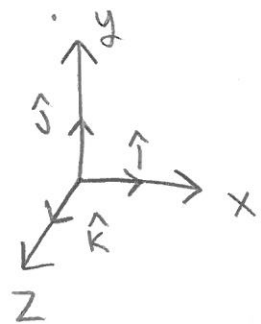


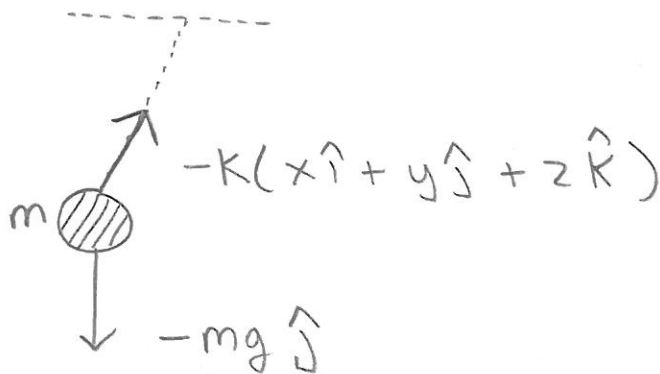
(Time to complete = 2 hours)

3) Mass hanging from a spring. Consider a point mass hanging from a zero-rest-length linear spring in a constant gravitational field.

a) Set up equations. Set up for numerical solution. Plot 2D projection of 3D trajectories.



FBD



We know that for a linear spring, $\vec{F} = -K r \hat{r}$

$$\Rightarrow \vec{F} = -K(x\hat{i} + y\hat{j} + z\hat{k})$$

For this problem, we will assume that gravity is acting in the $-\hat{j}$ direction.

LMB

$$\vec{F} = m \vec{a}$$

$$-K(x\hat{i} + y\hat{j} + z\hat{k}) - mg\hat{j} = m \vec{a}$$

3) a) (continued)

We also can write $\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$

$$\Rightarrow -k(x\hat{i} + y\hat{j} + z\hat{k}) - mg\hat{j} = m(a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$$

Now, we must equate like components:

$$\hat{i}\text{-direction: } -kx = ma_x$$

$$\hat{j}\text{-direction: } -ky - mg = ma_y$$

$$\hat{k}\text{-direction: } -kz = ma_z$$

First, we will solve this equation in \hat{i} -direction:

$$-kx = ma_x$$

$$a_x = \left(-\frac{k}{m}\right)x$$

We know that $a_x = \ddot{x}$

$$\Rightarrow \ddot{x} = \left(-\frac{k}{m}\right)x$$

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

3) a) (continued)

The characteristic equation is

$$r^2 + \frac{k}{m} = 0$$

$$\Rightarrow r = \pm \sqrt{\frac{k}{m}} i$$

We then have

$$* x = A \cos\left(\sqrt{\frac{k}{m}} t\right) + B \sin\left(\sqrt{\frac{k}{m}} t\right) *$$

Now, we will solve the equation in the \hat{j} -direction:

$$-ky - mg = ma_y$$

$$a_y = \left(-\frac{k}{m}\right)y - g$$

We also know that $a_y = \ddot{y}$

$$\Rightarrow \ddot{y} = \left(-\frac{k}{m}\right)y - g$$

$$\ddot{y} + \left(\frac{k}{m}\right)y = -g$$

3) a) (continued)

First, we will solve the characteristic equation for the homogenous case

$$\Rightarrow r^2 + \frac{k}{m} = 0$$

$$r = \pm \sqrt{\frac{k}{m}} i$$

$$\Rightarrow y_c = C \cos(\sqrt{\frac{k}{m}} t) + D \sin(\sqrt{\frac{k}{m}} t)$$

We also need the particular solution, which must be some constant E based on the fact that the right-hand side of the differential equation is a constant

$$\Rightarrow y_p = E$$

$$\ddot{y}_p + \left(\frac{k}{m}\right) y_p = -g \quad \checkmark$$

$$0 + \left(\frac{k}{m}\right) E = -g$$

$$\Rightarrow E = -\frac{mg}{k}$$

$$\Rightarrow y_p = -\frac{mg}{k}$$

3) a) (continued)

The full solution is the linear superposition of the complementary and particular solution

$$\Rightarrow y = y_c + y_p$$

$$* y = C \cos\left(\sqrt{\frac{k}{m}} t\right) + D \sin\left(\sqrt{\frac{k}{m}} t\right) - \frac{mg}{k} *$$

