"SOLUTIONS" ANDY RUINA

# Cornell
# MAE 4735/5735

# Final Exam

Dec 10, 2012

No calculators, books or notes allowed.

5 Problems, 150 minutes, no extra time (Cornell rules)

# How to get the highest score?

*Please* do these things:

? If, when working on a problem, you have any *questions* about what you should or should not assume or write, please read these directions again.

• Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.

$\vec{v}_{\text{vect}}$ Use correct **vector notation**.

A+ Be (I) neat, (II) clear and (III) well organized.

□ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).

>> Make appropriate `Matlab code` clear and correct.
You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
Small syntax errors will have small penalties.

Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.

→| **Justify** your results so a grader can distinguish an informed answer from a guess.

• If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).

≈ Work for **partial credit** (from 60–100%, depending on the problem)

- Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
- Reduce the problem to a clearly defined set of equations to solve.
- Provide Matlab code which would generate the desired answer (and explain the nature of the output).

**Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
Note the last page is **blank** for your use. Ask for more extra paper if you need it.

| | |
|---|---|
| Problem 7: | /25 |
| Problem 8: | /25 |
| Problem 9: | /25 |
| Problem 10: | /25 |
| Problem 11: | /25 |

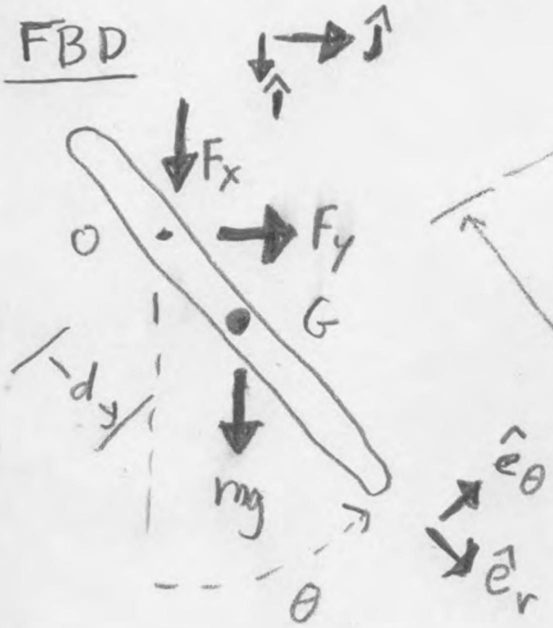
**7a)** 2D. A stick with length $\ell$, mass $m$ and moment of inertia $I$ about its center of mass G is suspended from a hinge on the stick a distance $d < \ell/2$ from G.

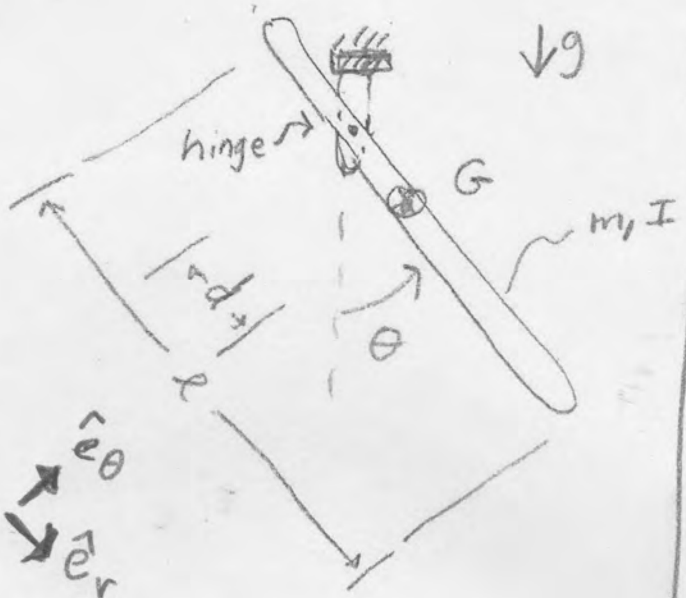
**(i)** Find the equations of motion.

