MAE 5735 : DYNAMICS & VIBRATIONS

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Lecture 1 : Wed 8/22/12 Phil. 403

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HW: Due Wed. in class

I. Intro to Mechanics

00. Basic assumptions
   a) Time & space work like we expect: space is flat, time is measurable & ~~ob~~observer-independent, etc
   b) Matter is identifiable, continuous, etc.

0. Force is the means of mechanical interaction (Newton I can be interpreted as a definition of force – identifying it by what happens in its absence)
   - Free Body Diagram: drawing of a system of interest and all external forces acting on it.

The 3 pillars of mechanics (1950's redefinition, "rational mechanics")

1. Materials Properties :
   Constitutive Laws

e.g.

$T = k\Delta l$	tension on a spring
$\vec{F} = -mg\hat{j}$	near-earth gravity
$F = -\frac{mM\vec{r}}{r^3}$	gravitation
$\vec{F} = -c\vec{v} = -cA\vec{v}$	linear drag
$\vec{F} = -cv\vec{v} = -cv^2\frac{\vec{v}}{\lvert\vec{v}\rvert}$	quadratic drag

* Our ideas about force are derived from observation of material properties

(3 pillars cot'd)

2. Geometry & Kinematics
(positions, lengths, angles, time)

e.g.

linear $\quad \vec{a} = \dfrac{d^2\vec{r}}{dt^2}$

polar rotation $\quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$

3. Laws of Mechanics

A. Linear Momentum Balance

$$\sum \vec{F}_{ext} = \dot{\vec{L}}$$

($L \equiv$ linear momentum

where $\dot{\vec{L}} \equiv \sum m_j \vec{a}_j \longleftrightarrow \int \vec{a}\, dm$

B. Angular Momentum Balance

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

moments due to all ext. forces

where $\dot{\vec{H}}_{/c} = \sum \vec{r}_{j/c} \times (m_j \vec{a}_j)$ discrete

$\longleftrightarrow \int \vec{r}_{/c} \times \vec{a}\, dm$

C. Energy & Power

Power in $= \dot{E}_{total}$ ← kinetic + potential + "internal" (heat, etc)

electrical, mechanical + heat flow

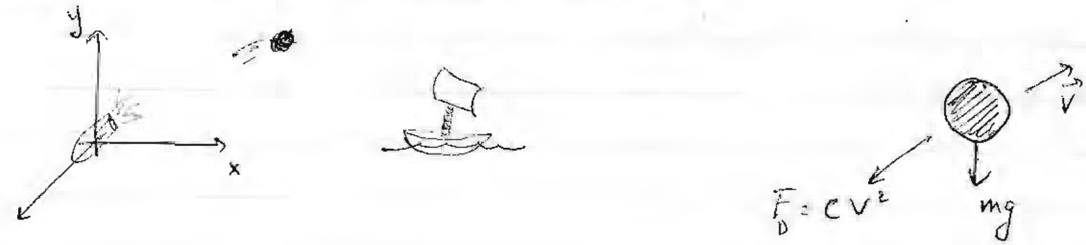
* It is widely agreed that the three "pillars" are all correct. Which one(s) are fundamental/axiomatic is a matter of some debate.

8/24 $\boxed{\text{DYNAMICS \& VIBRATIONS}}$

Lecture II:

PARTICLE DYNAMICS (e.g. ballistics)

e.g. ballistics



$\hat{\imath} = \underset{\sim}{e}_x = \underset{\sim}{e}_1$

$\hat{\jmath} = \underset{\sim}{e}_y = \underset{\sim}{e}_2$

$c = c_1 \rho A$

Linear momentum

$\vec{a}$ always means particle acceleration. The form it takes is a geometrical property (spherical, cylindrical, moving frame, etc.

$$\sum F = \dot{L}$$
$$= m\vec{a} = m(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath})$$
$$= m(\dot{v}_x\,\hat{\imath} + \dot{v}_y\,\hat{\jmath})$$

Now, the drag force is directed opposite the velocity, i.e. $-\frac{\vec{v}}{|\vec{v}|}$. So

$$F_D = cv^2\left(-\frac{\vec{v}}{|\vec{v}|}\right)$$

and

$$F_G = -mg\hat{\jmath}$$

So – paying close attention to signs and directions – our ODE is

$$\boxed{-mg\hat{\jmath} - cv\vec{v} = m\vec{a}}$$

This can't (probably) be solved analytically. So, we use numerical methods.

If at time t, we know $\vec{r}$ (position of cannonball) and $\vec{v}$ (velocity), then what is its position & velocity at time $t + \Delta t \iff t + h$? ($h = \Delta t$ "small")

$$\vec{r}(t+h) \approx \vec{r}(t) + h\vec{v}(t)$$
$$\vec{v}(t+h) \approx \vec{v}(t) + h\vec{a}(t)$$

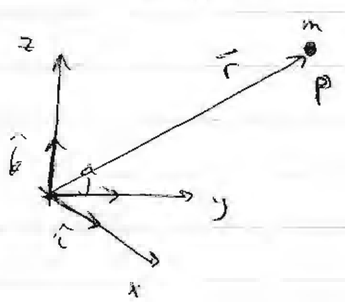
$$\vec{a} = \frac{-mg\hat{\jmath} - cv\vec{v}}{m}$$

Then, we iterate...

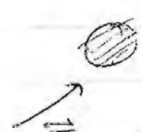
```
#define constants
   g = 10;
     :
   r = [0 0]';                  // ' transposes
   dmax = v0 *ts                // ts := time step
   for   i = 1:n
        F = ...
        :
        plot (x, y, 'r*')                  // 'r*' = red stars on plot
        axis ([0, dmax  0 dmax])
        axis ('square')
        hold on
        pause (.001)
   end
```

Fixed ("Newtonian") frames

$\mathcal{F}\ \mathcal{N}$



FBD



$\vec{F}_{tot} = m\vec{a}$

$$\vec{a} = \begin{cases} \vec{a}_{/\mathcal{F}} \\ \vec{a}_{P/\mathcal{F}} \\ \dfrac{{}^{\mathcal{F}}d\, \vec{v}_{P/\mathcal{F}}}{dt} \\ \dfrac{{}^{\mathcal{F}}d}{dt}\left(\dfrac{{}^{\mathcal{F}}d\,\vec{r}_{P/O}}{dt}\right) \end{cases}$$

multiple notations for expressing $\vec{a}$ with respect to the fixed frame $\mathcal{F}$

8/24
Dynamics

$$\vec{F} = \vec{0}$$

$\boxed{\text{Ex.}}$

$$= \vec{c} \; (\text{eg} \quad -mg\hat{k})$$

$$= \frac{-mMG}{r^3}\,\vec{r}_m \qquad \text{"inverse square"}$$

$$= -k\vec{r}$$

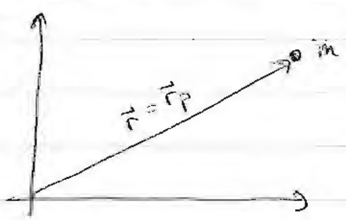
$$= -c\vec{v} \qquad \text{"linear drag"}$$

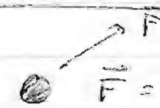
$$= -cv\vec{v} \qquad \text{"quadratic drag"}$$

8/27
DYNAMICS

PARTICLE MECHANICS THEOREMS (cotd.)

$\mathcal{F}$





$\vec{F} = m\vec{a}$

We observe:

$$\frac{d\vec{v}}{dt} = \vec{a}$$

$$\vec{F} = \frac{d\vec{v}}{dt}\,m$$

"// by the pillars of mechanics", would we call these observations constitutive laws or geometry/kinematics?

We define:

$$\vec{L} \equiv m\vec{v}$$

We derive (from above):

$$\boxed{\vec{F}_{tot} = \dot{\vec{L}}}$$

What else can we do? Well, lets dot $\vec{v}$ into the above:

$$\vec{F}\cdot\vec{v} = m\,\vec{a}\cdot\vec{v}$$

NB: $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \dot{\vec{v}}\cdot\vec{v} + \vec{v}\cdot\dot{\vec{v}}$
$= 2\vec{v}\cdot\dot{\vec{v}} = 2\dot{\vec{v}}\cdot\vec{v}$
$= 2\vec{a}\cdot\vec{v}$

So

$$\vec{F}\cdot\vec{v} = \tfrac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v})$$

$$= \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) \qquad // \; v = |\vec{v}|.$$

$$= \dot{E}_k$$

But $\vec{F}\cdot\vec{v}$ is the power "done" by Force $\vec{F}$, so

$$\boxed{P = \dot{E}_k}$$

"power is rate of change of KE"

Let's look now at power, $\boxed{P = \vec{F} \cdot \vec{v}}$

Aside: CONSERVATIVE FORCES

But first, an aside on <u>conservative forces</u>.

A conservative force is a function of position <u>only</u>, NOT velocity, time, etc:

$$\vec{F} = \vec{F}(\vec{r})$$

I. Conservative forces can always be <u>written as the gradient of some scaler ~~field~~ function</u>:

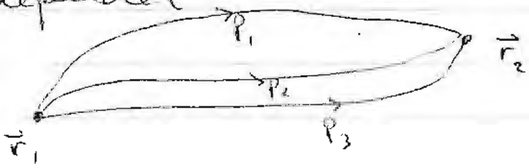
$$\vec{F} = -\vec{\nabla} V \qquad V \equiv \text{potential energy}$$

We sometimes call it E_p.

Some examples of famous potential functions...

$$E_p = \begin{cases} m\frac{c}{r} \\ \frac{1}{2}kr^2 \\ mgz \end{cases} \Rightarrow \vec{F} = \begin{cases} -\frac{c}{r^2}\hat{e}_r \\ -k\vec{r} \\ -mg\hat{k} \end{cases}$$

II. The total work done by a conservative force is path independent:



$W_1 = W_2 = W_3$
$= E_{p2} - E_{p1}$

III. The <u>curl = 0</u>

$$\vec{\nabla} \times \vec{F} = \vec{0}$$

IV $\oint dW = 0 \qquad dW = \vec{F} \cdot d\vec{r}$

The work done over any closed loop = 0.

Ex. Find E_p associated with: $\boxed{\vec{F} = x\hat{j} - y\hat{i}}$

$$\vec{F} = -\vec{\nabla} E_p$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) E_p$$
$$= \frac{\partial E_p}{\partial x}\hat{i} + \frac{\partial E_p}{\partial y}\hat{j}$$

$$\frac{\partial E_p}{\partial y} = x \qquad \frac{\partial E_p}{\partial x} = -y$$

$\cancel{E_p = xy z - gx}$

$$E_p = \int \frac{\partial E_p}{\partial y}\, dy = xy + g(x)$$

$$\frac{\partial E_p}{\partial x} = y + \frac{dg}{dx} = -y \Rightarrow \frac{dg}{dx} = -2y \Rightarrow g(x) = -y^2$$

$$E_p = xy - y^2 \qquad E_{p_x} = y \qquad E_{p_y} = x - 2y$$

$$E_p = \int \frac{\partial E_p}{\partial x}\, dx = -yx + h(y)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{matrix} \hat{i}\,(\cancel{1-1})^{0} \\ -\hat{j}\,(0) \\ +\hat{k}\,(1+1) \end{matrix} = 2\hat{k} \neq 0 \Rightarrow \boxed{\text{NO POTENTIAL EXISTS}}$$

Back to our particle mechanics...

$$\vec{F} = m\vec{a}$$
$$\vec{F}\cdot\vec{v} = m\,\vec{a}\cdot\vec{v}$$

Mult. by dt:
$$\vec{F}\cdot\vec{v}\,dt = m\,\vec{a}\cdot\vec{v}\,dt = m\left(\tfrac{1}{2}\,\dot{\vec{v}}\cdot\vec{v}\right)dt = \tfrac{1}{2}mv^2\,dt$$
$$\vec{F}\cdot d\vec{r} = dE_k$$

$$\Rightarrow \boxed{\int \vec{F}\cdot d\vec{r} = \Delta E_k}$$ "Work - Energy Theorem"

$$\Leftrightarrow \text{(1-D)} \quad F\Delta x = \Delta E_k$$

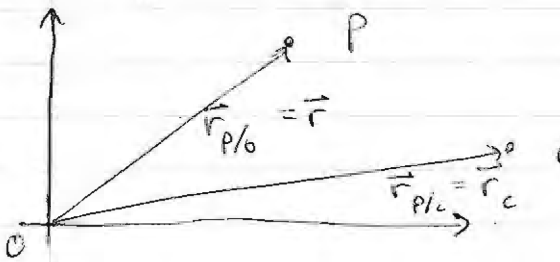
In the special case that $\vec{F}$ is conservative,

$$-\Delta E_p = E_k$$
$$\Delta(E_p + E_k) = 0$$

$$\boxed{\Delta E_{TOT} = 0}$$ "Conservation of Energy"

* In numerical calculations, things like conservation of energy can be used to check for errors in code or machine rounding / algorithmic error, etc.

Angular Momentum from $\vec{F} = m\vec{a}$



$$\vec{F} = m\vec{a}$$

$$\boxed{\vec{r} \times \vec{F} = \vec{r} \times m\vec{a}}$$ WRT origin O

$$\boxed{\vec{r}_{/c} \times \vec{F} = \vec{r}_{/c} \times m\vec{a}}$$ WRT point C

* $\vec{r}_{/c}$ is a position vector to a coordinate frame with origin at C, but with axes oriented the same as $\mathcal{F}$ (which has origin at O).

8/29

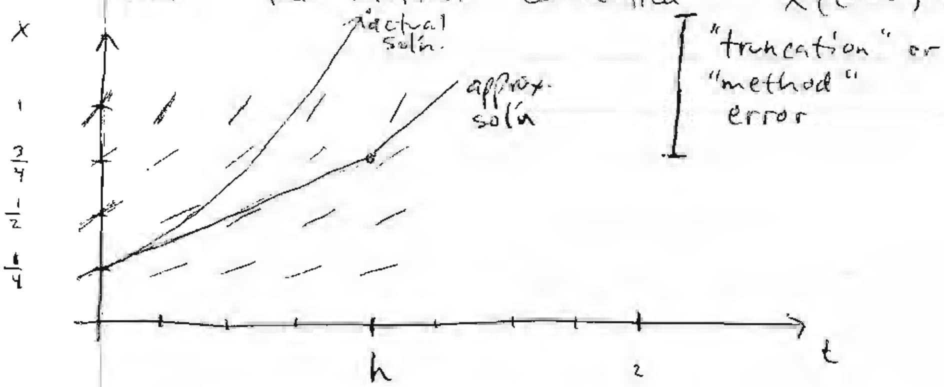
DYNAMICS LECTURE

Today: Euler vs. higher order methods
& Central force motion

Euler vs. Higher Order

ODE: "direction field"

Let's consider $\dot{x} = f(x)$, $f(x) = x$
with the initial condition $x(t=0) = 1/4$



$\dot{x} = x$ means that at any time t, the slope is just $x(t)$.

The actual soln is tangent to the slope field everywhere. The approximation is tangent only at discrete multiples of time step h.

*Aside: rounding error in the computer is random, and can be + or −. So $\varepsilon_{rounding}^2$ grows linearly with n, but $\varepsilon_{rounding}$ grows linearly with $\sqrt{n}$

Roundoff error $\approx \sqrt{n}\,\epsilon$ where $\epsilon \equiv$ roundoff error for each step
$n \equiv$ # of steps

e.g. $\sqrt{10^{12}} \cdot 10^{-16} \approx 10^{-10}$
$= 10^{12}$ steps
$= 10^{-16}$

The next best Euler-based method takes the "midpoint slope" rather than the left-hand slope over an interval h.

Simple Euler

```
x = 1;
ts = 1;
n = 10^5;
h = ts/n;

for i = 1:n
  ẋ = x;
  x = x + hẋ;
end

error = norm(exp(1) - x)
```

→

```
n = 10^5/2                      // decrease

for i = 1:n
  ẋ_temp = x;
  x_temp = x + (h/2) ẋ_temp;    // advance 1/2 step, find slope there
                                   then use it to take a full step.
  ẋ = x_temp
  x = x + hẋ
end
```

// this line

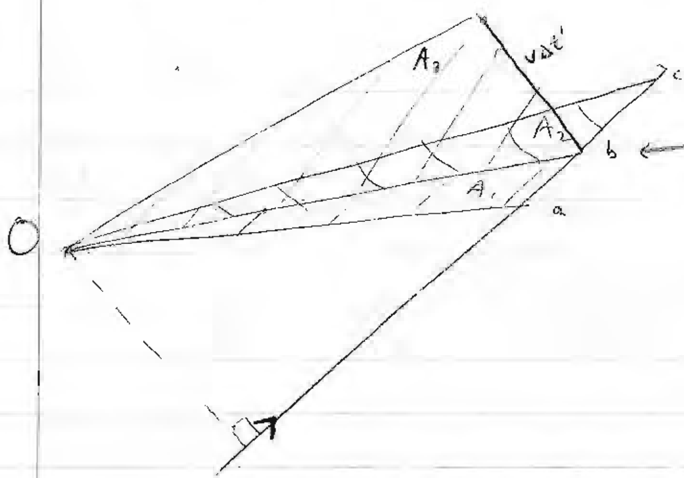
```
clf

g=10;  m=1;  c=0;
n=200;
ts = 3;
h = ts/n;
v_0 = 20;   θ = π/4;

dmax = .8 v_0/ts

r = [0 0]'
v = v_0 [cos θ  sin θ]';

for i=1:n
   F = mg [0 1]' - c*norm(v)·v;
   r_temp = r + v h/2 ;
   a_temp = F/m ;
   v_temp = v + a h/2 ;

   F = m·g·[0 -1] - c*norm(v_temp) · v_temp ;
   a_temp = F/m;
   r = r + v_temp *h ;
   v = v + a_temp *h ;

   x = r(1);   y = r(2);
   plot(x,y, 'r*')
   axis ('square')
   hold on          // hold on keeps points up on the plot window.
   pause (0.00001)
end
```

* To change to central force motion, we just change F to

$$F = -GmMr/(\text{norm}(r))^3$$

(and tweak n, ts, dmax, v, etc.)

