i} = M(\ddot{x} - d\ddot{\theta}\sin\theta - d\dot{\theta}^2\cos\theta)\hat{i} + M(d\ddot{\theta}\cos\theta - d\dot{\theta}^2\sin\theta)\hat{j}\}$

$Mg - R_x = M(\ddot{x} - d\ddot{\theta}\sin\theta - d\dot{\theta}^2\cos\theta)$

$R_x = M(g - \ddot{x} + d\ddot{\theta}\sin\theta + d\dot{\theta}^2\cos\theta)$

② LMB: $\sum \vec{F} = m\vec{a}$

$\{mg\hat{i} + R_x\hat{i} - kx\hat{i} + R_y\hat{j} - N\hat{j} = m\ddot{x}\hat{i}\}\cdot\hat{i}$

$R_x + mg - kx = m\ddot{x}$

$Mg - M\ddot{x} + Md\ddot{\theta}\sin\theta + Md\dot{\theta}^2\cos\theta + mg - kx = m\ddot{x}$

$(M+m)g + M(d\ddot{\theta}\sin\theta + Md\dot{\theta}^2\cos\theta) = (m+M)\ddot{x} + kx$

2 Equations:

$$\ddot{\theta}(d^2M + I) = dM\ddot{x}\sin\theta - dMg\sin\theta$$

$$(m+M)\ddot{x} + kx = (M+m)g + M(d\ddot{\theta}\sin\theta + Md\dot{\theta}^2\cos\theta)$$

Linearize:

$$\ddot{\theta}(d^2M + I) = \cancel{dM\ddot{x}\theta} - dMg\theta$$

$$(m+M)\ddot{x} + kx = (M+m)g + \cancel{Md\ddot{\theta}\theta}$$

$$\left.\begin{aligned} \ddot{\theta} &= \frac{-dMg}{d^2M + I}\,\theta \\ \ddot{x} + \frac{k}{M+m}x &= g \end{aligned}\right\} \text{decoupled}$$

$$d = \frac{l}{2}, \quad I = \frac{Ml^2}{12}$$

$$w^2 = \frac{\frac{\cancel{l}}{2}\cancel{M}g}{\frac{l^{\cancel{2}}}{4}\cancel{M} + \frac{\cancel{M}l^{\cancel{2}}}{12}} = \frac{\frac{1}{2}}{l\left(\frac{3}{12} + \frac{1}{12}\right)}g = \frac{\frac{1}{2}}{l\,\frac{1}{3}}g = \frac{3}{2}\frac{g}{l}$$

Modes:

$$\begin{bmatrix}\theta \\ x\end{bmatrix} = \begin{bmatrix}1 \\ 0\end{bmatrix} \qquad w = \sqrt{\frac{3}{2}\frac{g}{l}}$$

$$\begin{bmatrix}\theta \\ x\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix} \qquad w = \sqrt{\frac{k}{M+m}}$$

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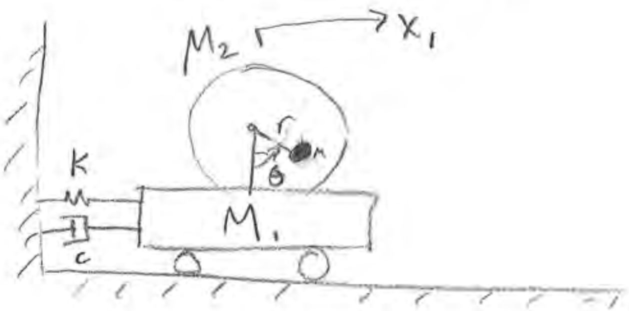
Problem 38

A cart of mass M_1 supports a motor spinning with angular velocity $\dot{\theta}$. The motor is off balance, represented by a small mass m at radius r. The cart is attached to a wall by a spring k and a damper c. The balanced motor has mass M_2 and inertia I

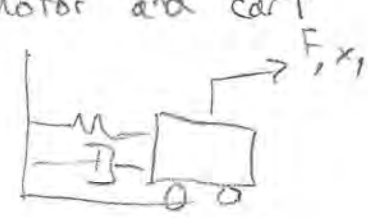


Find the equations of motion. What is the torque necessary to keep $\dot{\theta}$ constant? Ignore gravity for this calculation.