**(ii)** For given $\theta, \dot{\theta}, m, I, \ell$ and $d$ find the force of the hinge on the pendulum (using any base vectors you like).

**(iii)** For given $\ell, m, I$ and gravitational acceleration $g$ for what $d$ is the period of small oscillation minimized. If you can reduce this to finding the root of a polynomial or transcendental equation, that is good enough, you need not find the root. [ Hints: Find the equations of motion → Solve them → Find the period of small oscillations → Minimize.]

FBD




call "d", "s".

$\omega^2 = \dfrac{mgs}{I + ms^2}$

Minimize ⇒

$d\omega^2/ds = 0$ ⇒

$(I + ms^2) - s \cdot 2ms = 0$

$ms^2 = I$

$s = \sqrt{I/m}$

$\boxed{d = \sqrt{I/m}}$

$\underline{AMB_{/O}}$ $\qquad \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$

$\{ -mgd\sin\theta\,\hat{k} = \vec{r}_{G/O} \times m\vec{a}_G + I\ddot{\theta}\hat{k}$

$\vec{r}_{G/O} = d\hat{e}_r \qquad [d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_r] = \vec{a}_G$

$= (md^2 + I)\ddot{\theta}\hat{k} \}$

(i)

$\{\}\cdot\hat{k} \Rightarrow \quad \boxed{\ddot{\theta} = \dfrac{-mgd}{I + md^2}\sin\theta} \Rightarrow \omega^2 = \dfrac{mgd}{I + md^2}$ ①

$\underline{LMB}$ $\qquad \sum \vec{F} = m\vec{a}$

$\underbrace{F_x\hat{\imath} + F_y\hat{\jmath}}_{\vec{F}} + mg\hat{\imath} = m(d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_r)$ (ii)

$\boxed{\vec{F} = -mg\hat{\imath} - \dfrac{m^2 g d^2}{I + md^2}\sin\theta\,\hat{e}_\theta - m d\dot{\theta}^2\hat{e}_r}$

**7b)** A block with mass $M$ slides without friction on a flat level surface. The top surface has slope $\gamma$. A smaller block with mass $m$ slides without friction on the sloped top of the lower block.

**(i)** Find two scalar equations from which you could, if you liked, solve for $\ddot{s}$ and $\ddot{x}$ in terms of some or all of $x, \dot{x}, s, \dot{s}, \gamma, m, M$ and $g$ [that is, you need not invert the mass matrix].

**(ii)** Some of the statements below are true, some are false, some may be partially true. Say which, and say why (with equations and/or words) clearly enough so that a skeptic would be convinced.

(a) system potential energy is conserved

NO. $s \neq 0 \Rightarrow$ P.E. changes.

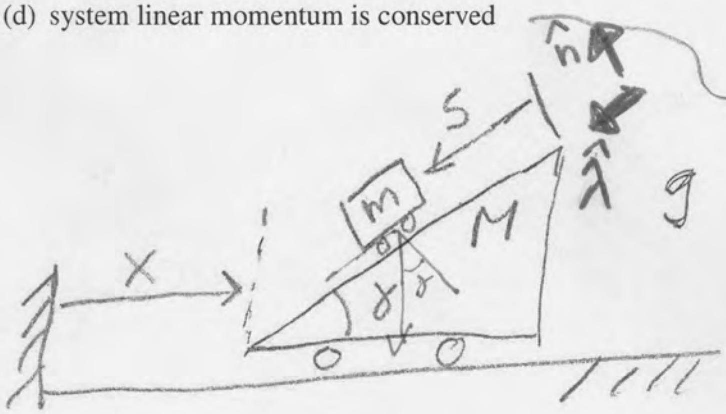
(b) system kinetic energy is conserved

NO. $E_K(t=0) = 0$. Then it increases

(c) system total energy is conserved

Yes. All forces are conservative or workless constraints

(d) system linear momentum is conserved




Partially

see system FBD

$\{\vec{F}_{tot} = m_{tot}\vec{a}\} \cdot \hat{\imath}$

$\Rightarrow m_{tot} a_x = 0 \Rightarrow$ $\boxed{L_x \text{ conserved}}$

$L_y$ not

System FBD

$\{LMB\} \cdot \hat{\imath} \Rightarrow 0 = \boxed{(M+m)\ddot{x} - \ddot{s}\cos\gamma\, m}$ (1)