Here's a proof from Newton's Principia. Feynman also did it in his Messenger Lecture #2 (on YouTube).



Consider a particle moving along straight path.

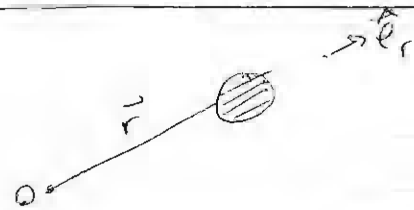
$$A_1 = A_2 = bv\frac{\Delta t}{2}$$

$$A_2 = A_3 = b'\frac{h'}{2}$$

... I drew it poorly, but anyway. The statement $A_1 = A_3$ is equivalent to conservation of linear momentum ~~(equal areas swept out in equal time)~~, and it follows purely from geometry & cons. of linear momentum.

8/31 : Came to LECTURE late...

$\boxed{\text{Central force cont'd ; multiple particles}}$



$$\vec{F} = F\hat{e}_r \qquad \hat{e}_r = \frac{\vec{r}}{|\vec{r}|}$$

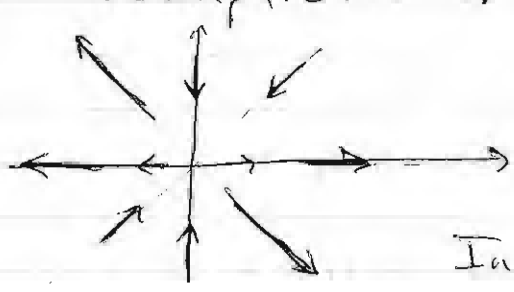
AMB
$$\vec{M} = \dot{\vec{H}}$$
$$\vec{r}\times\vec{F} = \dot{\vec{H}}$$

But $\vec{F} = F\hat{e}_r$ and $\vec{r} \parallel \hat{e}_r \Rightarrow \vec{r}\times\hat{e}_r = 0$

So
$$\dot{\vec{H}} = 0 \Rightarrow \vec{H} = \text{CONST.}$$

Common assumption: $\vec{F} = F\hat{e}_r$

e.g.



$\vec{F} = F(\vec{r})\hat{e}_r$

here $F \neq F(r)$ only!

In general, $\vec{F} = F(\vec{r}, \vec{v}, t)$

Two famous examples:

$$\vec{F} = -\frac{GmM}{r^2}\hat{e}_r \quad , \quad \vec{F} = -k\vec{r} = -kr\hat{e}_r$$

NB: $\vec{F} = F(r)\hat{e}_r \Rightarrow F$ is CONSERVATIVE

$$\Leftrightarrow E_p = -\int_{r_0}^{r} F(r')\,dr'$$

e.g. $F = -\frac{GmM}{r^2}$, $r_0 = \infty \Rightarrow \boxed{E_p = \frac{-GMm}{r}}$

e.g. $\vec{F} = -kr$, $r_0 = 0 \Rightarrow \boxed{E_p = \frac{1}{2}kr^2}$

Why does $\vec{F} = F(r)\hat{e}_r \Rightarrow F$ conservative?

Well,
$$\int_{\vec{r}_1}^{\vec{r}_2} dW = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} F(r)\,\hat{e}_r \cdot d\vec{r}$$

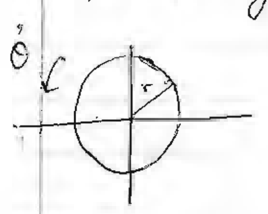
$dr = d\vec{r}\cdot\hat{e}_r$ = change in radius

$\boxed{\text{TWO FACTS}}$

rue ∀ orbits / central force motion

①	$\vec{H}$ = CONST	② $E_{TOT} = E_k + E_p$ = CONST

3rd, less general fact for <u>circular orbits</u>:

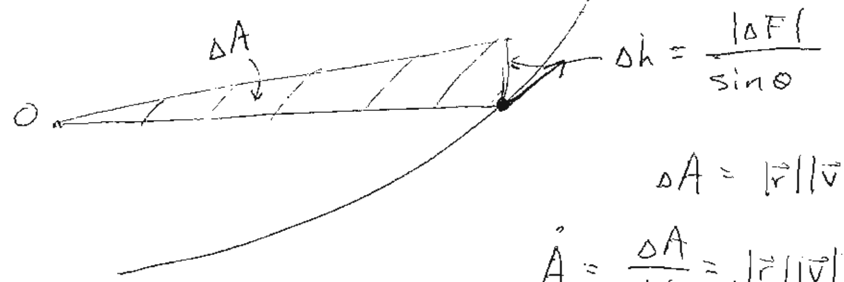


$$\vec{r} = r\cos(\dot{\theta}t)\,\hat{i} + r\sin(\dot{\theta}t)\,\hat{j}$$

$$\vec{a} = \ddot{\vec{r}} = -\dot{\theta}^2\vec{r} \Rightarrow a = \dot{\theta}^2 r = \frac{v^2}{r}$$

So $\boxed{\vec{F} = m\vec{a} = F(r) = \frac{mv^2}{r}}$ (3) for circular orbits ONLY

Aside: Alternate AMB derivation:



$\Delta h = \frac{|\Delta \vec{r}|}{\sin\theta}$ $\qquad \Delta\vec{r} = \vec{v}\cdot\Delta t$

$$\Delta A = |\vec{r}||\vec{v}|\sin\theta\,\Delta t$$

$$\dot{A} = \frac{\Delta A}{\Delta t} = |\vec{r}||\vec{v}|\sin\theta = |\vec{r}\times\vec{v}|$$

define $\vec{H} \equiv m\vec{r}\times\vec{v}$

$\vec{H}_{/o}$

then $\dot{\vec{H}} = m\left[\cancel{\dot{\vec{r}}\times\vec{v}} + \vec{r}\times\dot{\vec{v}}\right] = \vec{r}\times m\vec{a}$

$$\dot{\vec{H}} = \vec{r}\times\vec{F}$$

assume $\vec{F} = F(r)\hat{e}_r$, then $\dot{\vec{H}} = \vec{r}\times\vec{F} = \vec{r}\times(F(r))\hat{e}_r = 0$

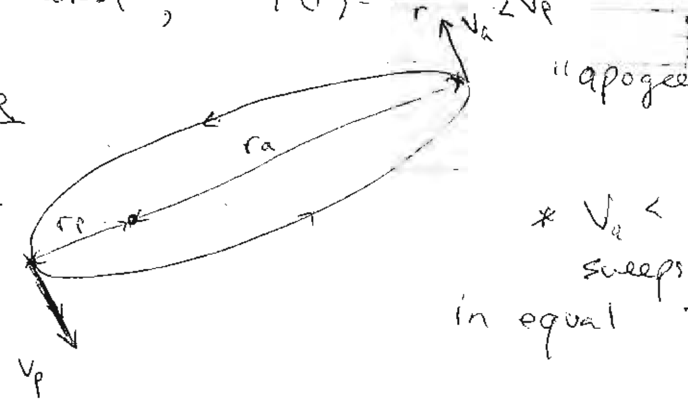
since $\vec{r}\times\hat{e}_r = 0$

So $\dot{\vec{H}} = 0 \Rightarrow \vec{H}$ conserved.

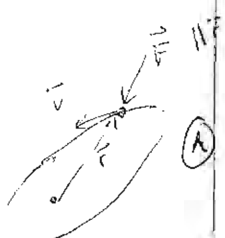
---

ex's: Algebraic solutions of central force motion using our three handy results: $\vec{H}$ = CONST, $E_{TOT} = E_p + E_k$ = CONST, $F(r) = \frac{mv^2}{r}$

e.g.: Inverse Square

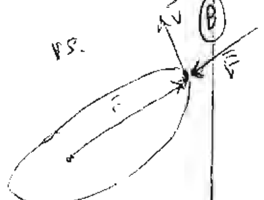


* $V_a < V_p$ because $\vec{r}$ sweeps out equal areas in equal time.



(A)

vs.



(B)

~~NB in (A),~~ $|\vec{H}_a| = |\vec{H}_p| \Leftrightarrow \boxed{r_a v_a = r_p v_p}$ ① AMB

② Energy B

$\Rightarrow E_{TOT,a} = E_{TOT,p} \Leftrightarrow \boxed{\frac{1}{2}mv_a^2 - \frac{mMG}{r_a} = \frac{1}{2}mv_p^2 - \frac{mMG}{r_p}}$

and $\boxed{\frac{mMG}{r^2} = \frac{mv^2}{r}}$ ③

A whole class of problems can be solved with these three.

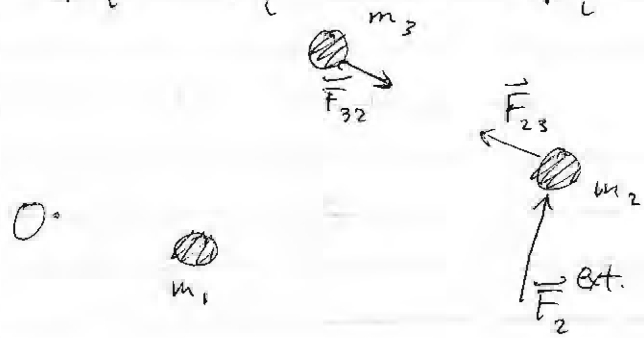
MULTI - PARTICLE SYSTEMS

We understand these as ~~a~~ sets of individual particles, and we write our $\vec{F} = m\vec{a}$ equations for each.

$$\vec{F} = m\vec{a} \longrightarrow \vec{F}_i = m_i \vec{a}_i$$

It is useful to separate internal vs. external forces:

$$\vec{F}_i = \vec{F}_i^{\,\text{internal}} + \vec{F}_i^{\,\text{external}}$$



$\vec{F}_{23} \equiv$ force <u>on</u> m_2 <u>by</u> m_3

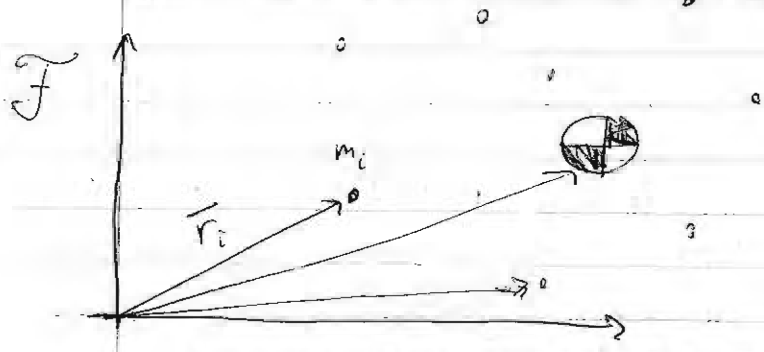
The principle of action & reaction tells us that

$$\vec{F}_{23} = \vec{F}_{32}$$

These internal forces are called "pair-wise equal and opposite." Newton thought that ALL forces were like like this. But they're not: e.g. atoms in a solid crystal..

DYNAMICS LECTURE : 9/5

Systems of Particles (continued)



Center of Mass

$\equiv$ G, CoM, CM, ⊕, etc.

$\equiv$ "average position" of all the material

$$\boxed{\vec{r}_G = \sum \frac{\vec{r}_i m_i}{m_{TOT}} \qquad m_{TOT} = \sum_i m_i}$$

Indicial Notation

$$m_{TOT}\,\vec{r}_G = \sum \vec{r}_i\, m_i$$

$$m_{TOT}\,\vec{v}_G = \sum \vec{v}_i\, m_i$$

$$m_{TOT}\,\vec{a}_G = \sum \vec{a}_i\, m_i$$

For continuum,

$$m_{TOT}\,\vec{r}_G = \int_m \vec{r}_i\, dm$$

etc.

What claims can we make about systems of particles?
We're interested in $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/c}$, $\dot{\vec{H}}_{/c}$, E_k, $\dot{E}_k$

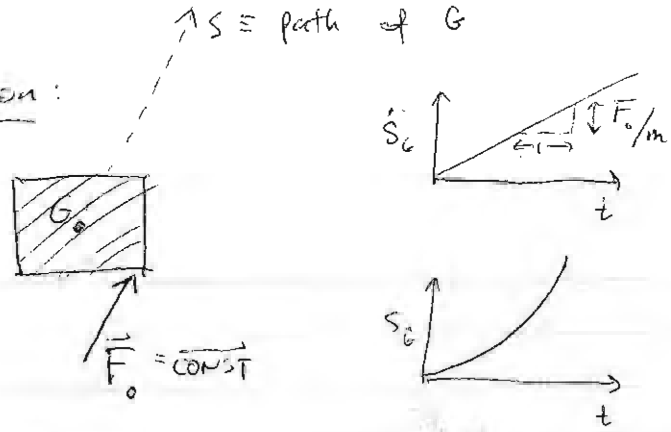
Linear Momentum

LMB: $\sum \vec{F}^{ext} = \sum m_i \vec{a}_i = \dot{\vec{L}}$

$$\sum \vec{F}^{ext} = m_{TOT}\,\vec{a}_G$$

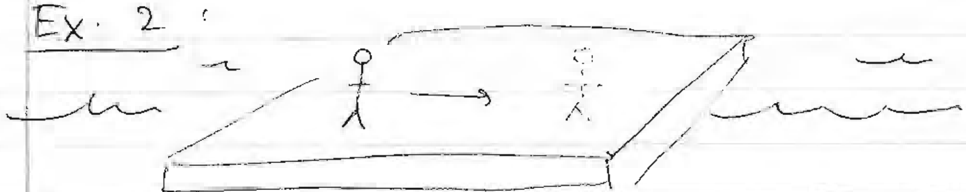
In words: The CoM of a system moves precisely like a particle with the same total mass and being acted upon by the same total force.

Illustration:



Although the plate certainly rotates, the CoM moves precisely like a single particle of mass m_{TOT} moving under constant force $\vec{F}_0$ applied at G.

Ex. 2:



(If there's friction, CoM moves with the walker)

Person walks on frictionless boat: CoM of system (= boat + person) doesn't move.

Ex. 3: Locomotion

System initially at rest, $\sum \vec{F}^{ext} = \vec{0}$

LMB: $\vec{0} = \dot{\vec{L}}$

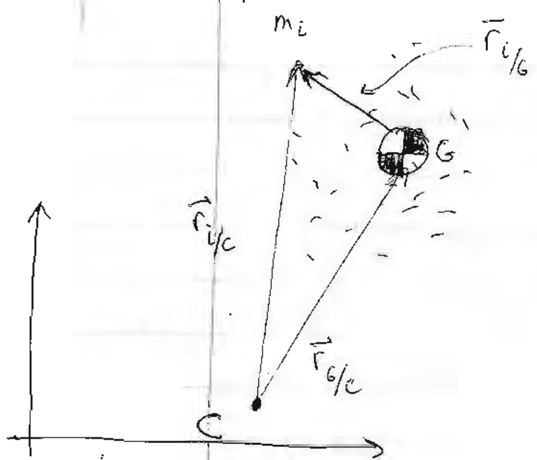
$$\Rightarrow \vec{L} = \overrightarrow{\text{CONST}} = \vec{0} \text{ (by initial conditions)}$$

$$\Rightarrow \vec{v}_G = \vec{0}$$

$$\Rightarrow \vec{r}_G = \overrightarrow{\text{CONST}} \Rightarrow \text{CoM stationary}$$

Systems of Particles cont'd.

ANGULAR MOMENTUM



The total AM WRT C is:

$$\boxed{\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{v}_i}$$

We want to simplify this ~~by~~ with ideas about CoM:

$$\vec{r}_{i/c} = \vec{r}_{G/c} + \vec{r}_{i/G}$$

* Call

$$\vec{v}_{i/c} = \vec{v}_{G/c} + \vec{v}_{i/G}$$

$$\vec{H}_{/c} = \sum \left[\left(\vec{r}_G + \vec{r}_{i/G} \right) \times \left(\vec{v}_G \times \vec{v}_{i/G} \right) m_i \right]$$

= 4 terms under a (distributable) sum

$$= \sum \vec{r}_G \times \vec{v}_G \, m_i \qquad (a)$$

$$+ \sum \vec{r}_{i/G} \times \vec{v}_G \, m_i \qquad (b)$$

$$+ \sum \vec{r}_G \times \vec{v}_{i/G} \, m_i \qquad (c)$$

$$+ \sum \vec{r}_{i/G} \times \vec{v}_{i/G} \, m_i \qquad (d)$$

Let's simplify......