Problem 38 Solution

Separate the unbalanced part of motor from motor and cart

LMB 1



$$(M_1 + M_2 - m)\ddot{x}_1 + c\dot{x}_1 + kx_1 = F$$

$$x_2 = r\sin\theta \quad , \quad \dot{x}_2 = r\dot{\theta}\cos\theta \quad , \quad \ddot{x}_2 = r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \Rightarrow \ddot{x}_2 = -r\dot{\theta}^2\sin\theta$$

LMB 2

$$m\ddot{x}_2 = -F$$

$$mr\dot{\theta}^2\sin\theta = F \qquad \theta = \dot{\theta}t \qquad \Rightarrow \dot{\theta} \text{ is constant}$$

$$(M_1 + M_2 - m)\ddot{x}_1 + c\dot{x}_1 + kx = mr\dot{\theta}^2\sin(\dot{\theta}t)$$

Finding torque

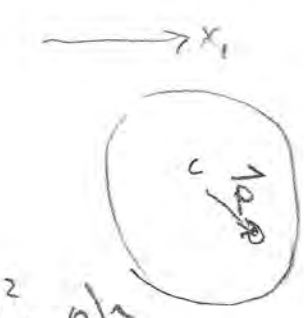
$$\ddot{x}_2 = r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

$$\ddot{\theta} = \frac{x_2 + r\dot{\theta}^2\sin\theta}{r\cos\theta}$$

$$\vec{P}_{/c} = r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\dot{\vec{P}}_{/c} = r\dot{\theta}\cos\theta\,\hat{i} - r\dot{\theta}\sin\theta\,\hat{j}$$

$$\ddot{\vec{P}}_{/c} = (r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{i} - (r\ddot{\theta}\sin\theta + r\dot{\theta}^2\cos\theta)\hat{j}$$



$$\vec{a}_p = \ddot{\vec{P}}_{/c} + \ddot{x}_1\hat{i}$$

AMB/c

$$\vec{\tau}_m = m(\vec{p}_{/c} \times \vec{a}_p) + I\ddot{\theta}\hat{k}$$

$$\ddot{\theta} = 0$$

$$\vec{\tau}_m = m(r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}) \times (-r\dot{\theta}^2\sin\theta\,\hat{i} - r\dot{\theta}^2\cos\theta\,\hat{j} + \ddot{x}_1\hat{i})$$

$$\vec{\tau}_m = m(-\ddot{x}_1 r\cos\theta)\hat{k}$$

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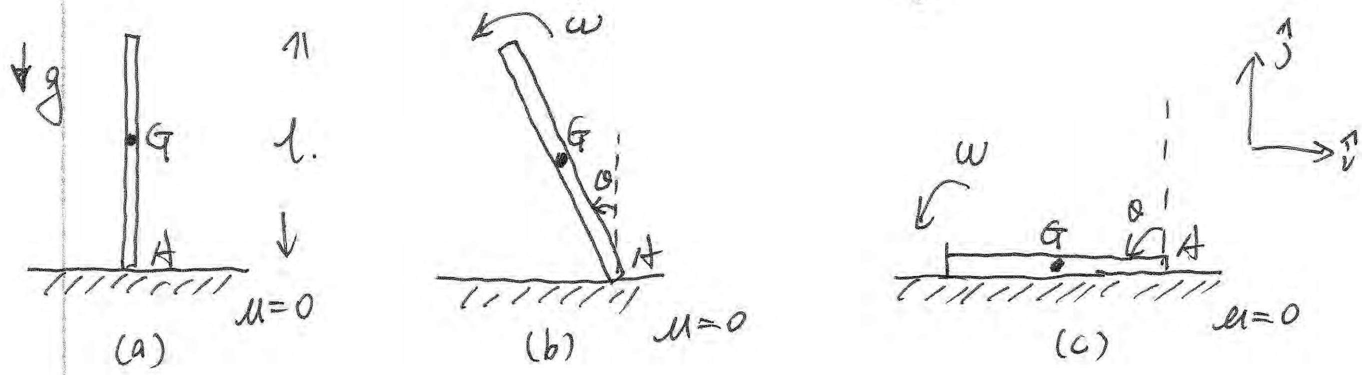
MHE 5735

38. Write a final Exam question.



2D. A uniform bar with mass M and length l. Initially stand still until there is a small perturbation so that it begins to fall. Ignore friction, Choose θ as general coordinates. G is the mass center.