Small block FBD


$\underline{LMB} \Rightarrow N\hat{n} - mg\hat{\jmath}$

$= m(\underbrace{\ddot{x}\hat{\imath} + \ddot{s}\hat{\lambda}}_{\vec{a}})$

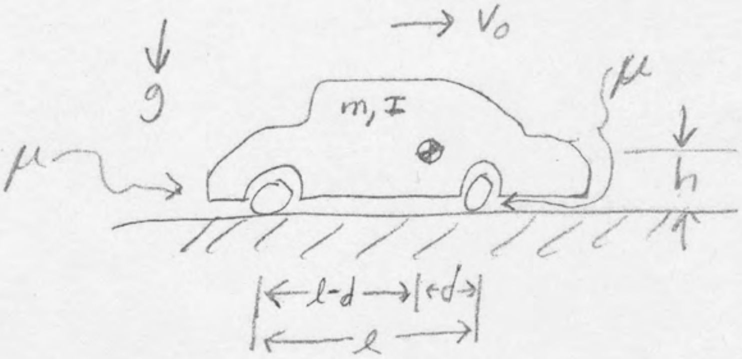
$\{LMB\} \cdot \hat{\lambda} \Rightarrow$

$-mg\underbrace{\hat{\jmath}\cdot\hat{\lambda}}_{-\sin\gamma} = m(-\ddot{x}\cos\gamma + \ddot{s})$

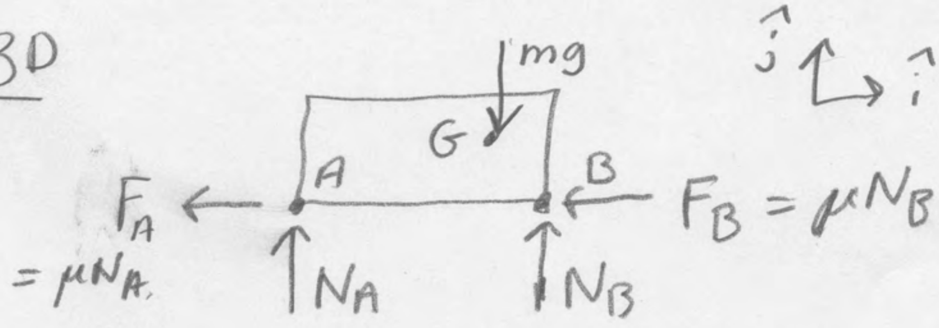
$\boxed{g\sin\gamma = -\ddot{x}\cos\gamma + \ddot{s}}$ (2)

$\boxed{\text{(1) \& (2) are 2 eqs for } \ddot{x} \text{ \& } \ddot{s}}$

**8a)** A car with mass $m$ moves at speed $v_0$ and suddenly slams on the brakes. All four wheels skid with friction coefficient $\mu$ (and friction angle $\phi$ with $\tan\phi = \mu$). Assume the suspension is rigid. In terms of some or all of $d, \ell, h, \mu, \phi, m, I$ and gravity $g$ find how long it takes for the car to come to a stop.