(a) Now $\vec{r}_G$ & $\vec{v}_G$ don't depend on index i,

so we have $\sum_i \vec{r}_G \times \vec{v}_G \, m_i = \vec{r}_G \times \vec{v}_G \underbrace{\sum_i m_i}_{m_{TOT}}$

so (a) $\longrightarrow$ $\vec{r}_G \times \vec{v}_G \, m_{TOT}$

(b) $\sum \vec{r}_i \, m_i$ is the position of CoM, measured relative to the CoM. That means it's <u>zero</u>. Algebra can prove it, too.

so (b) $\longrightarrow$ $\vec{O}$

(c) $\sum \vec{r}_{G/c} \times \vec{v}_{i/o} \, m_i = \vec{r}_{G/c} \times \underbrace{\left(\sum m_i \vec{v}_{i/G} \right)}_{\vec{O}}$

so (c) $\longrightarrow$ $\vec{O}$ because the sum of momenta measured WRT CoM is .. zero?

$\overline{a}_{CM} = 0$ iff $\sum \vec{F}_{ext} = 0$

(d) : No luck!

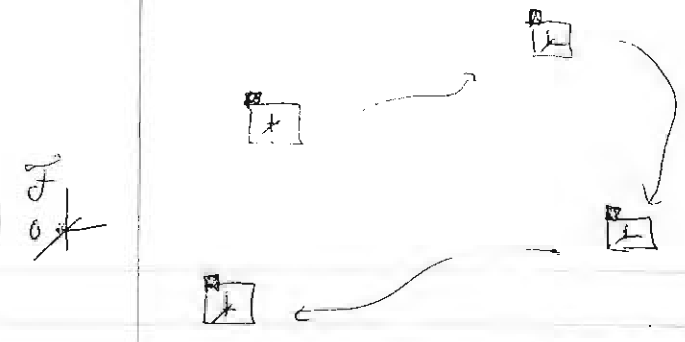
Thus,

$$\vec{H}_{/c} = \vec{r}_{G/c} \times \vec{v}_G \, m_{TOT} + \sum \vec{r}_{i/G} \times \vec{v}_{i/G} \, m_i$$

$$* \ \vec{v}_{i/G} = \vec{v}_{i/o} - \vec{v}_{G/o}$$

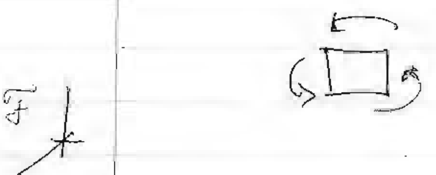
$$\vec{H}_{/c} = \vec{H}_{G/c} + \underbrace{\vec{H}_{/G}}_{\equiv \vec{H}_{/rel}}$$

So, relative to C, the total AM is the AM we'd get if we imagined a single particle at G; PLUS the "relative AM", i.e. the AM of the system WRT $\vec{r}_G$ (CoM frame)



Since the body isn't rotating WRT its CoM, $\vec{H}_{rel} = 0$.

But it rotates WRT origin of $\mathcal{F}$, so $\vec{H}_{G/c}$ is nonzero and computed just like AM for one particle @ G.



$\vec{H}_{G/c} = 0$ (no CoM motion WRT $\mathcal{F}$

Interesting Facts

If $C = G$ instantaneously, then

$$\vec{H}_{/G} = \sum \vec{r}_{i/G} \times m_i \vec{V}_{i/G}$$

and $$\underbrace{\sum \vec{M}^{ext}_{/G}}_{\text{external torques}} = \dot{\vec{H}}_{G} = \sum \vec{r}_{i/G} \times m_i \vec{a}_{i/G}$$

Angular Locomotion

No external torques $\iff \sum \vec{M}_{/G} = 0$

$$\Rightarrow \dot{\vec{H}}_{/G} = \vec{0} \Rightarrow \vec{H}_{G} = \overrightarrow{\text{CONST}} = \vec{0} \quad \left(\text{by initial conditions}\right)$$

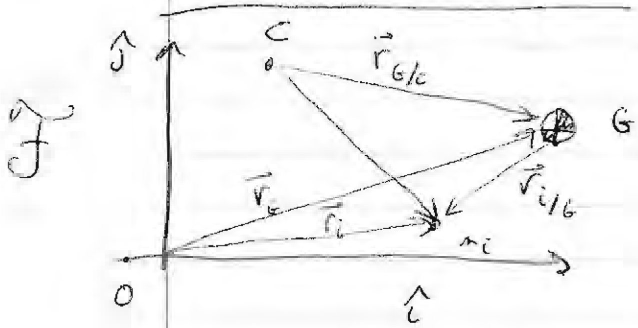
$$\boxed{\sum \vec{r}_{i/G} \times m_i V_{i/G} = \vec{0}}$$

* This isn't a conservation law in the same way that $\sum \vec{L}_i^{ext} = 0$ is, because there does not exist a quantity ~~sto~~ $\beta(t)$ s.t. $\frac{d}{dt}\beta(t) = \sum \vec{r}_{i/G} \times m_i \vec{V}_{i/G}$

LECTURE 9/7 :

1. More multi-particles
2. Intro to (motivation for) constraints
3. Video (Skylab), Matlab

MULTI-PARTICLE Cont'd.



<u>Recall:</u> I $\quad \dot{\vec{L}} = \sum m_i \vec{a}_i = m_{TOT}\, \vec{a}_G$

II $\quad \dot{\vec{H}}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{a}_i = \vec{r}_{G/c} \times m_{TOT}\, \vec{a}_G + \vec{r}_{i/G} \times m_i \vec{a}_{i/G}$

$= \dot{\vec{H}}_{G/c} + \dot{\vec{H}}_{/G}$

* $\dot{H}_{/c}$ is the fundamental quantity, what we're taking the derivative of is less important

III $\quad E_k = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_{TOT} v_G^2 + \frac{1}{2} \sum m_i \vec{v}_{i/G}^{\,2}$

* These theorems are <u>exact</u>, to the extent that mechanics as a science is exact.

<u>Implications for Mechanics</u>

<u>LMB</u> : $\sum \vec{F}^{ext} = m_{TOT}\, \vec{a}_G$

<u>AMB</u> : $\sum \vec{M}^{ext}_{/c} = \vec{r}_{G/c} \times m_{TOT}\, \vec{a}_G + \sum \vec{r}_{i/G} \times \vec{a}_{i/G}\, m_i$

* This is usually but <u>not</u> <u>always</u> the best way to describe a system; sometimes it's easier to just catalog all the parts individually (e.g. multi-link systems where CoM moves weirdly)
Ex: 2-part pendulum

where $\vec{a}_{i/G} = \vec{a}_i - \vec{a}_G$

* AMB is also a useful simplification, but not universally so...
e.g. multi-object systems

* We can make one of several assumptions/postulates to move from our "laws" (I)–(III) to AMB & LMB:

1) Pairwise equal-opposite internal forces
2) All internal forces sum "somehow" to $\vec{0}$; not necessarily in eq-opp. pairs
3) All internal forces are conservative $\Leftrightarrow$ can be written as the gradient of some potential function.

* (2) is the best. (1) breaks down sometimes, so does (3)

AMB cot'd : valid if:

(a) C is a fixed point in space, or
(b) C=G is moving (but inertial)
   * C=G can be noninertial if at the time of interest IFF $\vec{a}$ is colinear with "particle's velocity" ($\dot{\vec{r}}_{G}$)

Energy

$$\sum \vec{F}_i \cdot \vec{v}_i = \dot{E}_k$$

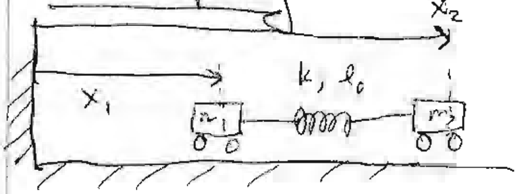
all forces; external AND internal

MAIN MORAL: If you know the force laws for your system, you can find the motion of a system by $\vec{F}_i = m_i \vec{a}_i$ for every particle ($i = 1 \ldots n$)

Moving on.....

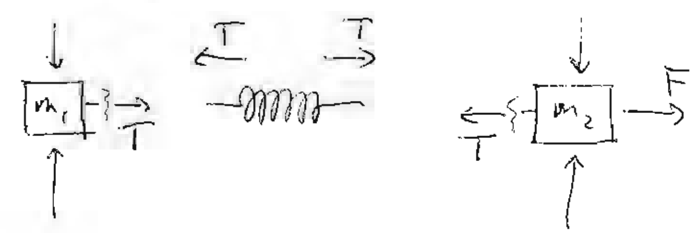
$$\boxed{\text{INTRO: CONSTRAINTS \& VIBRATIONS}}$$

ex: 1D spring



$\Leftrightarrow$

FBD's



LMB

(1.) $m_1 \ddot{x}_1 = T$

(2) $m_2 \ddot{x}_2 = F - T$

Material Property

$T = k(x_2 - x_1) - \ell_0$

*define $x_2 - x_1 = \ell$

Eq.s of Motion

$$\dot{x}_1 = v_1$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_1 = T/m_1$$
$$\dot{v}_2 = (F - T)/m_2$$

$* \; T = k(x_2 - x_1) - \ell_0$

Aside: matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k\ell_0 + 0 \\ -k\ell_0 + F \end{bmatrix}$$

* See email for Matlab example of numerical solution for this system

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9/10 LECTURE:

1. Skylab
2. Constraints: simple example

Skylab $\quad \sum \vec{F}^{ext} = \vec{0} \Rightarrow LMB \Rightarrow \vec{r}_G = \text{CONST}$

and system initially still

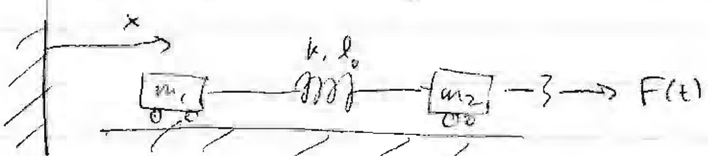
$$AMB \Rightarrow \vec{r}_{i/c} \times \vec{v}_i m_i = \vec{0}$$

* There's no quantity that, when differentiated, gives $\vec{r}_{i/c} \times \vec{v}_i m_i$

$\cancel{\Rightarrow}$ rotation $= 0$

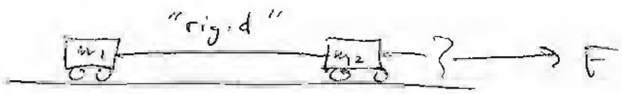
(The guy can rotate by windmilling his arms and stuff. in space)

CONSTRAINTS (continued)



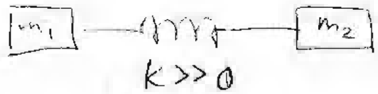
2 ODEs, 2 degrees of freedom.

But what if...



There are four approaches to this problem...

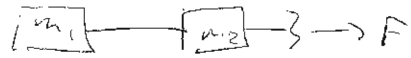
1. Treat m_1 and m_2 as separate objects:



Pros: - could be accurate
- simple to implement: $\bar{F} = m\bar{a}$ for each particle

Cons: - too complicated, perhaps (we may not be interested in the detailed motion of each mass — "parsimonious models")
- we're modeling something we don't fully understand — we arbitrarily assign a "large" k, without knowing how big it should be...
- We get stiff ODEs, which lead to very high frequency vibration modes. In order to capture that motion we need a numerical time step that's $< \frac{1}{2} f$. That sends computation time through the roof.

4 ways to model a stiff system...



2. Constraint model

We use $k \to \infty$ (rather than "k large")

$$\Rightarrow x_2 - x_1 = l_0 \quad \forall t$$

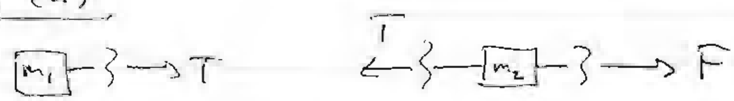
(a) DAEs : conceptually most direct
"differential algebraic eqns"

(b) Eliminate constraint forces by algebra

(c) Finesse the constraint forces (e.g. Lagrange's or Hamilton's equations)

Ex: 2 (a)

FBDs:



NB: same FBDs (& hence mechanical equations) as for the spring, but a different constitutive law for T

LMB 1: $m_1 \ddot{x}_1 = T$

LMB 2: $m_2 \ddot{x}_2 = F - T$

Kinematic Constraint $\quad x_2 - x_1 = l_0$

$$\Rightarrow \ddot{x}_2 - \ddot{x}_1 = 0$$

So we have 3 eqns w/ 3 unknowns ($\ddot{x}_1$, $\ddot{x}_2$, T) that we must solve $\forall t$

$$\begin{aligned} m_1 \ddot{x}_1 \qquad\quad - T &= 0 \\ m_2 \ddot{x}_2 + T &= F \\ \ddot{x}_1 - \ddot{x}_2 \qquad &= 0 \end{aligned}$$

$$\Longleftrightarrow \begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

~~"mass matrix"~~ $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \equiv$ "mass matrix", M $\quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \equiv$ "Jacobian, J"

2. (a)

This takes the general form

$$N \begin{bmatrix} \text{accel.} \\ \text{const.} \\ \text{forces} \end{bmatrix} = \begin{bmatrix} \text{"List of known things"} \end{bmatrix}$$

where $N = \begin{bmatrix} M & J \\ J^T & 0 \end{bmatrix}$

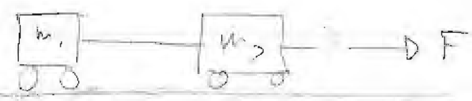
If we put this in as the rhs() term in Matlab's ODE solver, it will return

$\ddot{x}_1$, $\ddot{x}_2$, T (we can throw T away if we want)

Pros of 2 (a)

- Reasonably (but not ~~optimally~~ maximally) efficient
- Doesn't explicitly enforce the constraint. So numerical errors can accumulate and lead to $x_2 - x_1 \neq l_0$

LECTURE 9/12/12 : CONSTRAINTS continued.



Two perspectives for FBD

(a) separate objects :

(b) system :

LMB 1 : $T = m_1 \ddot{x}_1$ LMB 2 : $F - T = m_2 \ddot{x}_2$

Constraint eq: $x_2 - x_1 = l_0 \quad \forall t$

Now, method 2(a) gave us a system of DAEs to solve at every instant to get $\ddot{x}_1$, $\ddot{x}_2$ and T.

2 (b)

In method 2 (b), we'll manipulate the equations to eliminate the constraint forces, T.

$$+\begin{cases} m_1 \ddot{x}_1 = T \\ m_2 \ddot{x}_2 = F - T \end{cases}$$

$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = F$ but $\ddot{x}_1 = \ddot{x}_2$ from our constraint,

$(m_1 + m_2)\ddot{x}_1 = F$ ← so

* How is this different from just treating the masses as a single system

2 (c) Finesse the constraint forces (e.g. by Lagrange or Hamiltonian)

Four ways to do this, without referring to the constraint force T:

I. LMB for the system

$$\sum \vec{F}^{ext} = \dot{\vec{L}}$$

$$F = m_{TOT}\ddot{x}_G \qquad \ddot{x}_G = \ddot{x}_1, \text{ so}$$

$$\boxed{\ddot{x}_1 = \frac{F}{m_{TOT}}}$$

II. Lagrange's Equations

$$\mathcal{L} = E_k - E_p = \tfrac{1}{2} m_{TOT}\dot{x}_G^2 - \left[-\int_0^x F(x')\,dx'\right]$$

LE's: $$\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0}$$

* This simple form applies only when all forces are conservative.

$\frac{\partial \mathcal{L}}{\partial x} = F(x)$, $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_{TOT}\dot{x}_G$, $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_{TOT}\ddot{x}_G$, so $\boxed{F(x) - m_{TOT}\ddot{x}_G = 0}$

This only works for $\underline{\underline{\text{one}}}$ degree of freedom!

III. Cons. of Energy

if $F = F(x)$, then $E_p = -\int_0^x F(x')\,dx'$

$$E_{TOT} = E_k + E_p = \text{CONST.}$$

$$\dot{E}_{TOT} = 0 = \frac{d}{dt}\left[\frac{1}{2} m_{TOT} \dot{x}_1^2 + -\int_0^x F(x')\,dx'\right]$$

$$\Rightarrow \quad 0 = m_{TOT}\,\dot{x}_1\ddot{x}_1 - F(x)\,\dot{x}$$

When $\dot{x} \neq 0$, we get

$$\boxed{m_{TOT}\,\ddot{x}_1 = F(x)}$$

* for $\dot{x} = 0$, we need an alternate derivation...