(a) Describe in word the shape of trajectory of G. provide enough reasoning.

(b) Determine the Motion of the bar with all or some of $\theta, \dot{\theta}, \ddot{\theta}, g, l, M$.

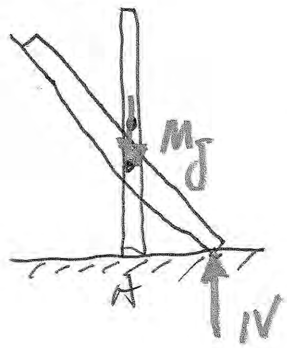
(c) What is the reaction force N of the ground when the bar is just about to reach the ground.

(d) (Extra credit) ~~If~~ For $\mu \neq 0$, will the reaction force of (c) smaller, bigger or equal to that when $\mu = 0$. provide enough reasoning (μ is big enough so that there will be no sliding)

Solution.

(a)

$\boxed{\text{FBD}}$



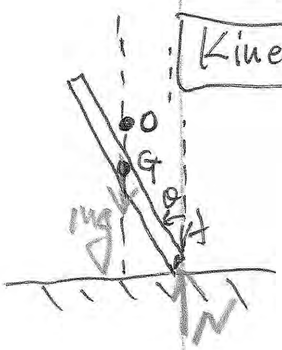
In the whole process, $\vec{F} \cdot \hat{i} = 0$

LMB: $\vec{F} \cdot \hat{i} = m a_{x} \Rightarrow a_x = 0$

Initially $v_x = 0$

so the trajectory of G should be a vertical line

(b) AMB/A



$\boxed{\text{Kinematic.}}$

$$\vec{r}_G = \frac{l}{2}(\cos\theta - 1)\hat{j}$$

$$\vec{v}_G = \frac{l}{2}(-\dot{\theta}\sin\theta)\hat{j}$$

$$\vec{a}_G = \frac{l}{2}\left[-\cos\theta \cdot \dot{\theta}^2 - \sin\theta \cdot \ddot{\theta}\right]\hat{j}$$

$$\vec{r}_{G/A} = \frac{l}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$\boxed{\text{AMB}/A}$

$$\vec{r}_{G/A} \times (-mg\hat{j}) = \vec{r}_{G/A} \times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$$

where $I^G = \frac{1}{12} m l^2$

$$\frac{l}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times (-mg\hat{j}) = \frac{l}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times \left(m\frac{l}{2}\right)\left[-\cos\theta \cdot \dot{\theta}^2 - \sin\theta\ddot{\theta}\right]\hat{j} + \frac{1}{12} m l^2 \ddot{\theta}\hat{k}$$

$$\Rightarrow \boxed{6g\sin\theta = 3l\dot{\theta}^2\sin\theta\cos\theta + 3l\ddot{\theta}\sin^2\theta + l\ddot{\theta}}$$

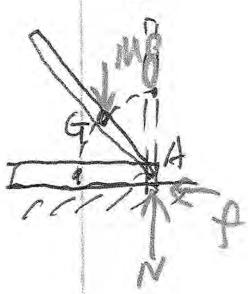
(c) when $\theta = \frac{\pi}{2}$. $\Rightarrow \ddot{\theta} = \frac{3g}{2l}$.

$\{LMB\} \cdot \hat{i} \Rightarrow N - mg = m a_{Gy}$

$a_{Gy} = -\frac{l\ddot{\theta}}{2}$ when $\theta = \frac{\pi}{2}$

$N = -\frac{3}{4}mg + mg = \frac{1}{4}mg$

(d)



AMB$/_A$.