FBD




$\boxed{\vec{a} = a\hat{\imath}}$

LMB

$$\left\{ \sum \vec{F} = m\vec{a} \right\}$$

$$\{\ \}\cdot\hat{\imath} \Rightarrow -\mu N_A - \mu N_B = ma$$

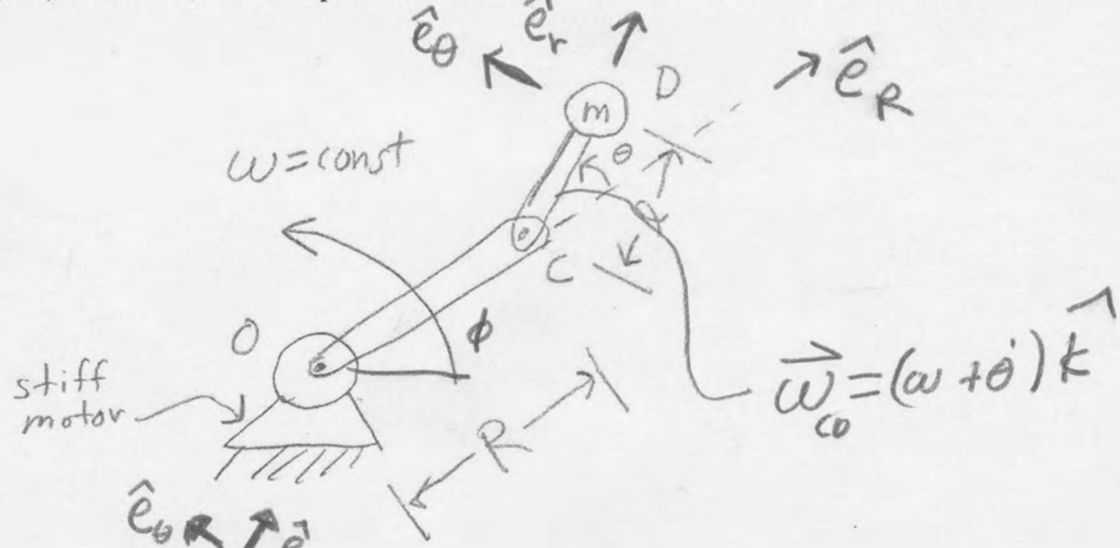
$$-\mu(N_A + N_B) = ma$$

$$\{\ \}\cdot\hat{\jmath} \Rightarrow N_A + N_B - mg = 0$$

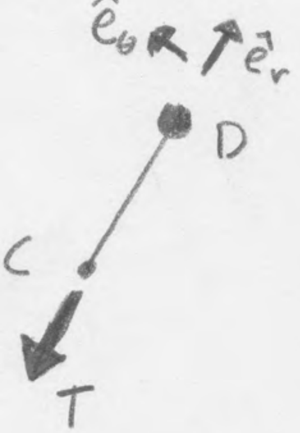
$$\Rightarrow \boxed{a = -\mu g}$$

$$V = V_0 + at = 0 \Rightarrow \boxed{t = \frac{V_0}{\mu g}}$$

**8b)** No gravity. One link of a pendulum has radius $R$ and is powered by a stiff motor to rotate at a fixed rate $\omega$. The second link has length $\ell$, is massless, and has a point mass $m$ at the end. Find $\ddot{\theta}$ in terms of some or all of $\theta, \dot{\theta}, R, \ell$ and $\omega$.




FBD




AMB$_{/C}$: $\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$$\vec{0} = \underbrace{\vec{r}_{D/C}}_{\ell \hat{e}_r} \times m\vec{a}_D, \qquad \vec{a}_D = \vec{a}_C + \vec{a}_{D/C}$$

$$\vec{a}_C = -R\omega^2 \hat{e}_R, \qquad \vec{a}_{D/C} = \ell(\ddot{\theta} + \underbrace{\dot{\omega}}_{0})\hat{e}_\theta - \ell(\underbrace{\dot{\theta}+\omega}_{\omega_{CO}})^2 \hat{e}_r, \qquad \dot{\omega}_{CO}$$

$$\left\{ \vec{0} = m\left[ -\underbrace{\hat{e}_r \times \hat{e}_R}_{-\sin\theta \hat{k}} \ell R\omega^2 + \ell^2 \ddot{\theta} \underbrace{\hat{e}_r \times \hat{e}_\theta}_{\hat{k}} \right] \right\}$$