Finally, we will solve the equation in the k-direction:

$$-kz = ma_z$$

We have $a_z = \ddot{z}$

$$\Rightarrow m\ddot{z} = -kz$$

$$\ddot{z} + \left(\frac{k}{m}\right)z = 0$$

The characteristic equation is

$$r^2 + \frac{k}{m} = 0$$

$$\Rightarrow r = \pm \sqrt{\frac{k}{m}} i$$

3) a) (continued)

The solution is then

$$* z = F \cos\left(\sqrt{\frac{k}{m}} t\right) + G \sin\left(\sqrt{\frac{k}{m}} t\right) *$$

In this way, we can construct a position vector

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$



$$\vec{r}(t) = \left[A \cos\left(\sqrt{\frac{k}{m}} t\right) + B \sin\left(\sqrt{\frac{k}{m}} t\right) \right] \hat{i} + \left[C \cos\left(\sqrt{\frac{k}{m}} t\right) + D \sin\left(\sqrt{\frac{k}{m}} t\right) - \frac{mg}{k} \right] \hat{j} + \left[F \cos\left(\sqrt{\frac{k}{m}} t\right) + G \sin\left(\sqrt{\frac{k}{m}} t\right) \right] \hat{k}$$

For numerical solution, see attached Matlab code and attached plots of 2D projections of 3D trajectories. I will choose to display the motion in the x-y plane. ✓

```
%By  
%MAE 5735  
%HW 2, Problem 3a
```

```
%This program finds a numerical solution to the problem pertaining to the  
%motion of a point mass hanging from a zero-rest-length linear spring in a  
%constant gravitational field. We are only considering motion in the x-y  
%plane.
```

```
clc  
clear all  
close all
```

```
%Givens  
g = 1; %Graviational acceleration in m/s^2  
m = 1; %Mass in kg  
k = 1; %Spring constant in N/m
```

```
%Initial vectors of interest  
r = [0 0]'; %Initial position vector  
v = [1 0]'; %Initial velocity vector
```

```
%Implementation of Euler's method  
step = 0.001; %Step size in seconds used in the Euler method  
time_span = 10; %Total time in seconds used to calculate trajectory  
for j = 1:(time_span/step)  
    F = -k*r-(m*g)*[0 1]'; %Total force due to restoring force of spring  
                        %and gravity  
    a = F./m; %Acceleration vector  
    r = r+(step)*v; %Updating the position vector  
    v = v+(step)*a; %Updating the velocity vector  
    plot(r(1),r(2),'ro') %Plotting the trajectory  
    hold on
```

```
end  
axis square  
xlabel('x [m]')  
ylabel('y [m]')  
title('Hanging mass trajectory for t = 2 s using Euler method with step size 0.001 s  
)
```

3) b)

By playing around with initial conditions, find the most wild motion you can find. Make one or more revealing plots.