$\vec{r}_G = \frac{l}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = \vec{r}_{G/A}$

$\vec{a}_G = \frac{l}{2}(-\ddot{\theta}\cos\theta\,\hat{i} + \dot{\theta}^2\sin\theta\,\hat{i} - \ddot{\theta}\sin\theta\,\hat{j} - \dot{\theta}^2\cos\theta\,\hat{j})$

$\vec{r}_{G/A} \times (-mg\,\hat{j}) = \vec{r}_{G/A} \times m\vec{a}_G + I^G\ddot{\theta}\,\hat{k}$

$\Rightarrow \boxed{\frac{l}{3}\ddot{\theta} = \frac{g}{2}\sin\theta}$ motion is different

in $\theta = \frac{\pi}{2}$ $\ddot{\theta} = \frac{3g}{2l}$, $\vec{a}_G = \frac{l}{2}(\dot{\theta}^2\hat{i} - \ddot{\theta}\hat{j})$

$N = -\frac{3}{4}mg + mg = \frac{1}{4}mg$

$\boxed{\text{N will be the same}}$.

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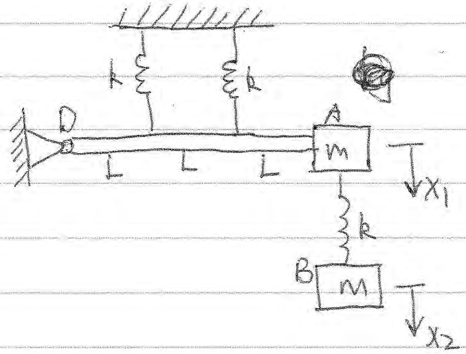
HW #38

Problem



Look at the figure on the left. The stiffness of the three springs are all k. The rod has a mass of m. The Questions are based on small motions neer $x_1 = x_2 = 0$. Neglect g.

1. Write out the equations of motion in the form of $M\ddot{\vec{x}} + K\vec{x} = 0$
2. Find the two frequencies of normal modes without using the method of finding the eigenvalues of $M^{-1}K$.
3. Change the left spring with a damping, find the equations of motion in the form of $M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = 0$
4. Write Matlab code that would give x_1 and x_2 at T. Assume any initial conditions that you like.

<1> The rotational inertia of the rod about O is:
$J = \frac{1}{3}m(3L)^2 = 3mL^2$

<2> $\sqrt{109} = 10.44$

Solution:

1. Using Langrange Equations to solve this problem

$E_k = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}J\left(\frac{\dot{x}_1}{3L}\right)^2 \qquad J = \frac{1}{3}m(3L)^2 = 3mL^2$

$E_{p_k} = \frac{1}{2}k\left(\frac{x_1}{3}\right)^2 + \frac{1}{2}k\left(\frac{2}{3}x_1\right)^2 + \frac{1}{2}k(x_2 - x_1)^2$

$\cancel{E_{p_G} = -mgx_1 - mgx_2 - mg\frac{x_1}{2} - C}$

So $L = E_k - E_p = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}J\left(\frac{\dot{x}_1}{3L}\right)^2 - \frac{1}{2}k\left(\frac{x_1}{3}\right)^2 - \frac{1}{2}k\left(\frac{2}{3}x_1\right)^2 - \frac{1}{2}k(x_2 - x_1)^2 \cancel{+ mgx_1 + mgx_2 + mg\frac{x_1}{2} + C}$

$= \frac{2}{3}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{5}{18}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 \cancel{+ \frac{3}{2}mgx_1 + mgx_2 + C}$

$\frac{\partial L}{\partial x_1} = -\frac{5}{9}kx_1 + k(x_2 - x_1) \cancel{+ \frac{3}{2}mg}$

$\frac{\partial L}{\partial \dot{x}_1} = \frac{4}{3}m\dot{x}_1$

$\frac{\partial L}{\partial x_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) \Rightarrow \frac{4}{3}m\ddot{x}_1 + \frac{5}{9}kx_1 - kx_2 + kx_1 = 0$

$\Rightarrow m\ddot{x}_1 + \frac{7}{6}kx_1 - \frac{3}{4}kx_2 = 0 \quad \cdots ①$

$\frac{\partial L}{\partial x_2} = -k(x_2 - x_1) \qquad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$

$\frac{\partial L}{\partial x_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) \Rightarrow m\ddot{x}_2 + kx_2 - kx_1 = 0 \quad \cdots ②$

Write ① and ② in a matrix form.