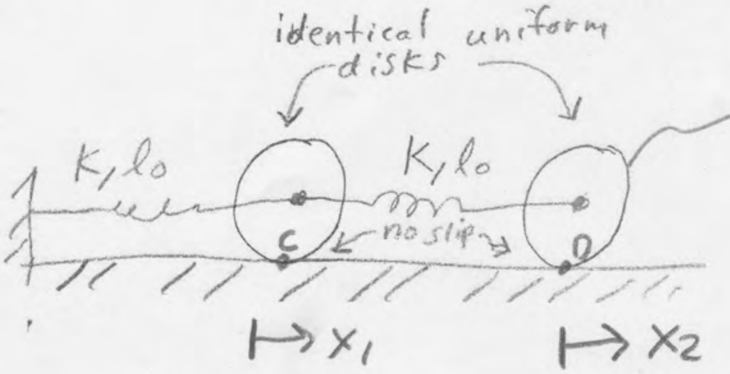
$$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \frac{-R}{\ell}\omega^2 \sin\theta}$$

simple pendulum equation

Note: $\dot{\phi} = \omega \Rightarrow \phi = \omega t \Rightarrow \ddot{\theta} = \theta''\omega^2$ where $()' = d()/d\phi \Rightarrow \theta''\omega^2 = \frac{-R}{\ell}\omega^2 \sin\theta$

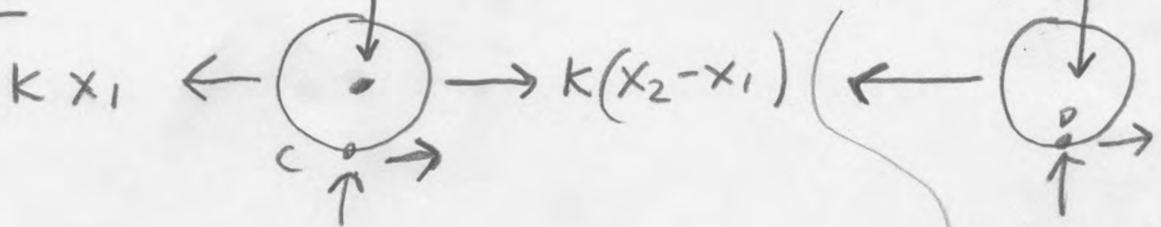
$\Rightarrow$ For small $\theta$ $\theta'' = -\frac{R}{\ell}\phi$. So, if $R = \ell \Rightarrow \theta = A\cos\phi + B\sin\phi \Rightarrow$ $\boxed{\text{one oscillation per revolution}}$

**9)** Two uniform round disks with mass $m$ roll without slipping on a flat plane. They are connected to each other and to the left wall with two springs, both with stiffness $k$ and rest length $\ell_0$. Find one normal mode and it's corresponding frequency.




$I = mR^2/2$

## FBDs




$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$\ddot{\theta}_1$, $mR^2/2$

$$-kx_1 \cancel{R} + k(x_2 - x_1)\cancel{R} = -m\ddot{x}_1 \cancel{R} - \left(\frac{\ddot{x}_1}{R}\right) I$$

$$\boxed{-2kx_1 + kx_2 = \frac{3}{2} m\ddot{x}_1}$$

$$\sum \vec{M}_{/D} = \dot{\vec{H}}_{/D}$$

$$-kR(x_2 - x_1) = -m\ddot{x}_2 R - \frac{\ddot{x}_2}{R} I$$

$$\boxed{kx_1 - kx_2 = \frac{3}{2} m\ddot{x}_2}$$

Assume modes shape: $\vec{v}$

Equal masses $\Rightarrow$ equal $k_{eff}$

$k_{eff_1} = k_{eff_2}$

force, force, disp, disp

$$-\left(\frac{-2k \cdot 1 + kv}{1}\right) = -\left(\frac{k \cdot 1 - kv}{v}\right)$$

$$-2v + v^2 = 1 - v$$

$$v^2 - v - 1 = 0$$

$$v = \frac{1 \pm \sqrt{1+4}}{2}$$

use $v = \frac{1+\sqrt{5}}{2}$

$$\omega^2 = \frac{-k(v-2)}{\text{"}m\text{"}} = \frac{-k\left(\frac{-3}{2} + \frac{\sqrt{5}}{2}\right)}{(3/2)m}$$

$$\boxed{\vec{v} = \begin{bmatrix} 1 \\ (1+\sqrt{5})/2 \end{bmatrix} \text{ (mode)} \qquad \text{freq. } \omega = \sqrt{\frac{k}{m}}\sqrt{1 - \sqrt{5}/3}}$$