* See attached plots of 2D projections of 3D trajectories *

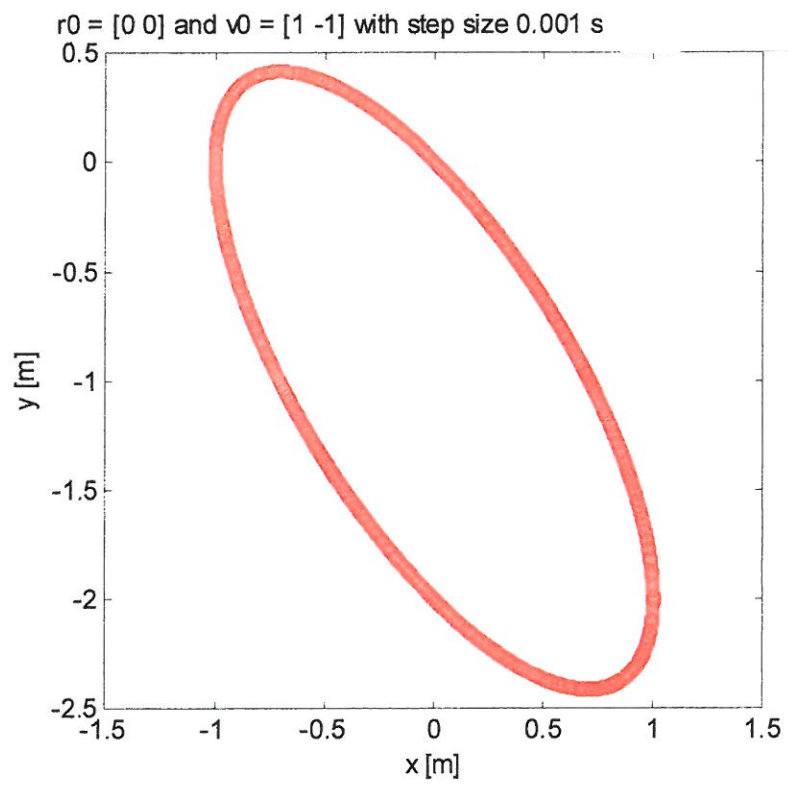
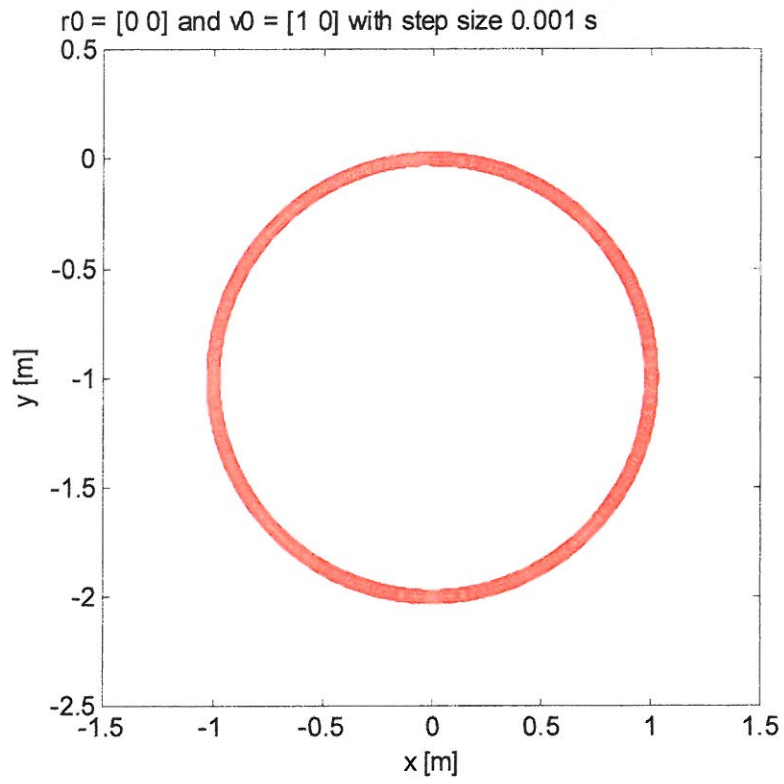
These plots were generated by varying the initial conditions in the previous Matlab code .

NOTE: For my numerical solution, I selected the following values for necessary constants :