IV. Power Balance

Also only valid for 1 DOF

The power of ALL forces acting on a system equals the rate of change of the system's $\underline{\text{kinetic}}$ energy.
↳? or total?

$$P^{all} = \dot{E}_k \qquad F\dot{x}_2 = \frac{d}{dt}\left(\frac{1}{2} m_{TOT} \dot{x}_G^2\right) = m_2\,\ddot{x}_G\,\dot{x}_G$$

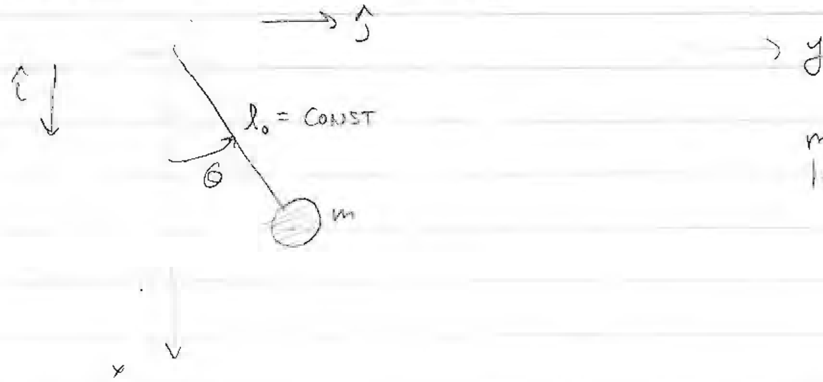
$$\Rightarrow \quad F = m_{TOT}\,\ddot{x}_1 \qquad \text{since } \dot{x}_1 = \dot{x}_2 = \dot{x}_G \text{, etc.}$$

Okay! So we talked about three methods for these types of problems:

1. Separate objects
2. Constraint forces ↳ 2a, 2b, 2c
3. Finesse constraints to eliminate them
   ↳

More Complex Example

Simple Pendulum



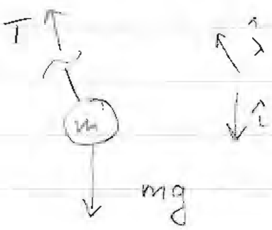
massless rod of length l_0 = CONST

Again, multiple approaches to this problem.

Method # I Replace rod with a "stiff" spring

* This yields STIFF ODE's $\Rightarrow$ need small Δt $\Rightarrow$ ~~too~~ lots of runtime
* Extra vibrations are introduced that we may not care about

II. Write DAEs : FBD



LMB : $\sum \vec{F} = \dot{\vec{L}}$

$$mg\,\hat{i} + T\,\hat{\lambda} = m\vec{a} = m(\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j})$$

where $\hat{\lambda}$ is directed from m to the origin.

$$\hat{\lambda} = \frac{x\,\hat{i} + y\,\hat{j}}{\sqrt{x^2+y^2}}$$

$$\hat{i} : \quad mg - T\left(\frac{x}{\sqrt{x^2+y^2}}\right) = m\ddot{x}$$

$$\hat{j} : \quad -T\left(\frac{y}{\sqrt{x^2+y^2}}\right) = m\ddot{y}$$

Our unknowns are $\{\ddot{x}, \ddot{y}, T\}$. At every t, we know $\{x, \dot{x}, y, \dot{y}, m, l_0, g\}$. We have 2 EQN & 3 UNKNOWNS. What's our third EQN?

The constraint, of course! $\boxed{x^2 + y^2 = l_0^2}$ $\leftarrow$ KINEMATIC CONSTRAINT

To make use of an constraint (algebraic) equation, we take it's time derivatives to make it a differential equation.

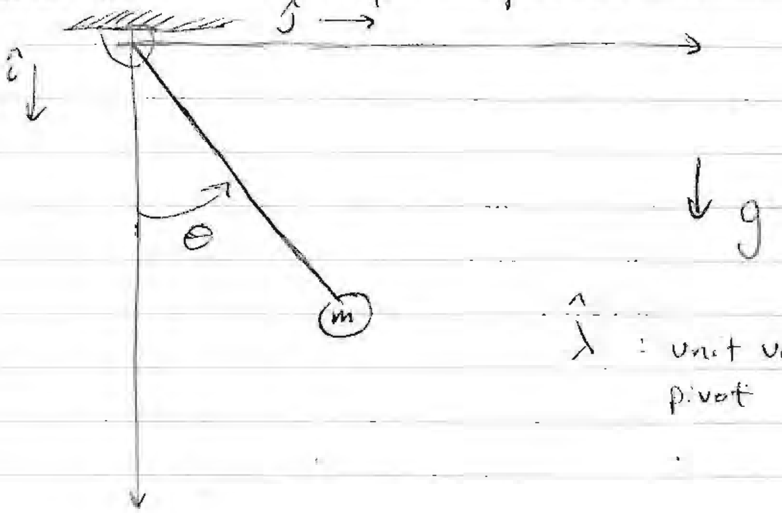
$$\frac{d^2}{dt^2}(x^2+y^2) = \frac{d^2}{dt^2} l_0 = 0$$

$$x\ddot{x} + y\ddot{y} = -(\dot{x}^2+\dot{y}^2)$$

9/14 LECTURE: CONSTRAINTS & PENDULA

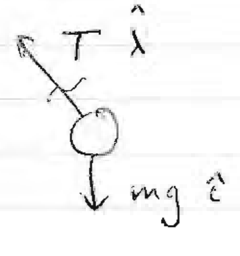
RECALL: Simple pendulum

$l_0 = \text{const}$



$\hat{\lambda}$: unit vector from mass (m) to pivot point (O, the origin)

FBD



This is the second example we've seen of a system with a kinematic constraint (that $l_0 = \text{const}$).

Let's look at 4 methods for treating this problem. We did #1 last time: substitute a "stiff" spring in for the rod.
#2 was to use the LMBs and constraint to get a set of DAEs.

Method

$\boxed{\#2.1}$

$$\hat{i} : \quad m\ddot{x} = mg - T\left(\frac{x}{\sqrt{x^2+y^2}}\right) \qquad ①$$

$$\hat{j} : \quad m\ddot{y} = -T\left(\frac{y}{\sqrt{x^2+y^2}}\right) \qquad ②$$

Kinematic Constraint:

$$x^2 + y^2 = l_0^2$$

$$\Leftrightarrow \dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y} = 0 \qquad ③$$

①, ② and ③ are DAEs for $\ddot{x}$, $\ddot{y}$, T in terms of $x, \dot{x}, y, \dot{y}, l_0, m, g$.

In matrix form...

$$\begin{bmatrix} m & 0 & -x/l_0 \\ 0 & m & y/l_0 \\ x & y & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ T \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ -(\dot{x}^2 + \dot{y}^2) \end{bmatrix}$$

this follows the general form $N = \begin{bmatrix} [M] & [J] \\ [J^T] & 0 \end{bmatrix}$

We can solve this system at every instant in time. With $\dot{x} = v_x$, $\dot{y} = v_y$, $\ddot{x} = \dot{v}_x$, $\ddot{y} = \dot{v}_y$

$\Rightarrow$ 5 first order ODEs + alg. eqn. for T,

~~[illegible]~~ giving x, y, v_x, v_y, T.

Method

#2.2 Take EQs. ① $*y$ − ② $*x$ to eliminate T.
Then use constraint EQ. ③ to eliminate, say, $\ddot{y}$

$\Rightarrow$ This is doable, but results in a giant mess of algebra for $\ddot{x}$.

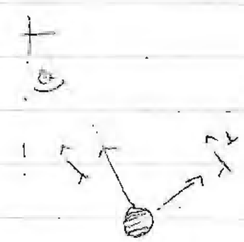
Method

#2.3 Finesse! ... ~~to~~ reframe the problem such that the constraint force "disappears."

There are many ways to do this. Here are a few:

ex.(a) LMB in direction $\perp$ to $\hat{\lambda}$. Call it $\hat{\lambda}^{\perp}$

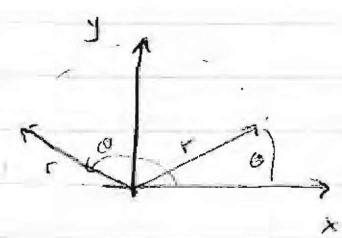
$$\hat{\lambda}^{\perp} = \frac{y\hat{\imath} - x\hat{\jmath}}{\sqrt{}}$$



$$\hat{\lambda}^{\perp} \cdot \left[mg\hat{\imath} + T\hat{\lambda} = m\vec{a} \right]$$

$$mgy + 0 = y\ddot{x}m - x\ddot{y}m$$

<u>Aside</u>: polar coordinates.

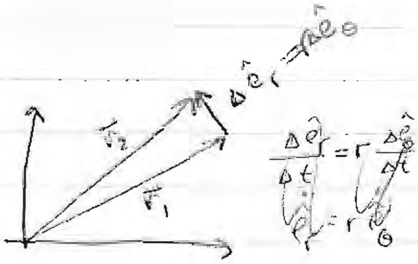


$$\vec{r} = r\hat{e}_r \iff \hat{e}_r = \frac{\vec{r}}{|\vec{r}|}$$

$\hat{e}_r$ and $\hat{e}_\theta$ change in time:

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$



$$\Rightarrow \dot{\hat{e}}_r = \frac{d}{dt}\hat{e}_r = \dot{\theta}\,\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r$$

Let $\vec{r}_2 = r_2\hat{e}_r'$
$\vec{r}_1 = r_1\hat{e}_r$

$$\vec{r} = \boxed{\begin{aligned}\vec{r} &= r\hat{e}_r \\ \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\end{aligned}}$$

$$\boxed{\vec{a} = \hat{e}_r(\ddot{r} - r\dot{\theta}^2) + \hat{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta})}$$

$$\ddot{\vec{r}} = \vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta = \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

Method

$\boxed{\#2.3}$ continued.

ex (b) LMB in polar coordinates, in the $\hat{e}_\theta$ direction

$$\hat{e}_\theta \cdot \{LMB\} \Rightarrow \hat{e}_\theta \cdot \left\{ -T\hat{e}_r + mg\hat{\imath} = m\vec{a} \right\}$$

$$mg(\hat{\imath} \cdot \hat{e}_\theta) = (r\ddot{\theta} + 2\dot{r}\dot{\theta})m$$

$$-\cancel{m}g\sin\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cancel{m}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g\sin\theta}{l_0} = 0}$$

ex(c) LMB, polar, in $\hat{e}_r$ direction

ex (d) Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

$$\mathcal{L} = T - V = E_k - E_p$$

$$= \frac{1}{2}m\dot{\theta}^2 l_0^2 - (-l_0\cos\theta \cdot mg)$$

$$= \frac{1}{2}m\dot{\theta}^2 l_0^2 + l_0\cos\theta \cdot mg$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\dot{\theta}l_0^2 \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml_0^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -l_0 mg\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 = \cancel{m}\ddot{\theta}l_0^{\cancel{2}} - \cancel{l_0}\,\cancel{m}g\,\cancel{\cos\theta}\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{g}{l_0}\sin\theta}$$

Method

#2-3 (finesse)

ex (e) AMB : $\vec{M}_0 = \dot{\vec{H}}_{/0}$

$r\hat{e}_r \times mg\hat{\imath} = (\vec{r}_{/0} \times m\vec{a})$ where $\vec{r} = r\hat{e}_r$ and $\vec{a}$ is polar

$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l_0}\sin\theta}$

ex (f) Conservation of Energy

$\frac{d}{dt}E_{TOT} = 0$, $E_{TOT} = \underbrace{-mg l_0 \cos\theta}_{E_p} + \underbrace{m l_0 \dot{\theta}^2}_{E_k}$

ex (g) Power Balance

$P = \dot{E}_k$

$mg\hat{\imath}\cdot(r\dot{\theta}\hat{e}_\theta + \dot{r}\hat{e}_r) = m\dot{\theta}\ddot{\theta}$

$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l_0}\sin\theta}$

Conclusion : There are MANY different "finesse" methods to get around the problem of the tension force. There are even variations within methods, e.g. when & how to apply the kinematic constraints.

To solve hard problems, one must be familiar with ALL methods and know when & how to choose between them.

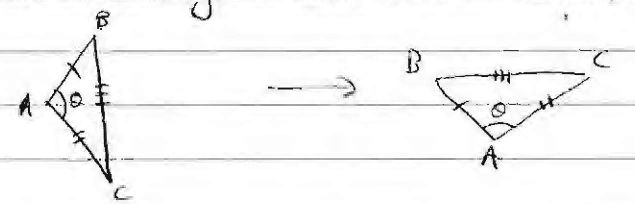
LEED? PV?
Ramping time

Mon. 9/17

TODAY: RIGID OBJECTS, ROTATION, ROTATING FRAMES

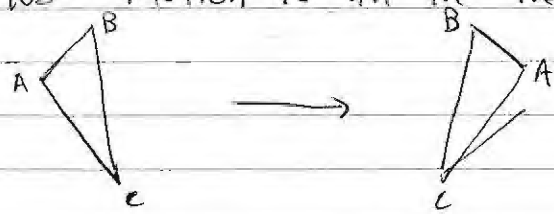
(2D) A <u>rigid body</u> is a collection of material where all distances between pairs of material points are constant in time & all positions are <u>continuous</u> in time.

$\Rightarrow$ all angles are also constant in time



is okay

• NB: Motion is all in the 2D plane: NO REFLECTIONS



Not allowed!

$\Rightarrow$ all lines rotate the same amount θ (from $t=0$ to $t=t$)

So, how can we do mechanics on a rigid object?

points	edges
	1
	3
	5
	7
	9
	11
	13
	15
	17
	⋮
i	$2n-3$

The naive approach...

Write $\vec{F}_i = m\vec{a}_i$ for each particle

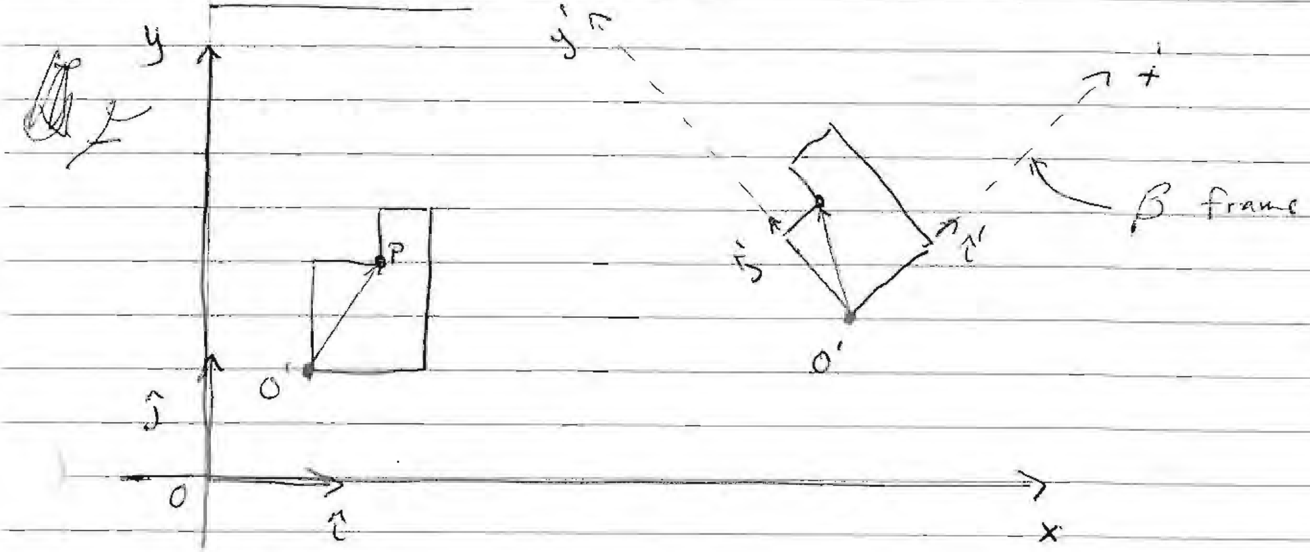
Add enough constraints to keep object rigid $\Rightarrow l_{ij}^2 = \text{CONST}$
(Imagine trusses connecting particles)

We need 3 numbers to describe the "pose" of a rigid object in 2D: it has 3 DoF.
Each constraint removes 1 DoF, so we need $2n-3$ constraint equations

$\Rightarrow \begin{cases} 2n & 2^{nd} \text{ order ODEs} \\ 2n-3 & \text{constraints} \end{cases} \Rightarrow 4n-3$ DAEs to solve

Now, this "constrain every particle" approach takes a LOT of setup and a LOT of computation to solve. Nobody actually does this. We need a shortcut. And that shortcut is...

ROTATION



$\mathcal{F}$: fixed frame : (x, y)
β : body frame : (x', y')
θ : rotation of β relative to $\mathcal{F}$

We want to keep track of all points WRT the origin of frame β as it moves & rotates in time. Let's pick a typical point, call it P.