**10a)** This problem concerns the classical 1 DOF spring-mass system with $m, k$ and $c$.

**(a)** Make a clear picture of the system.

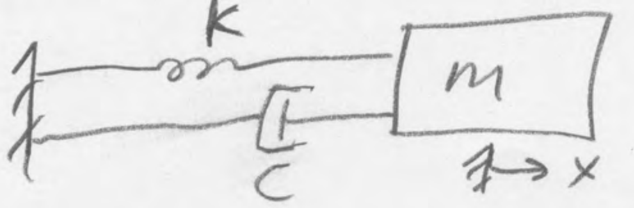
**(b)** Using any mechanics method you like, find the equations of motion: $m\ddot{x} + c\dot{x} + kx = 0$.

**(c)** Reduce this to standard form: $\ddot{x} + 2\omega\eta\dot{x} + \omega^2 x = 0$

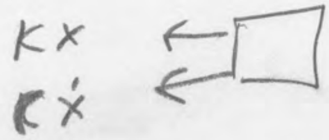
**(d)** Define both $\omega$ and $\eta$ with equations. Explain the meaning of both terms with words.

**(e)** Find an equation for the damped natural frequency $\omega_d$.

(a)




$\rightarrow X$




$\left\{\sum \vec{F} = m\vec{a}\right\} \cdot \hat{\imath} \Rightarrow -kx - c\dot{x} = m\ddot{x}$

$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = 0}$ (b)

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

define $\omega^2 = k/m$

if no damping $\Rightarrow$ freq. of osc. $\Rightarrow$ $x = A\cos(\omega)t$

$\omega$ = undamped natural freq.

$\ddot{x} + \frac{c}{m}\frac{\sqrt{k/m}}{\sqrt{k/m}}\dot{x} + \omega^2 x = 0$

$\ddot{x} + 2\frac{c}{2\sqrt{km}}\omega\dot{x} + \omega^2 x = 0$

$\ddot{x} + 2\omega\xi\dot{x} + \omega^2 x = 0$

$\xi = \frac{c}{\sqrt{2km}} = \frac{c}{c_{crit}} = \boxed{\frac{\text{damping}}{\text{critical damping}}}$

Assume $x = e^{\lambda t}$

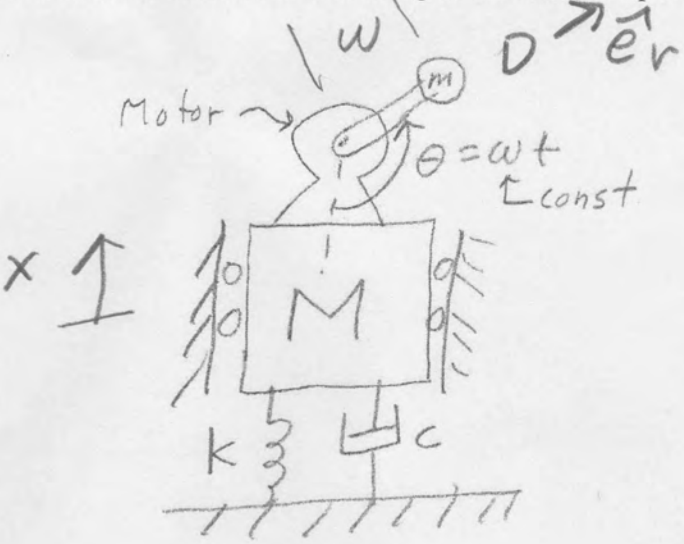
$\Rightarrow \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

$\Rightarrow \lambda = \frac{-c/m \mp \sqrt{\left(\frac{c}{m}\right)^2 - 4k/m}}{2}$

$\Rightarrow$ $x = e^{-\frac{c}{2m}t}\left[\cos\left(\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\right)t + i\sin(\quad)t\right]$