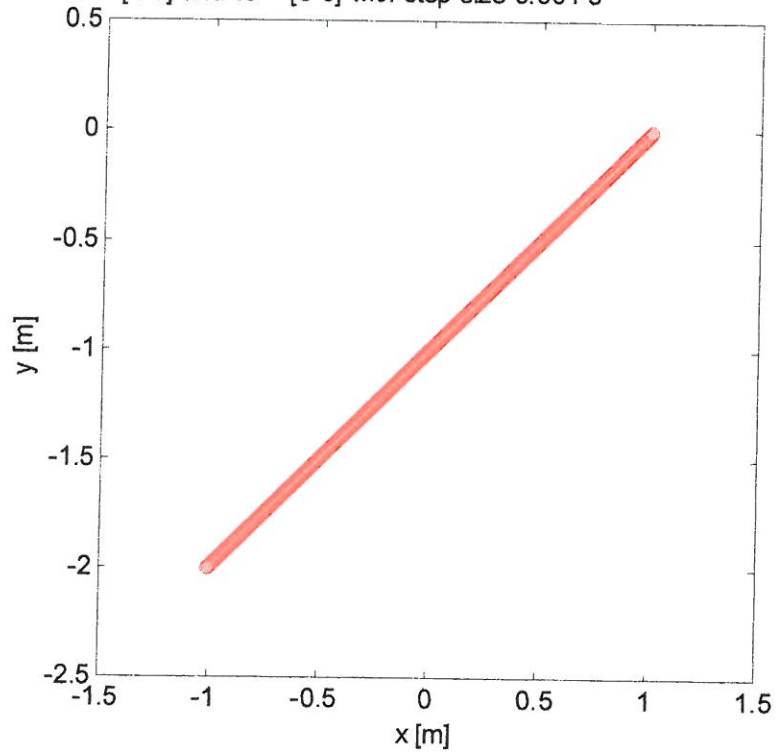
$$g = 1 \text{ m/s}^2$$

$$m = 1 \text{ Kg}$$

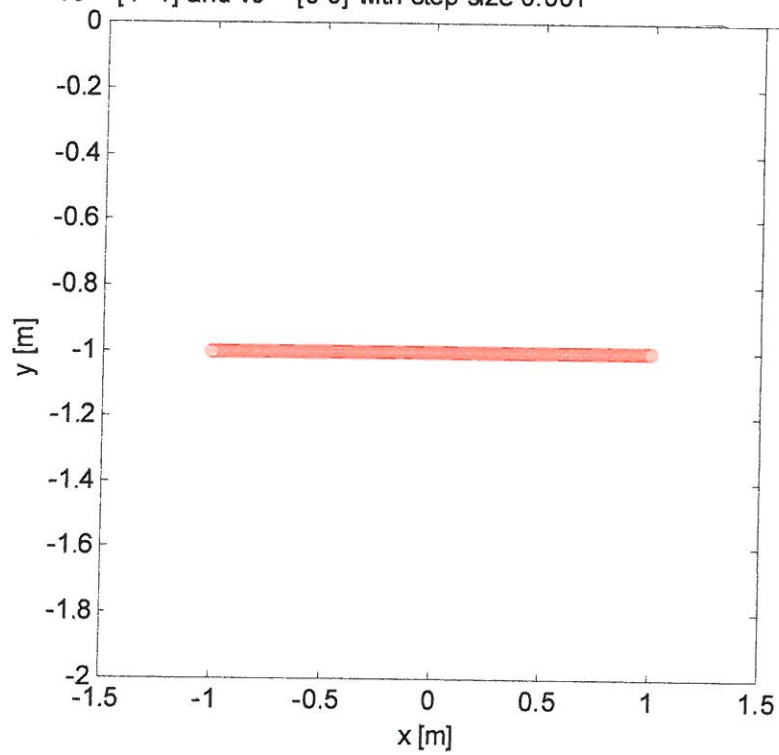
$$K = 1 \text{ N/m}$$



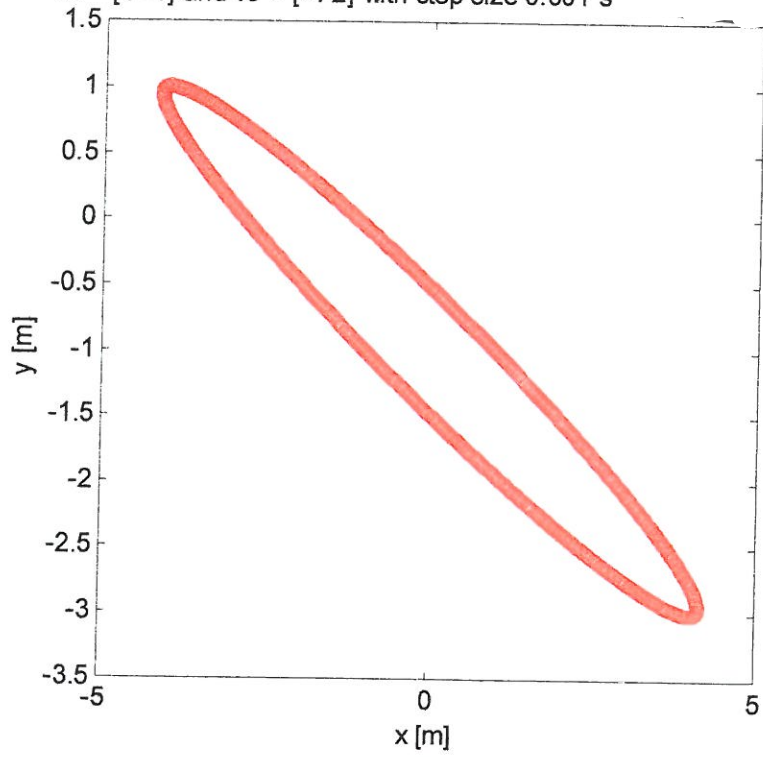
$r_0 = [1 \ 0]$ and $v_0 = [0 \ 0]$ with step size 0.001 s