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

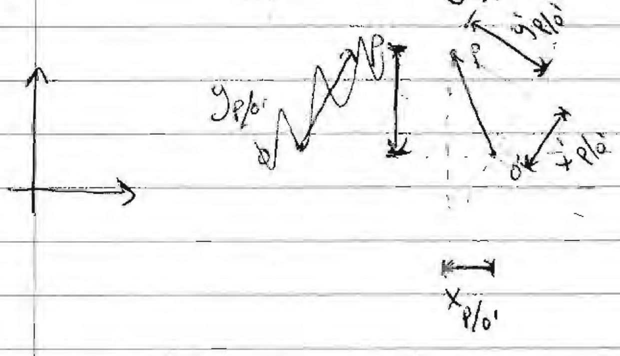
The vector $\vec{r}_{P/O'}$ is the thing we care about. What about velocities & accelerations?

$$\vec{v}_{P/O'} = \overset{\mathcal{F}}{\dot{\vec{r}}}_{P/O'} = \frac{{}^{\mathcal{F}}d}{dt}\,\vec{r}_{P/O'}$$ ← the derivative is calculated in fixed frame $\mathcal{F}$

$$= \frac{d}{dt}\left[x_{P/O'}\,\hat{\imath} + y_{P/O'}\,\hat{\jmath}\right]$$

Aside: $\vec{r}_{P/O'} = \vec{r}_{P/O'}$

$$x_{P/O'}\,\hat{i} + y_{P/O'}\,\hat{j} = x'_{P/O'}\,\hat{i}' + y'_{P/O'}\,\hat{j}'$$



... this is just saying we can write a vector in either frame. i.e.

$$\underline{u} = u_i \underline{e}_i = u'_i \underline{e}'_i$$

So, back to describing velocity:

$$\vec{v}_{P/O'} = \dot{x}_{P/O'}\,\hat{i} + \dot{y}_{P/O'}\,\hat{j}$$

OR

$$\vec{v}_{P/O'} = \frac{d}{dt}\left(x'_{P/O'}\,\hat{i}' + y'_{P/O'}\,\hat{j}' \right)$$

Now, $x'_{P/O'}$ is the x-position of P WRT origin of β, O'. But β moves with the object, so this distance doesn't change. $\dot{x}'_{P/O'} = \text{CONST}$

So we have

$$\vec{v}_{P/O'} = \dot{x}_{P/O'}\,\hat{i} + \dot{y}_{P/O'}\,\hat{j}$$

OR

$$\vec{v}_{P/O'} = x'_{P/O'}(\vec{\omega} \times \hat{i}') + y'_{P/O'}(\vec{\omega} \times \hat{j}')$$

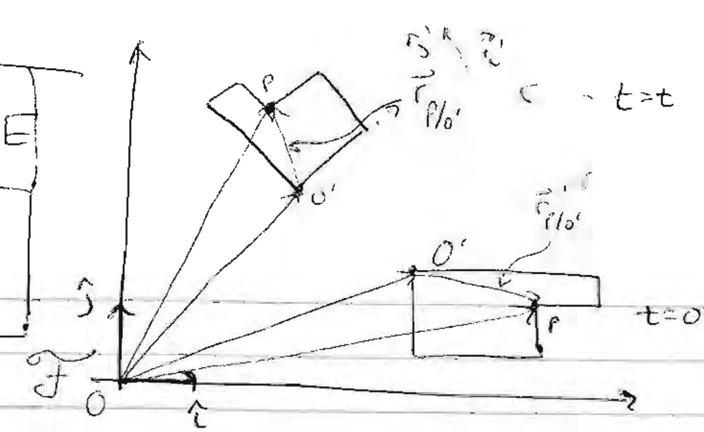
where $\vec{\omega} = \dot{\theta}\,\hat{k}$

Why? $\longrightarrow$ $\hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ $\iff$ $\underline{e}'_i = P_{ij}\underline{e}_j$

$\hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

9/18 LECTURE

2D Rotation
AMB for rigid objects



Key interest

$$\vec{r}_{P/O'}, \quad \vec{v}_{P/O'}, \quad \vec{a}_{P/O'} \longleftarrow = \frac{{}^{\mathcal{F}}d}{dt}\vec{v}_{P/O'}$$

$$\vec{v}_{P/O'} = \frac{{}^{\mathcal{F}}d}{dt}\vec{r}_{P/O'}$$

$$\hat{\imath}' = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \Rightarrow \qquad \dot{\hat{\imath}}' = \vec{\omega}\times\hat{\imath}'$$
$$\hat{\jmath}' = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \qquad\qquad \dot{\hat{\jmath}}' = \vec{\omega}\times\hat{\jmath}'$$

We can write our vector in either basis:

$$\vec{r}_{P/O'} = \vec{r}_{P/O'}$$

$$x_{P/O'}\,\hat{\imath} + y_{P/O'}\,\hat{\jmath} = x'_{P/O'}\,\hat{\imath}' + y'_{P/O'}\,\hat{\jmath}'$$

NB: $\begin{cases} x_{P/O'} \neq x'_{P/O'} \\ y_{P/O'} \neq y'_{P/O'} \end{cases} \Rightarrow \left[\vec{r}_{P/O'}\right]_{xy} \neq \left[\vec{r}_{P/O'}\right]_{x'y'}$

The components in the two bases are NOT equal, i.e. $u_i \neq u_i'$

In $u_i' = \hat{e}_i' \cdot \vec{u}$

* Generally, $u_i = \hat{e}_i \cdot \vec{u}$

So, we can find our $x_{P/O'}$ in terms of $x'_{P/O'}$ & $y'_{P/O'}$

$$x_{P/O'} = \hat{\imath}\cdot\left(x'_{P/O'}\,\hat{\imath}' + y'_{P/O'}\,\hat{\jmath}'\right)$$
$$= x'_{P/O'}\left(\hat{\imath}\cdot\hat{\imath}'\right) + y'_{P/O'}\left(\hat{\jmath}\cdot\hat{\jmath}'\right)$$

Again in general,

$$\left[\vec{Q}\right]_{xy} = \begin{bmatrix} \hat{e}_1 \cdot \hat{e}_1' & \hat{e}_1 \cdot \hat{e}_2' \\ \hat{e}_2 \cdot \hat{e}_1' & \hat{e}_2 \cdot \hat{e}_2' \end{bmatrix} \begin{bmatrix} Q_{x'} \\ Q_{y'} \end{bmatrix}$$

or $$\left[\vec{Q}\right]_{xy} = \left[R\right]\left[\vec{Q}\right]_{x'y'}$$

$$Q_i = \sum_{i=1}^{2} \hat{e}_i \cdot \hat{e}_j' x_j'$$

rotation matrix $\equiv$ $$R = \left[\left[\hat{i}'\right]_{xy} \Big| \left[\hat{j}'\right]_{x'y'} \right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

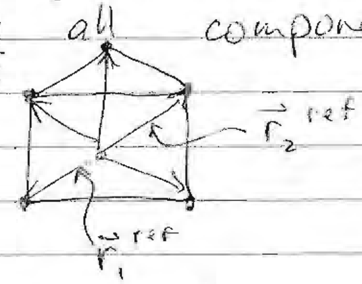
So $$\boxed{\left[\vec{r}_{P/O'}\right]_{xy} = \left[R\right]\left[\vec{r}_{P/O'}\right]_{x'y'}}$$

where $$\left[\vec{r}_{P/O'}\right]_{x'y'} = \left[\vec{r}^{\,ref}_{P/O'}\right]_{xy}$$

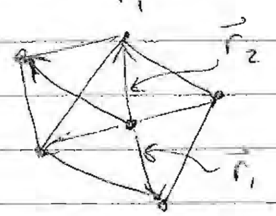
$\Rightarrow$ Given $\begin{cases} \text{drawing of vectors} \\ \text{rotation matrix, } R \end{cases}$

$\Rightarrow$ We can draw the new (rotated) vectors by multiplying all component lists by R.

Ex.



$\Rightarrow$ rotate



$$\vec{r}_1 = \left[R\right]\left[\vec{r}_1^{\,ref}\right]_{xy}$$

$$\Rightarrow \left[\left[\vec{r}_1^{\,ref}\right]_{xy} \; \left[\vec{r}_2^{\,ref}\right]_{xy} \; \cdots \right]\left[R\right] = \left[\left[\vec{r}_1\right]_{xy} \; \left[\vec{r}_2\right]_{xy} \; \cdots \right]$$

What about velocity & acceleration? of $P/_{O'}$

$$\vec{V}_{P/O'} = \vec{\omega} \times \vec{r}_{P/O'}$$

$$\vec{a}_{P/O'} = \dot{\vec{\omega}} \times \vec{r}_{P/O'} - |\vec{\omega}|^2 \vec{r}_{P/O'}$$

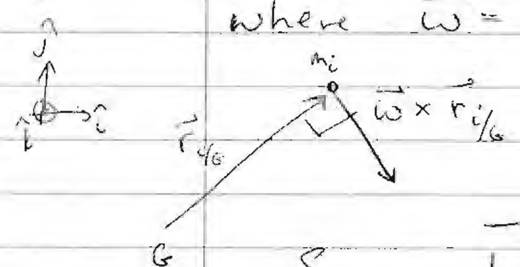
is this O or O'?

Okay, now let's talk about AMB.

Look at: $$\vec{H}_{/G} = \sum \vec{r}_{i/G} \times (m_i \vec{v}_i)$$

$$= \sum \vec{r}_{i/G} \times (\vec{\omega} \times \vec{r}_{i/G}) m_i$$

where $\vec{\omega} = \omega \hat{k}$



So $$\vec{H}_{/G} = \sum |\vec{r}_{i/G}|^2 m_i \, \omega \hat{k}$$

Define $$I^G_G = \sum |\vec{r}_{i/G}|^2 m_i$$ moment of inertia WRT CoM

then $$\vec{H}_{/G} = I^G \omega \hat{k}$$ AM wrt CoM

So $$\dot{\vec{H}}_{/G} = I^G \dot{\omega} \hat{k}$$ RoC of AM wrt CoM

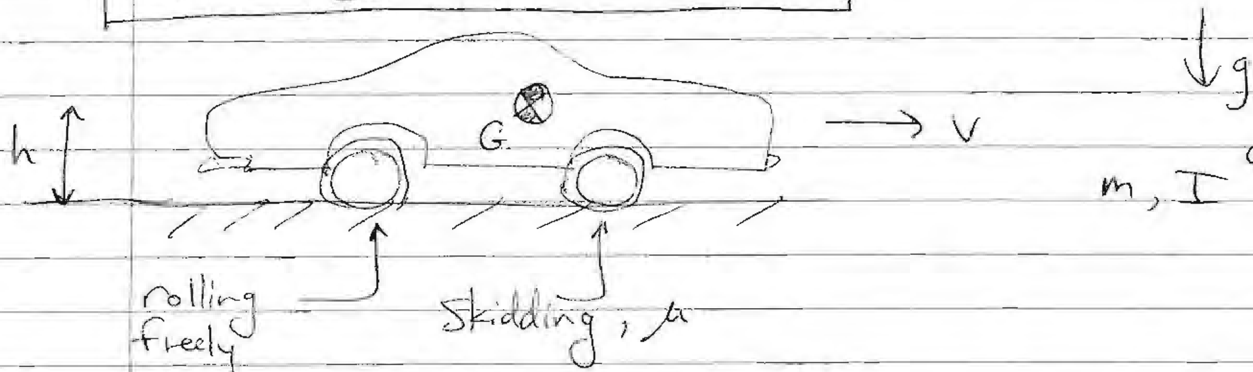
So $$\boxed{\begin{aligned}\dot{\vec{H}}_{/c} &= \vec{r}_{G/c} \times \vec{a}_G \, m_{TOT} + I^G \dot{\omega} \hat{k} \\ \sum \vec{M}_{/c} &= \dot{\vec{H}}_{/c}\end{aligned}}$$

So the rate of change of AM relative to some point C is the sum of:

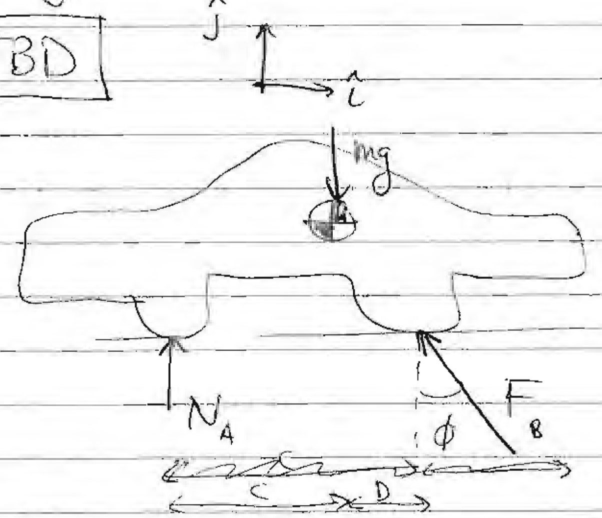
point-mass rotation of CoM
- a term involving AM ~~relative to~~ of the "bulk motion" of the CoM WRT point C

"spinniness" of system in CoM frame
- a term describing the "internal" rotation of the object

$\boxed{\text{Ex: BRAKING CAR}}$

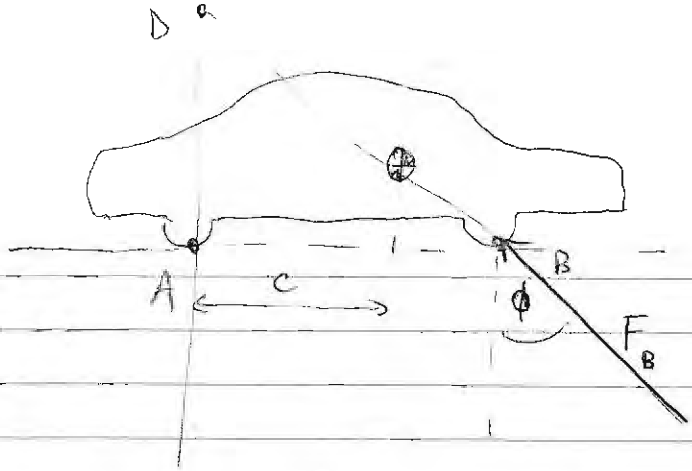


$\boxed{\text{FBD}}$



$\tan \phi = \mu$

We wish to find acceleration - velocity - position <u>without</u> using the forces



Take AMB about D

$$\sum \vec{M}_{/D} = \dot{\vec{H}}_{/D}$$

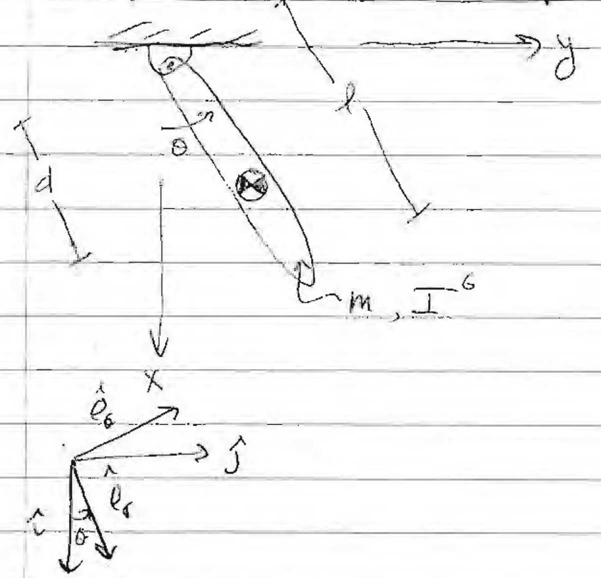
$$-mgc\hat{k} = \vec{r}_{G/D} \times (m\vec{a}_G) + I^G \dot{\omega}\hat{k}$$

But the suspension is rigid, so $\dot{\omega} = 0$

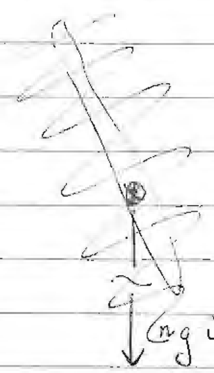
$$-mgc\hat{k} = \left(c\hat{i} - \left(\frac{c+d}{\tan\phi}(-h)\right)\hat{j}\right)$$

9/24 LECTURE: 2-D RIGID OBJECT EXAMPLES

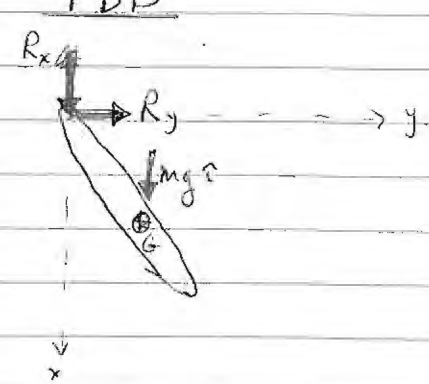
"Physical" or "compound" pendulum



$g\downarrow$



FBD



$$\underline{AMB_{/O}} \qquad \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times mg\hat{\imath} = \begin{cases} \sum \vec{r}_{i/O} \times m_i \vec{a}_i \\ \int \vec{r}_{/O} \times \vec{a}\, dm \end{cases}$$

* Since O is stationary, $\vec{H}_{/O}$ is well defined. If O moves, we can still use $\dot{\vec{H}}_{/O}$, but we can't say that it's the rate of change of any physical quantity

(A) $$\boxed{\underbrace{\vec{r}_{G/O} \times (mg\,\hat{\imath})}_{\sum \vec{M}_{/O}} = \underbrace{\vec{r}_{G/O} \times m\vec{a}_G + I^G \ddot{\theta}\,\hat{k}}_{\dot{\vec{H}}_{/O}}}$$

NB: Ruina says we MUST be able to derive & use this version of AMB.

Okay, we want to express this whole system in terms of one "minimal coordinate." Here we'll use θ because it's clean & simple, but in general we have a lot of freedom in choosing a minimal coordinate. It may be x, y, r, θ, or some $u = f(x, y, \theta, r, \dots)$ that may have no immediate physical meaning, but ~~has~~ is mathematically convenient.