$\boxed{\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}$

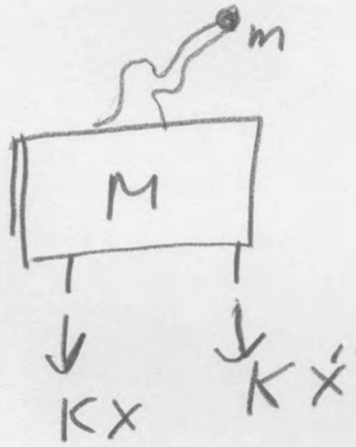
**10b)** A platform and motor with total mass $M$ are supported by a spring $(k, \ell_0)$ and a dashpot $(c)$. The motor spins at constant rate $\omega$ so that $\theta = \omega t$. The motor spins an eccentric mass $m$ which is a distance $d$ from the motor shaft. Find the equations of motion for the mass $M$ consisting of the motor and platform (relative to the static equilibrium position).




$$\boxed{\hat{e}_r \cdot \hat{\imath} = -\cos \omega t}$$

FBD

leave off gravity because motion is rel. to equilib. so, by superposition it drops out




LMB:

$$\sum \vec{F} = m_D \vec{a}_D + M \ddot{x} \hat{\imath}$$

$$-kx - c\dot{x} = m_D \vec{a}_D \cdot \hat{\imath} + M \ddot{x}$$

$$\vec{a}_D = \ddot{x}\hat{\imath} - \ell \omega^2 \hat{e}_r$$

$$= m_D \ddot{x} - m_D \ell \omega^2 \cos \omega t$$

$$\boxed{(M+m)\ddot{x} + c\dot{x} + kx = m \ell \omega^2 \cos \omega t}$$

**11)** A system has 7 degrees of freedom paramaterized by the components of the 7-element vector $\vec{x}$. The equations of motion, for small motion, are:

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$$

where the matrices $M$ and $K$ are given and symmetric.

**(i)** Assume that $C = 0$. Write MATLAB commands to find a constant vector $\vec{v}$ and $\omega$ so that

$$\vec{x}(t) = \sin(\omega t)\vec{v}$$

is a solution of the governing equations. Just one normal mode solution is desired.

**(ii)** Clearly define at least one non-zero damping matrix $C$ so that, with an appropriate change of variables the equations can be re-written as a set of decoupled scalar equations of the form:

$$\ddot{r}_i + 2\omega_i\eta_i\dot{r}_i + \omega_i^2 r_i = 0.$$

As for all problems, justify your result.

**(iii)** Clearly define the most general damping matrix $C$ for which, with an appropriate change of variables the equations can be re-written as a set of decoupled scalar equations (as written above).

i) M = ...

K = ...

(i)

```
[V D] = eig(M\K);
v = V(:,6)            % v
w = [D(6,6)]^(1/2)    % w
```

ii) If $\{M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}\}$

$M^{-1/2}\vec{q} = \vec{x}$

$M^{-1/2}\cdot\{\} \Rightarrow \left[\ddot{\vec{q}} + M^{-1/2}CM^{-1/2}\dot{\vec{q}} + \underbrace{M^{-1/2}KM^{-1/2}}_{\tilde{K}}\vec{q} = \vec{0}\right]$

if $V = [\vec{v}_1 | \vec{v}_2 \ldots]$ evectors of $\tilde{K}$

and $\vec{q} = V\vec{r}$

$V^{-1}\cdot[\;] \Rightarrow \ddot{\vec{r}} + \underbrace{V^{-1}M^{-1/2}CM^{-1/2}V}_{\hat{C}}\dot{\vec{r}} + \Lambda\vec{r} = \vec{0}$

$\Lambda$ diagonal

If $\boxed{\hat{C} = \alpha K + \beta M \text{ its diagonal}}$

most general $C$ For diag. $\hat{C}$

Assume $\hat{C}$ = diagonal $\Rightarrow$ $\boxed{C = M^{1/2}V\hat{C}V^{-1}M^{1/2}}$