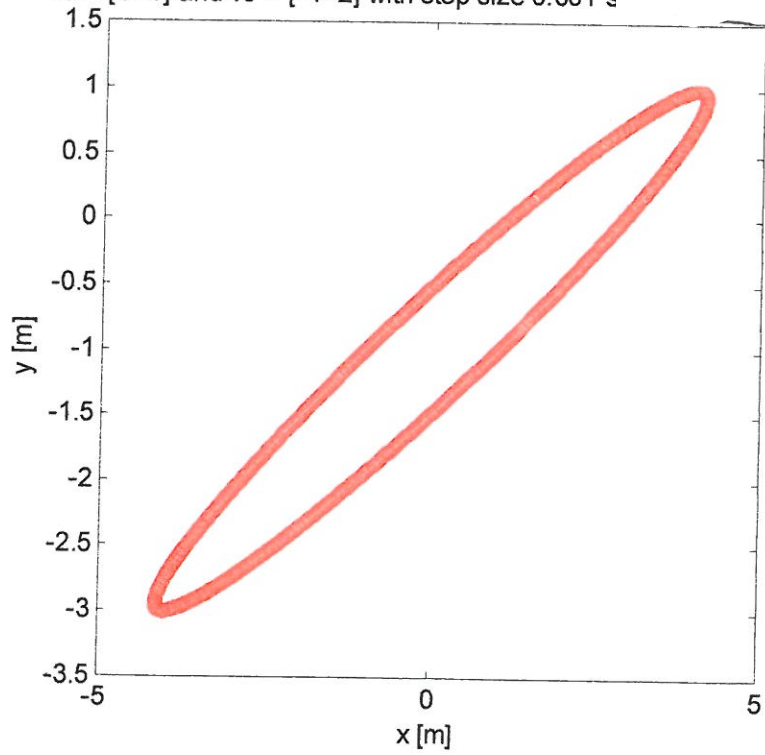
$r_0 = [1 \ -1]$ and $v_0 = [0 \ 0]$ with step size 0.001



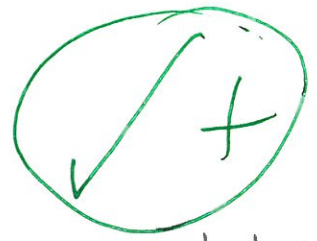
$r_0 = [1 \ -1]$ and $v_0 = [-4 \ 2]$ with step size 0.001 s



$r_0 = [1 \ -1]$ and $v_0 = [-4 \ -2]$ with step size 0.001 s



3) b) (continued)



As can be seen from the previous plots, various 2D trajectories that are achievable are straight lines, circles, and ellipses. From this set, I would say that the most wild motion I can find is an ellipse.

Additionally, I checked that the features observed for these trajectories are indeed properties of the system and not due to numerical errors. I decreased the time step-size and saw no appreciable change in the trajectories. This led me to conclude that numerical errors were negligible. ✓

(Not to mention your analytical solution.)

3) c)

Using analytical methods justify your answer to part (b).

From part (a), I have derived analytically that the trajectory is

$$\vec{r}(t) = \left[A \cos\left(\sqrt{\frac{k}{m}}t\right) + B \sin\left(\sqrt{\frac{k}{m}}t\right) \right] \hat{i} + \left[C \cos\left(\sqrt{\frac{k}{m}}t\right) + D \sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{mg}{k} \right] \hat{j} + \left[F \cos\left(\sqrt{\frac{k}{m}}t\right) + G \sin\left(\sqrt{\frac{k}{m}}t\right) \right] \hat{k}$$

Since I am only considering 2D projections of the trajectory onto the x-y axes, I can remove the \hat{k} -component and be left with

$$\vec{r}(t) = \left[A \cos\left(\sqrt{\frac{k}{m}}t\right) + B \sin\left(\sqrt{\frac{k}{m}}t\right) \right] \hat{i} + \left[C \cos\left(\sqrt{\frac{k}{m}}t\right) + D \sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{mg}{k} \right] \hat{j}$$

3) c) (continued)

We can compare the analytical relation with the parametric form of an ellipse to gain some insights. The parametric form of an ellipse is

$$\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j}$$

where

$$x(t) = x_c + a \cos t \cos \phi - b \sin t \sin \phi$$
$$y(t) = y_c + a \cos t \sin \phi + b \sin t \cos \phi$$

and (x_c, y_c) is the center of the ellipse, the parameter t varies from 0 to 2π , and ϕ is the angle between the x -axis and the major axis of the ellipse.

$$\Rightarrow \vec{r}(t) = [x_c + a \cos t \cos \phi - b \sin t \sin \phi] \hat{i} + [y_c + a \cos t \sin \phi + b \sin t \cos \phi] \hat{j}$$

Comparing this parametric form of an ellipse to the previously derived analytical formulation, it is easy to see that with an appropriate choice of initial conditions, these formulations describe the same shape.

3) c) (continued)

=> Ellipse trajectory is possible

Also, looking at the analytic formulation, it is clear that a circular trajectory is also possible since a circle is just a special form of an ellipse. Finally, the straight-line motion that was observed is also possible from the analytic solution since you can just imagine "squishing" an ellipse until it becomes a straight line. Nevertheless, out of all of these possible trajectories the "most wild" one is arguably an ellipse.