Aside on DAEs

$$\ddot{x}_G = mR_x + mg$$

$$\ddot{y}_G = mR_y$$

$$\ddot{\theta} = \left(\vec{r}_{O/G} \times (R_x\hat{\imath} + R_y\hat{\jmath})\right) \cdot \frac{1}{I^G}$$

Kinematic Constraints

O (·)

version ① $\vec{r}_A = \vec{r}_O$ (the pivot point is at the origin)

version ② $(x_G\hat{\imath} + y_G\hat{\jmath}) - d\cos\theta\,\hat{\imath} - d\sin\theta\,\hat{\jmath} = \vec{0}$

(the CoM moves in a circle of radius d)

* So, the equivalent constraints ① & ② give two constraints, ~~such that~~ since they're vector (2-D) equations.

* To solve, we take $\frac{d^2}{dt^2}$ of the constraints for 5 ODEs in $\{\ddot{x}_G, \ddot{y}_G, \ddot{\theta}, \}$

$AMB_{/0}$ cont'd

$\vec{r}_{G/0} = (d)\,\hat{e}_r$

$\vec{a}_{G} = (d)(\ddot{\theta}\,\hat{e}_\theta - \dot{\theta}^2\,\hat{e}_r)$ since $r_G = d$, and $\dot{r}_G = \ddot{r}_G = 0$

Ⓐ becomes: $d\hat{e}_r \times (mg\hat{\imath}) = (d\hat{e}_r) \times m(-d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\,\hat{e}_\theta) + I^G\ddot{\theta}\,\hat{k}$

$$-dmg\sin\theta\,\hat{k} = (md^2 + I^G)\ddot{\theta}\,\hat{k}$$

$\{\ \}\cdot\hat{k}$: $-dmg\sin\theta = (md^2 + I^G)\ddot{\theta}$

$$\Longleftrightarrow \quad \ddot{\theta} + \frac{mgd}{md^2 + I^G}\sin\theta = 0$$

OR, equivalently

$$\boxed{\ddot{\theta} = \frac{-mgd\sin\theta}{md^2 + I^G}}$$

What about R_x and R_y, the force on the pendulum by the ceiling?

Well, given $\ddot{\theta}$ we can do a LMB, $\sum\vec{F} = m\vec{a}$

$\Rightarrow R_x$ & R_y

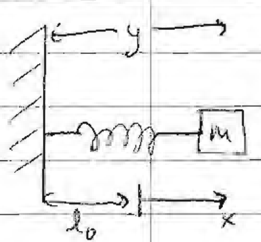
*Do this derivation...

9/26 : ① Intro to Vibrations

② Sleigh

Recall

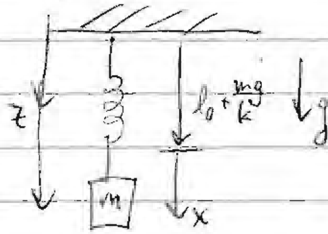
ex



$\ddot{y} = \frac{-(y-l_0)k}{m}$

$\boxed{\ddot{x} = -\frac{k}{m}x}$

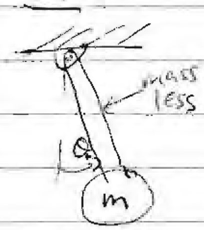
ex



$\ddot{z} = \frac{-k(z-(l_0-mg))}{m}$

$\boxed{\ddot{x} = -\frac{k}{m}x}$

ex

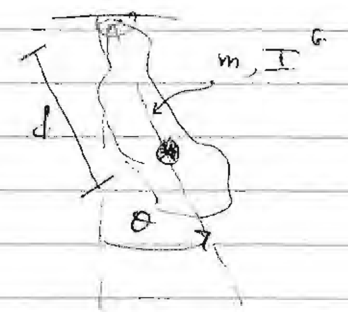


$\ddot{\theta} = -\frac{g}{l}\sin\theta$

$\theta \ll 1$

$\Rightarrow \boxed{\ddot{\theta} \approx -\frac{g}{l}\theta}$

ex



$\ddot{\theta} = \frac{-gmd}{I+md^2}\sin\theta$

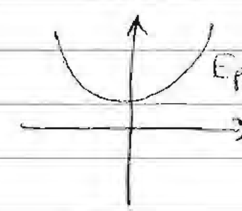
$\boxed{\ddot{\theta} \approx -c\theta}$, $\theta \ll 1$

In general...

1 DoF system parameterized by q, with

$E_k(\dot{q}=0) = 0$

$E_p(q=0)$ a local minimum



$E_{TOT} = \text{CONST} \Rightarrow \dot{E}_k = -\dot{E}_p$

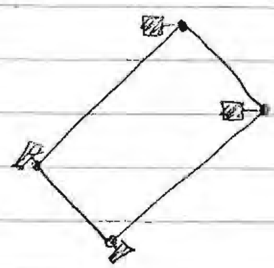
$$\frac{d}{dt}\left(\frac{m_{eff}}{2}\dot{q}^2\right) = -\frac{d}{dt}\left(E_{p,0} + \frac{k_{eff}}{2}q^2\right)$$

ANY 1 DoF system that meets these criteria, (parameterized in a coordinate q that locally minimizes E_p and that gives a $E_k \sim \dot{q}^2$) is described (in the neighborhood of this E_p minimum) as a...

$$\boxed{\begin{array}{c}\text{HARMONIC OSCILLATOR} \\ \ddot{x} = -cx\end{array}}$$

CHAPLYGEN SLEIGH

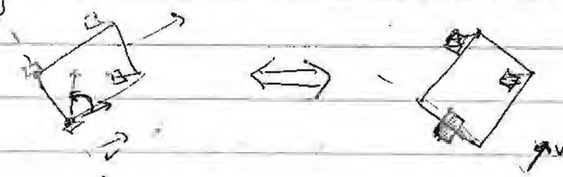
(the grocery cart problem)



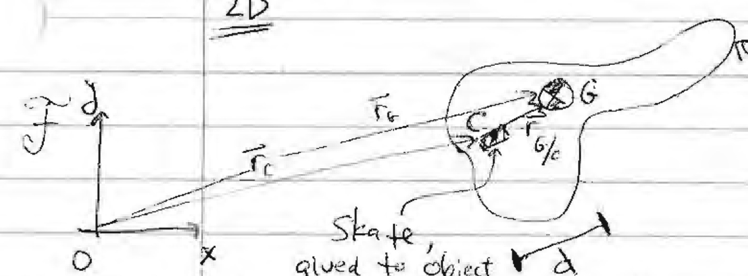
* "castor" wheels on the front, modeled as frictionless pads.

* Rear wheels roll w/o slipping, & are massless ⇒ modeled as skates (a.k.a. sleighs)

The rear wheels can move forward, or rotate, but cannot move laterally. All motion of the rear must be ⊥ to the rear axis

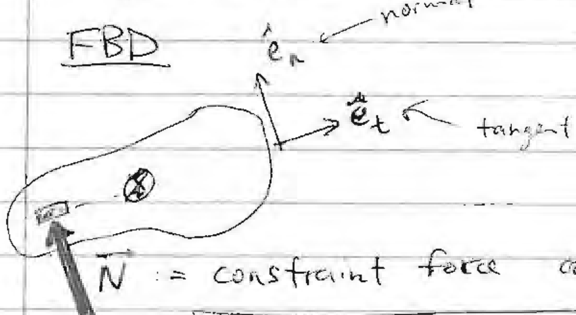


The general problem (capturing the fundamental dynamics):
2D



How does the sleigh move, given some initial push?

FBD



$\vec{N}$ = constraint force confines the skate to move "forward"

AMB

* The constraint force acts on point C, so it exercises no moment (torque) ~~on the~~ about C.

$$\sum_k \vec{M}_{/c} = \vec{\dot{H}}_{/c}$$

$$\vec{0} = \vec{r}_{G/c} \times m\vec{a} + I^G \ddot{\theta}\hat{k}$$

$\vec{r}_{G/c} = (d)\hat{e}_t$

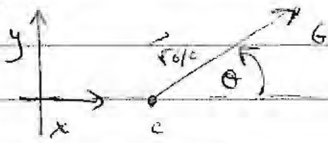
AMB

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$0 = \vec{r}_{G/c} \times \vec{a}_G m + I^G \ddot{\theta} \hat{k}$$

$$\vec{a}_G = \frac{d}{dt}\vec{v}_G = \frac{d}{dt}\left(\vec{v}_c + \vec{v}_{G/c}\right)$$

$$= \frac{d}{dt}\left(v\hat{e}_t + (\dot{\theta}\hat{k} \times \vec{r}_{G/c})\right)$$



where $\vec{v}_{G/c} = \dot{\theta}\hat{k} \times \vec{r}_{G/c}$

angular velocity of G in a circular centered at C

So

$$\vec{a}_G = \frac{d}{dt}(v\hat{e}_t) + \frac{d}{dt}(\dot{\theta}\hat{k} \times d \cdot \hat{e}_t)$$

$$= (\dot{v}\hat{e}_t + v\dot{\hat{e}}_t) + \left[\ddot{\theta}\hat{k} \times (d \cdot \hat{e}_t) + \dot{\theta}\hat{k} \times (d\,\dot{\hat{e}}_t)\right]$$

But $\dot{\hat{e}}_t = \dot{\theta}\hat{e}_n$, and $\hat{k} \times \hat{e}_t = \hat{e}_n$ and $\hat{k} \times \hat{e}_n = -\hat{e}_t$

$$\boxed{\vec{a}_G = \underbrace{(\dot{v}\hat{e}_t + v\dot{\theta}\hat{e}_n)}_{\text{circular motion accel.}} + \underbrace{(d \cdot \ddot{\theta}\hat{e}_n - \dot{\theta}^2 \cdot d\,\hat{e}_t)}_{}}$$

accel. of G relative to C, two points on a rigid object ⇔ circles, G around C.

So......

AMB: $\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$

$\hat{e}_t \times \hat{e}_t = 0$

$\hat{e}_t \times \hat{e}_n = \hat{k}$

$$\vec{0} = \vec{r}_G \times \vec{a}_G m + I^G \ddot{\theta}\hat{k}$$

$$\vec{0} = (d \cdot \hat{e}_t) \overset{m}{\times} \left[(\underbrace{\dot{v}\hat{e}_t}_{0} + v\dot{\theta}\hat{e}_n) + (d \cdot \ddot{\theta}\hat{e}_n - \underbrace{\dot{\theta}^2 \cdot d \cdot \hat{e}_t}_{0})\right] + I^G \ddot{\theta}\hat{k}$$

$$\vec{0} = d \cdot v \cdot \dot{\theta} m \hat{k} + md^2\ddot{\theta}\hat{k} + I^G\ddot{\theta}\hat{k}$$

$\cdot\hat{k}$: $0 = mdv\dot{\theta} + md^2\ddot{\theta} + I^G\ddot{\theta}$

$$\boxed{\ddot{\theta} = \frac{-mdv}{I + md^2}\dot{\theta}}$$

9/28 : CHAPLYGIN SLEIGH (cot'd)

AMB (last time)

$$\boxed{\ddot{\theta} = \frac{-md}{I^G + md^2} v \dot{\theta}}$$

LMB

$$\sum \vec{F} = m\vec{a}$$

$$\overset{\text{circular motion...}}{\vec{a}_G = \vec{a}_C} + \overset{\text{circles of G around C...}}{\vec{a}_{G/C}}$$

$$\{ N\hat{e}_n = m\vec{a}_G \} \quad , \quad \vec{a}_G = (\dot{v}\hat{e}_t + v\dot{\theta}\hat{e}_n) + (d\ddot{\theta}\hat{e}_n - \dot{\theta}^2 d\hat{e}_t)$$

$$\{\}\cdot\hat{e}_t \Rightarrow 0 = m\dot{v} - m\dot{\theta}^2 d$$

$$\boxed{\dot{v}_c = \dot{\theta}^2 d}$$

$$\{\}\cdot\hat{e}_n \Rightarrow \boxed{N = m\left[v\dot{\theta} + \ddot{\theta}d\right]} = m\left[v\dot{\theta} + d\left(\frac{-mvd}{I^G + md^2}\dot{\theta}\right)\right]$$

This problem is symmetric in x, y and θ, i.e. if we switch $x \leftrightarrow y$ or ~~shift~~ rotate our frame by θ, we still see the same dynamics. So our ODE should be independent of our "symmetric" variables x, y, θ.

Therefore, we seek an ODE in v & $\dot{\theta}$.

---

* Ruina talks a lot about AM about different points. For the prelim, review the problem of deriving the validity conditions for points C about which to measure AM

---

* # DoF = # of #s needed to define the "configuration" of the system, where "configuration" can mean the positions (e.g. x, y, θ) or the velocities (e.g. v_x, v_y, v_θ). If the numbers for $\vec{r}$ ~~and~~ # for $\vec{v}$, ~~are different~~, the system is NONHOLONOMIC and # DoF is not well defined.

To put this into Matlab...

$$z = \begin{bmatrix} x_c \\ y_c \\ \theta \\ \omega \\ v \end{bmatrix}$$

function zdot = rhs (t, z, p)

$\dot{x}_c = v_c \cos\theta$

$\dot{y}_c = v_c \sin\theta$

$\dot{\theta} = \omega$

$\dot{\omega} = \dfrac{-md}{I^G + m\omega d^2} \omega v$

$\dot{v} = \omega^2 d$

$$\Rightarrow \text{zdot} = \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \\ \dot{\omega} \\ \dot{v} \end{bmatrix}$$

end

Some analytic solutions

ex: $\ddot{\theta} = 0$, $\dot{\theta} = 0$, $v = 0$
$x = C_1$, $y = C_2$, $\theta = C_3$ (No motion)

ex: $\omega = 0$, $v = v_0$, $\theta = C_3$
$x = v_0 \cos C_3 + C_4$
$y = v_0 \sin C_3 + C_5$ (Const. [straight] velocity)

Let's linearize about our "static" const. ~~vel~~ $\vec{v}$ sol'n:

~~$v = v_0 + \theta$~~

$v = v_0 + \hat{v}$

$\omega = 0 + \hat{\omega}$

ODEs: $\dot{v} = \dot{\theta}^2 d \Rightarrow \cancel{\dot{v}_0}^{\,0} + \dot{\hat{v}} = d\hat{\omega}^2 \leftarrow$ neglect $O(\hat{\omega})$ terms

$\Rightarrow \boxed{\dot{\hat{v}} \approx 0}$

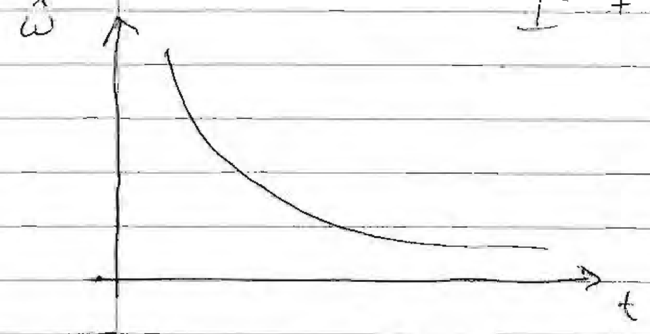
AMB/c: $(0 + \dot{\hat{\omega}}) = \dfrac{-md}{I + md^2} (v_0 + \hat{v}) * (0 + \hat{\omega}) \Rightarrow \boxed{\dot{\hat{\omega}} \approx \dfrac{-dm}{I + md^2} v_0 \hat{\omega}}$

So the "near-equilibrium" solution is

$$\dot{\hat{\omega}} \simeq \frac{-md}{I^G + md^2} V_0 \, \hat{\omega}$$

$$\dot{q} = -kq$$

$$\Rightarrow q(t) = Ce^{-kt} + D$$



<u>Aside on linearization</u>

$\dot{z} = f(z)$ about some soln point $z_0(t)$

$z = z_0 + \hat{z}(t)$

$\dot{\hat{z}} = f(z_0, \hat{z}) \approx [A(t)][\hat{z}]$ + "higher order terms"

So! The system is "attracted" to the constant straight line motion solution for small ~~distur~~ perturbations from it.

Meaning, you can start a grocery cart out a little bit away from equil., it will whip around a little bit then settle into $\vec{V} = \text{CONST}$.

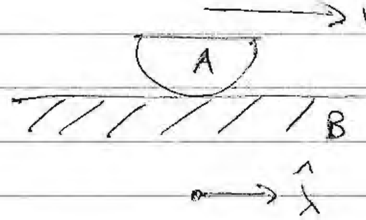
This is actually true for ANY solution except ONE, that exception being straight backward velocity.