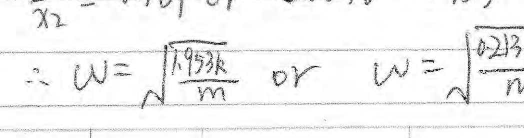
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{7}{6}k & -\frac{3}{4}k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

2. The motion equation is:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{7}{6}k & -\frac{3}{4}k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Which can be written as: $\begin{bmatrix} \frac{4}{3}m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{14}{9}k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

The system can be seen as the following system



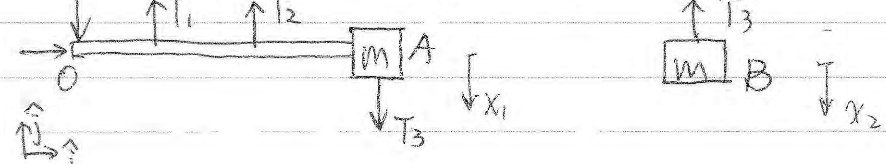
When the two masses are in sympathetic vibrations $\omega_1 = \omega_2 \Rightarrow \sqrt{\frac{k_{eff1}}{m_1}} = \sqrt{\frac{k_{eff2}}{m_2}}$

$\Rightarrow \frac{k_{eff1}}{m_1} = \frac{k_{eff2}}{m_2} \Rightarrow -\frac{-\frac{5}{9}kx_1 - kx_1 + kx_2}{x_1 \cdot m_1} = -\frac{-kx_2 + kx_1}{x_2 m_2}$

$\frac{x_1}{x_2} = 0.787$ or $\cancel{-0.276}$ $-0.953 \Rightarrow k_{eff2} = 1.953k$ or $0.213k$

$\therefore \omega = \sqrt{\frac{1.953k}{m}}$ or $\omega = \sqrt{\frac{0.213k}{m}}$

3. Draw the free-body-diagram of the rod-mass and the other mass.



where $T_1 = \frac{1}{3}kx_1$, $T_2 = \frac{2}{3}c\dot{x}_1$, $T_3 = k(x_2 - x_1)$

Write the AMB of the left system: $\vec{M}_{/O} = \dot{\vec{L}}_{/O}$, on the $\hat{k}$ direction

$$T_1L + 2T_2L - 3T_3L = -J\left(\frac{\ddot{x}_1}{3L}\right) - m(3L)^2\frac{\ddot{x}_1}{3L}$$

$$\Rightarrow 4m\ddot{x}_1 + \frac{1}{3}kx_1 + \frac{4}{3}c\dot{x}_1 - 3k(x_2 - x_1) = 0$$

$$\Rightarrow 4m\ddot{x}_1 + \frac{4}{3}c\dot{x}_1 + \frac{10}{3}kx_1 - 3kx_2 = 0 \quad \cdots \text{①}$$

Write the LMB of the right system:

$$m\ddot{x}_2 = -T_3 = k(x_1 - x_2)$$

$$\Rightarrow m\ddot{x}_2 + k(x_2 - x_1) = 0 \quad \cdots \text{②}$$

Write ① and ② in a matrix form

$$\begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{10}{3}k & -3k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

4. Let $z = [x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2]^T$

$$\dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \multicolumn{2}{c}{-M^{-1}K} & \multicolumn{2}{c}{-M^{-1}C} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \Rightarrow \dot{z} = Az$$

Matlab Code:

```
M=[4 0;0 1]; C=[4/3 0;0 0]; K=[10/3 -3;-1 1]; I=[1 0;0 1]; zero=[0 0;0 0]
A=[zero I; -M\K -M\C]; V0=[0 0]'; X0=[0 0]'; Z0=[X0; V0];
for t= 0:0.01:50
 z=expm(A.*t)*Z0;
 plot(t, z(1), t, z(2))   hold on
end
```