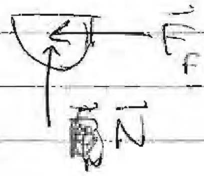
10/1

- Sleigh Demo
- Friction
- Matlab Stuff (errors, events, animation)
- Double Pendulum

Friction

$\vec{v} = \vec{v}_{A/B} \cdot \hat{\lambda}$



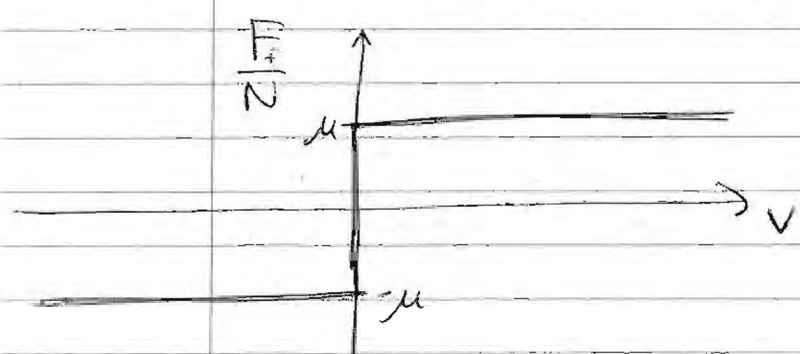


Conventionally, we write the friction force "$\vec{F}_f = \mu \vec{N}$"

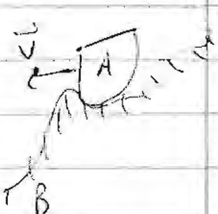
Really, we mean

$$F_f = \begin{cases} \mu N & , \ v > 0 \\ -\mu N & , \ v < 0 \end{cases}$$

and $-\mu N \leq F_f \leq \mu N \quad , \ v = 0$

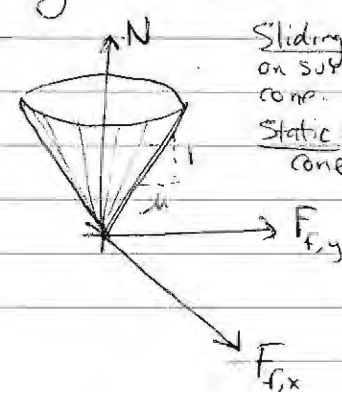


What about 2D or 3D? e.g. sliding on a hill



$\vec{F} = -\mu N \frac{\vec{v}}{|\vec{v}|}$ if $\vec{v} \neq \vec{0}$

$|F| \leq \mu N$ if $\vec{v} = \vec{0}$



Sliding: force on surface of cone.
Static: inside cone

MATLAB STUFF

~~As~~ tic ; := start a stopwatch
result = toc ; := end stopwatch & store value in result

options = odeset ('reltol', (value), 'abstol', (value))
→ sets stepwise (local) error tolerance
* GLOBAL error will in general be higher than local error, often by orders of magnitude

options = odeset ('events' , (function handle))
↳ The Matlab solver will call your specified function & check its output periodically.

Syntax:

```
function [value, done, dir] = f2 (t , z)

    Value = z(2) ;
    done  = 1 ;
    dir   = -1 ;

end
```

* Matlab will stop when Value hits zero. Done = 1 or Done = 0 are some options about whether to ~~end the~~ quit the ODE routine or just to store the value & ~~exit~~ continue

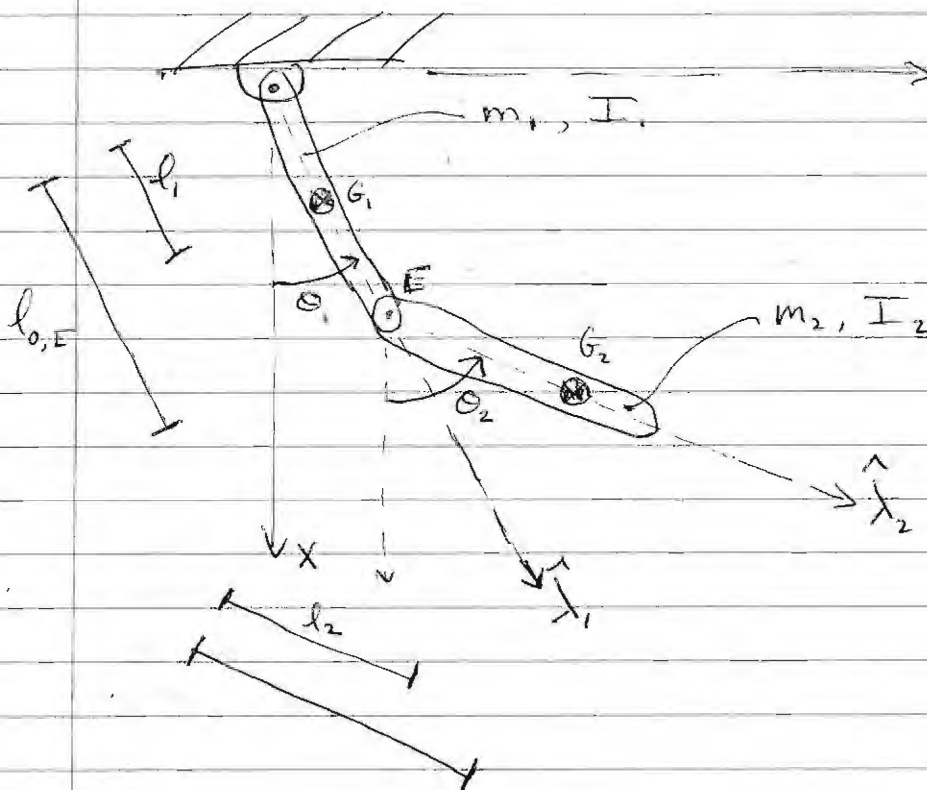
Animation

The simple way to do this is to just plot a whole bunch of times in quick succession.

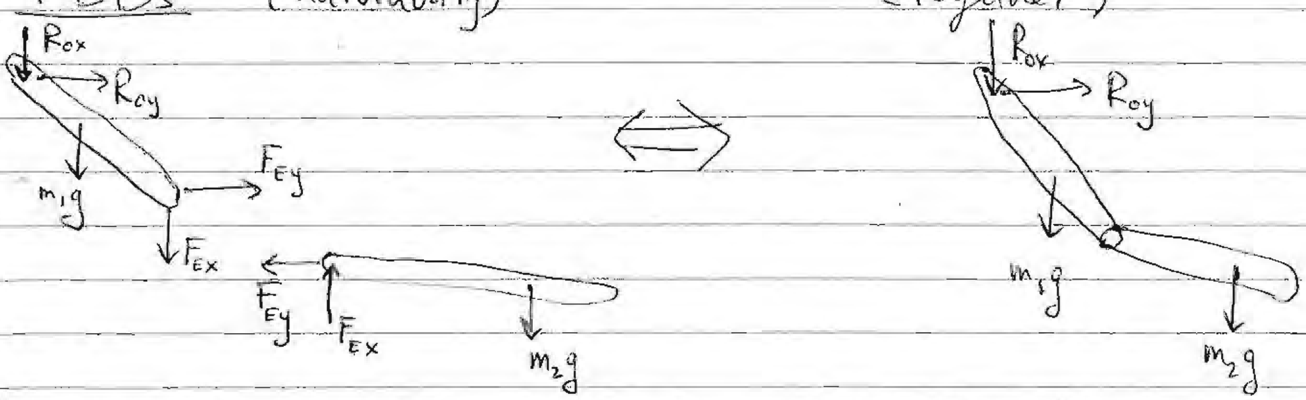
SEE WEB EXAMPLES FOR ALL THIS STUFF.

DOUBLE PENDULUM

The basic DYNAMICS algorithm...
Given the state of a system, specified by POSITIONS & VELOCITIES, we seek enough equations to solve for the unknowns we seek. Then we solve that system of equations for the time derivatives of ~~pe~~ $\bar{r}$ and $\vec{v}$ and use the solution to predict the evolution of the system.



FBDs (individually) (together)



* R is the force exerted on the top arm by the ceiling thingy.

SYSTEM AMB ABOUT O

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O} \longleftarrow \text{1 EQN in } \hat{k}$$

motor torque @ O

$$\left\{ \vec{r}_{G1/O} \times (m_1 g \hat{\imath}) + \vec{r}_{G2/O} \times (m_2 g \hat{\imath}) + M_O \hat{k} = \left(\vec{r}_{G1/O} \times (m_1 \vec{a}_{G1}) + I_1 \ddot{\theta}_1 \hat{k} \right) + \left(\vec{r}_{G2/O} \times (m_2 \vec{a}_{G2}) + I_2 \ddot{\theta}_2 \hat{k} \right) \right\}$$

$\boxed{10/3}$ ↓

We can also do an AMB about E for the "forearm"

$$\sum \vec{M}_{/E} = \dot{\vec{H}}_{/E} \longleftarrow \text{1 EQN in } \hat{k}$$

$$\left\{ \vec{r}_{G2/E} \times (m_2 g \hat{\imath}) + M_E \hat{k} = \vec{r}_{G2/E} \times (m_2 \vec{a}_{G2}) + I_2 \ddot{\theta}_2 \hat{k} \right\}$$

Let's calculate a scary-looking quantity, e.g.

$$\boxed{\begin{array}{l} \vec{M} = \vec{r} \times \vec{F} \\ \vec{a} = \dot{\vec{\omega}} \times \vec{r} \end{array}}$$

$$\vec{a}_{G2} = \vec{a}_E + \vec{a}_{G2/E}$$

where $\vec{a}_E = (\ddot{\theta}_1 \hat{k}) \times \vec{r}_{E/O} - \dot{\theta}_1^2 \vec{r}_{G/O}$ where $\vec{r}_{G/O} = d_1 \lambda_1$

$$\vec{a}_{G2/E} = (\theta_2 \hat{k}) \times \vec{r}_{G/E} - \dot{\theta}_2^2 \vec{r}$$

See Matlab solution!

10/5

- DOUBLE PENDULUM EQNS: ALTERNATIVES
- PENDULUM w/ MOVING BASE

Last time: minimal coordinates θ_1, θ_2
minimal eq'ns : $AMB_{/0}$, system
$AMB_{/E}$, forearm
$\Rightarrow$ 2 eqns & 2 unknowns $\ddot{\theta}_1$, $\ddot{\theta}_2$

We can solve these ODEs (coupled) for θ_1 & θ_2 then write

$$\vec{r}_{E/0} = l_1 \cos\theta_1 \hat{\imath} + l_2 \sin\theta_1 \hat{\jmath}$$

Here's an alternate method for finding, e.g., $\vec{a}_{G1}$ & $\vec{a}_{G2}$:

Start with

$$\vec{r}_{G1/0} = d_1 \cos\theta_1 \hat{\imath} + d_1 \sin\theta_1 \hat{\jmath}$$

$$\vec{V}_{G1/0} = -d(\sin\theta_1)\dot{\theta}_1 \hat{\imath} + \dots$$

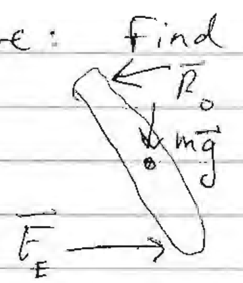
$$\vec{r}_{G2/0} = \dots$$

$$\vec{V}_{G2/0} = \dots$$

So we just take derivatives then solve the resulting eqns for $\ddot{\theta}_1$, $\ddot{\theta}_2$. We get the same eqns by this more direct but often longer route.

Second alternative: Find EoM for each arm.

Upper arm:



LMB $\sum \vec{F} = m\vec{a}$

$$\vec{R}_o + \vec{F}_E + mg\hat{i} = m_1 \vec{a}_{G1}$$

AMB about G1

$$\sum \vec{M}_{/G1} = \dot{\vec{H}}_{/G1}$$

$$\vec{r}_{o/G1} \times \vec{R}_o + \vec{r}_{E/G1} \times \vec{F}_E = I_1 \ddot{\theta}_1 \hat{k}$$

3 EQNS

Going this route involves playing with $\cdot$ & $\times$ to ~~get~~ add & subtract equations and eliminate $\vec{F}_E$ & $\vec{R}_o$. This results in the same LMB Eqns. we had earlier. We do the same process with the forearm.

All these methods involve lots of algebra. Maybe we can replace that with some derivatives, via the Lagrange method:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \text{for } i = 1, \dots, n_{\text{minimal coordinates}}$$

Take $q_1 = \theta_1$, $q_2 = \theta_2$

& define $\omega_1 = \dot{q}_1$, $\omega_2 = \dot{q}_2$ for clarity.

Then write $\mathcal{L} = T - V = \mathcal{L}(q_1, q_2, \omega_1, \omega_2)$

Lagrangeian Double Pendulum

$E_k \equiv \quad T = \frac{1}{2} m_1 V_{G1}^2 + \frac{1}{2} m_2 V_{G2}^2 + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2$

$E_p \equiv \quad V = -m_1 g d_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + d_2 \cos\theta_2)$

We also need to write out V_{G1}^2 & V_{G2}^2:

$$V_{G1}^2 = \vec{V}_{G1} \cdot \vec{V}_{G1} = d_1 \dot{\theta}_1^2$$

$$V_{G2}^2 = \vec{V}_{G2} \cdot \vec{V}_{G2} = \text{"a big mess"}$$

This method ~~fat~~ involves taking ~~at te~~ three derivatives, then simplifying a big algebraic mess. This is doable (in theory) on paper, just like the other methods, but the derivatives add another layer of complexity (& more risk of typos, bugs, etc).

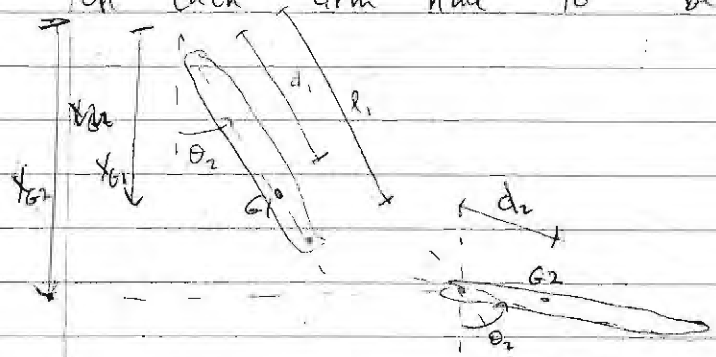
When doing this with ~~a~~ the help of computer algebra, you can use physical checks to find bugs, e.g.

- cons. of energy
- " " LM
- " " AM
- satisfaction of constraints
  - motion in the plane
  - lengths constant
- etc.

All these methods are nasty, in that they involve careful bookkeeping; either on paper or in pixels. We can do this more simply in DAEs by finding a bunch of simple relationships:

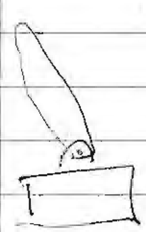
$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\theta}_2 \\ R_{ox} \\ R_{oy} \\ F_{Ex} \\ F_{Ey} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \text{stuff} \\ \\ \\ \\ \end{bmatrix}$$

Our constraint (one of them) is that the points @ the end of each arm have to be equal...



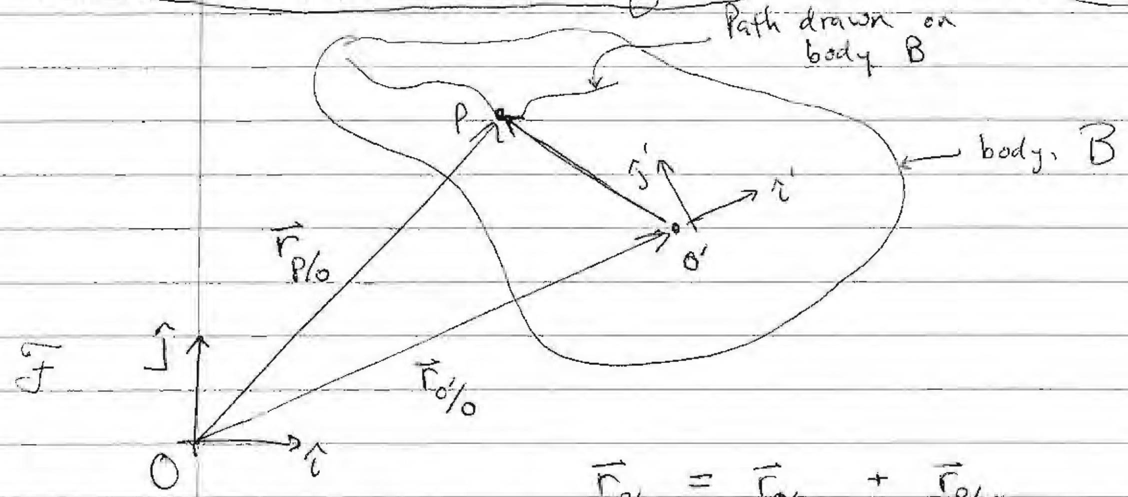
$$x_{G2} - d_2 \cos\theta_2 = x_{G1} + (l_1 - d_1)\cos\theta_1$$

Inverted pendulum w/ moving base point:



10/10 - Post-fall break lecture

- 5 term formula for acceleration...
- Matthieu eqn.



$$\boxed{\vec{a}_P = \ddot{x}_P \hat{I} + \ddot{y}_P \hat{J}}$$

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$\boxed{\begin{aligned} \vec{v}_{rel} &\equiv \vec{v}_{P/B} = \dot{x}'\hat{i}' + \dot{y}'\hat{j}' \\ \vec{a}_{rel} &\equiv \vec{a}_{P/B} = \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' \end{aligned}}$$

$$\vec{v}_P = \vec{v}_{O'} + \vec{v}_{rel} + \vec{\omega} \times \vec{r}_{rel}$$

$$\vec{v}_{P/\mathcal{F}} = \vec{v}_{O'/\mathcal{F}} + \underbrace{\vec{v}_{P/B} + \vec{\omega}_{B/\mathcal{F}} \times \vec{r}_{P/O'}}_{= \vec{v}_{P/O'}}$$

$$\boxed{\vec{a}_P = \vec{a}_{O'} + \vec{a}_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega} \times \vec{v}_{rel}}$$

So, this is our "five term acceleration formula" for the acceleration of a point P relative to fixed Newtonian frame $\mathcal{F}$, where P is moving along the surface of a rigid body B, and B itself is moving relative to $\mathcal{F}$. ~~(both, both some translational velocity and some~~

To see what the terms mean, we can imagine scenarios in which 4 of the 5 terms vanish.

Unpacking the 5 terms

1. $\bar{a}_{o'}$: acceleration of O' relative to $\tilde{J}$.

2. $\bar{a}_{rel}$: acceleration of P relative to O'

3. $\vec{\omega} \times (\vec{\omega} \times \bar{r}_{P/o'})$: "centripedal acceleration" ($-\omega^2 r$, in 2D)

4. $\dot{\vec{\omega}} \times \vec{r}_{P/o'}$: acceleration due to the rate of change of the angular velocity of B

5. $2\,\vec{\omega} \times \vec{v}_{rel}$: "coriolis acceleration"

So, the formula $\bar{a}_P = \ddot{x}_P\,\hat{i} + \ddot{y}_P\,\hat{j}$ is always valid. The five term formula is derived from it. The five terms are a shortcut, a way of making sense of the rather disorienting situations that involve all these relative ~~and~~ ~~relative~~ velocities and rotations.

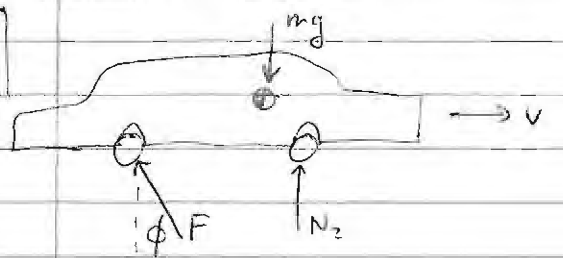
In principle, we can always write out $\vec{r}_{P/\tilde{J}}$ and take two time derivatives to find $\vec{a}_P$. This process can be very long and nitpicky, though, ~~and the~~ and the five term formula (if understood) can provide some intuition.

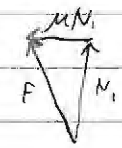
10/13

- Prelim recap
- Mathieu intuition
- Vibes intro.

Prelim stuff

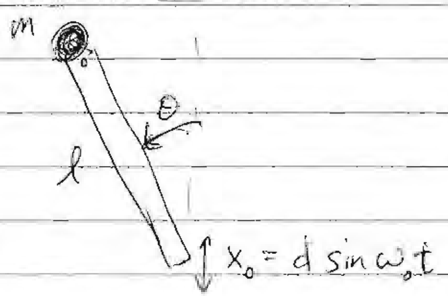
#1: Car





We need three equations, apparently? We can do pure AMB about three points, as long as they're not colinear. Or we can do a LMB + AMB hybrid.

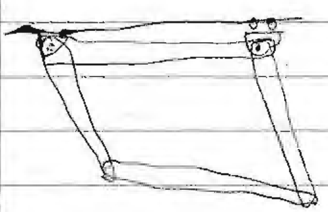
#2: balanced pendulum



$$\ddot{\theta} = \frac{g - \omega_0^2 d\sin\omega_0 t}{l}\sin\theta$$

... wiggling the base up and down is equivalent to the gravitational field oscillating.

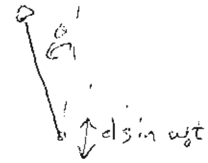
#3: 4-bar linkage



$$\underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{17\times 17}\begin{bmatrix} \text{accel's} \\ \text{const. forces} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Each bar has 3 DoF: x, y, θ

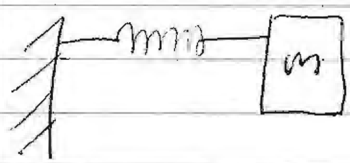
Constraints: Σangles = 360, lengths are fixed, ..



Mathieu intuition — why does the stick atop the wiggling base align with the vertical?

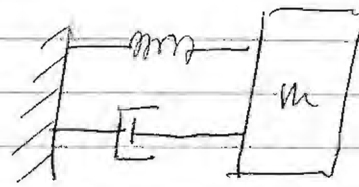
10/15 **VIBRATIONS**

1-DoF Vibrations



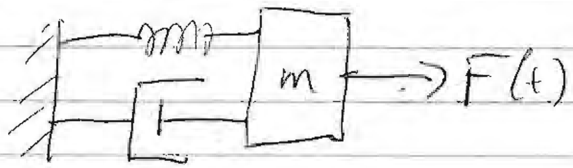
simplest: spring only

$\Rightarrow$



next up: spring + dashpot

$\Rightarrow$



general: spring + dashpot + driving force

We can model these systems and describe them by...

$$\boxed{\dot{\vec{z}} = A\vec{z}} \Rightarrow \boxed{\vec{z} = \vec{z}_0 e^{\lambda t}} \quad \lambda = \alpha + i\beta$$

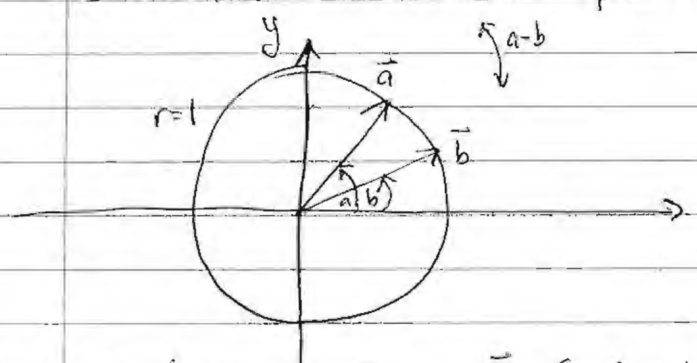
$$\vec{z} = \begin{bmatrix} \vdots \end{bmatrix}$$

* The solutions to this system of ODEs are boxed above. The exponentials are complex in general.

Useful vibration formulas & identities

$$\boxed{\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ \cos(a-b) &= \cos a\cos b + \sin a\sin b \\ \cos(\omega t-\phi) &= \cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi \\ \sin(a-b) &= \sin a\cos b - \cos a\sin b \end{aligned}}$$

Geometric mnemonic for identities



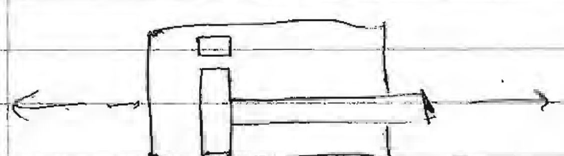
$\vec{a} = (\cos a, \sin a)$, $\vec{b} = (\cos b, \sin b)$

$$\vec{b}\times\vec{a} = |\vec{b}||\vec{a}|\sin(a-b) = 1\cdot 1\cdot\sin(a-b)$$

$$\vec{b}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos b & \sin b & 0 \\ \cos a & \sin a & 0 \end{vmatrix} = \hat{k}(\cos b\sin a - \sin b\cos a)$$

$$\Rightarrow \sin(a-b) = \sin a\cos b - \cos a\sin b$$

$\boxed{\text{Dashpot : wtf m8 ?}}$



$$T = c\,\dot{\ell}$$

"Linear dissipation (friction)"

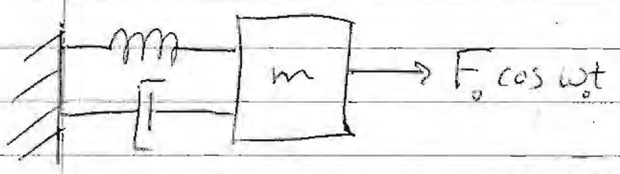
* Actual friction is generally not linear, but this model ~~results~~ gives nice linear ODEs that give closed-form solutions we're comfy with (i.e. exponentials)

* Matlab aside: if you see errors in your plots that grow linearly, there's a good chance that the cause is <u>rounding error</u>, which grows linearly in time.

LECTURE 10/17

<u>Today</u>: 1. Review
2. Intro to Normal Modes

Read/review: Inman CH. 2,3
(Lecture is skipping CH. 3 for now)



$$\boxed{m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega_0 t)}$$

Two main solutions: 1) $F_0 = 0 \Rightarrow x_h(t)$ — general homogeneous solution

2) $F_0 \neq 0 \Rightarrow x_p(t)$ — any particular solution
(steady state)

<u>Some vocab</u>:

Natural frequency: $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$, the frequency at which the system oscillates with no forcing & no damping

$$m\ddot{x} + kx = 0 \Rightarrow x = C\cos(\omega_n t) + D\sin(\omega_n t)$$

Forcing frequency ratio: $\boxed{r = \omega_0 / \omega_n}$ $\quad r = 1 \Rightarrow$ RESONANCE

Damping Ratio: $\boxed{\zeta = \dfrac{c}{c_{critical}} = \dfrac{c}{2\sqrt{mk}}}$

$\sqrt{b^2 - 4ac} = 0$
$\sqrt{c_{crit}^2 - 4mk} = 0$
$\sqrt{(c_{crit} - 2\sqrt{mk})^2} = 0$
$c_{crit} = 2\sqrt{mk}$

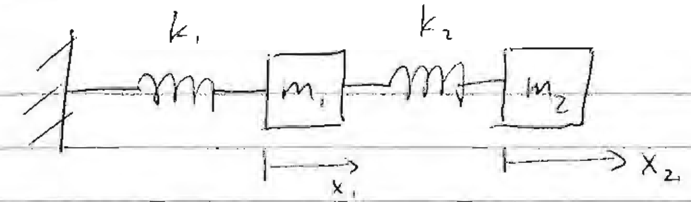
<u>Aside</u>: $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$ $\qquad \zeta = \frac{c}{2\sqrt{mk}} \quad \omega_n = \sqrt{k/m}$

$\zeta\omega_n$ form $\Longrightarrow$ $\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x}$ $\qquad \zeta\omega_n = \frac{c}{2\sqrt{mk}}\sqrt{\frac{k}{m}} = \frac{c}{2m}$

~~$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$~~

MDOF SYSTEMS (Multiple degrees of freedom)

Ex.



* We measure positions WRT an equilibrium position, but in practice finding this point is hard!

We claim that this is analogous to motion in two dimensions. To begin with, we'll neglect damping & forcing.

FBDs

$$T_1 = k_1 x_1 \qquad T_2 = k_2 (x_2 - x_1)$$

LMB

$$m_1 \ddot{x}_1 = k_2 (x_2 - x_1) - k_1 x_1$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1)$$

We can write this in matrix form:

$$\boxed{M \ddot{\bar{x}} + K \bar{x} = \bar{0}}$$

"mass matrix": $M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$, $K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$: "Stiffness matrix"

NOTE: M and K will both ALWAYS be symmetric which means they will always have a full spectrum of mutually orthogonal eigenvectors.

* however, the product $M^{-1}K$ need not be symmetric

$$\boxed{M\ddot{\vec{x}} + K\vec{x} = \vec{O}}$$

MDOF, no damping
no forcing

To play dumb, we can solve this numerically by writing

$$\star \begin{cases} \dot{\vec{x}} = \vec{v} \\ \dot{\vec{v}} = -M^{-1}K\vec{x} \iff \dot{\vec{v}} = -M\backslash K\vec{x} \end{cases}$$

This is solvable pretty easily if we're given some initial conditions $\vec{x}_0$ and $\vec{v}_0$.

We'd like an analytical solution to play around with, though. So we'll solve this (system of) ODE the same way you solve any ODE: <u>we guess.</u>

$$\vec{x} = \underbrace{\vec{u}}_{\text{CONST.}} e^{i\overbrace{\omega}^{\text{CONST}} t}$$

(no damping $\Rightarrow$ no exp. decay $\Rightarrow$ e$^{\text{pure imaginary}}$)

Let's plug it into our ODE & check:

$$\ddot{\vec{x}} = -\omega^2 \vec{u} e^{i\omega t} = -\omega^2 \vec{x}$$

$$M\ddot{\vec{x}} + K\vec{x} = \vec{O} \iff -\omega^2 M\vec{u}e^{i\omega t} + K\vec{u}e^{i\omega t} = \vec{O}$$

$$\Rightarrow -\omega^2 M\vec{u} + K\vec{u} = \vec{O}$$

mult. M^{-1} {} $\Rightarrow -\omega^2 I\vec{u} + M^{-1}K\vec{u} = \vec{O}$

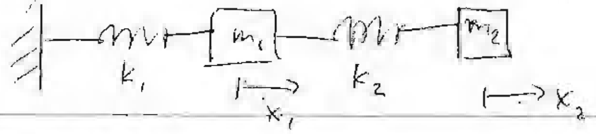
$$\Rightarrow (M^{-1}K - \omega^2 I)\vec{u} = \vec{O} \iff A\vec{v} = \lambda\vec{v}$$

This looks like a characteristic equation: $(A - \lambda I)\vec{u} = \vec{O}$
$\iff \det(A - \lambda I) = 0$

$\ast$ We can proceed, then, if we assume $M^{-1}K$ has n distinct eigenvectors & eigenvalues.

10/19 **VIBRATIONS CONT'D.**

From



M ("positive definite", always invertible)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

K ("positive semi-definite", usually invertible)

We guess

$$\vec{x} = e^{i\omega t}\,\vec{u}$$

$$\Rightarrow -\omega^2 M\vec{u} + K\vec{u} = \vec{0}$$

$$M^{-1}\{\} \Rightarrow \left[M^{-1}K - \omega^2 I\right]\vec{u} = \vec{0}$$

Matlab can solve equations of this form pretty easily:

$$[P, \lambda] = \mathrm{eig}(M^{-1}K)$$

$$P = \left[\begin{array}{c|c|c|c} \mathrm{eig} & \mathrm{eig} & & \mathrm{eig} \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{array}\right] \qquad (M \in \mathbb{R}^{n\times n})$$

$$\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \qquad \text{where } A\vec{v}_i = \lambda_i \vec{v}_i$$

<u>Fact</u>: From $\boxed{M^{-1}K\vec{u}_i = \omega_i^2\vec{u}_i}$, $M, K \in \mathbb{R}^{n\times n}$ we will find n linearly independent eigenvectors.

This will give us our general homogeneous solution,

$$\vec{X}_h = \sum A_i \vec{u}_i \cos\omega_i t + \sum B_i \vec{u}_i \sin\omega_i t$$

Because we have n <u>lin. indep.</u> $\vec{u}_i$, we can span an n-dim. position space at $t=0$ with $\sum A_i\vec{u}_i$, and ~~any~~ span an n-dim. velocity space at $t=0$ with $\sum B_i\vec{u}_i$. Thus, we can satisfy <u>any</u> initial conditions with appropriate choice of the $\{A_i, B_i\}$

$\vec{x}(t) =$

Ex. Consider $\vec{X}_0$, $\vec{V}_0 = 0$.

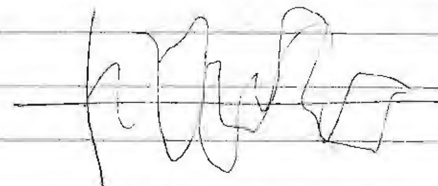
$\Rightarrow \cos(\omega_i(t=0)) = 1$ and $B_i = 0 \quad \forall i$

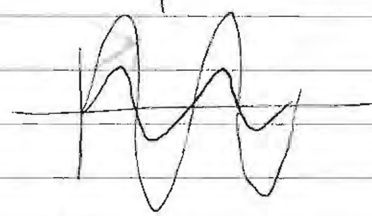
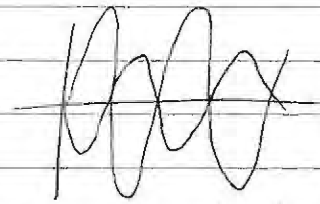
$$\sum A_i \vec{u}_i = \vec{X}_0 \Rightarrow \left\{ \vec{X}_0 = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}}_{\vec{A}} \right.$$

$\Rightarrow \vec{A} = P \backslash \vec{X}_0$ (Matlab)

$\Rightarrow$ $\underline{\text{Soln: } \vec{x} = \sum_i A_i \vec{u}_i \cos \omega_i t}$

We can also do this whole process in Matlab. Reins demonstrates it in class (see website for code).

Random ICs $\Rightarrow$  (messy)

"Special" ICs $\Rightarrow$  and 

in phase $\qquad$ $\pi/2$ out of phase

These are the "normal modes"

* figure(2) } opens a new figure window called figure 2
plot(...) } & plots thee . woot!

* M^(-1) $\Leftrightarrow$ inv(M) // inverts matrix M.

* diag(A) takes main diagonal of A & stores it in a vector (row or column?)

10/24: VIBRATIONS

Normal Modes (review)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{M\ddot{\vec{x}} + K\vec{x} = 0}$$

- M & K are usually symmetric. M is always positive definite. K is positive semi-definite.

This is the canonical ODE system. It ~~tends~~ admits solutions of the form $\vec{x}(t) = \vec{u}_i e^{i\omega_i t}$

Def: M is <u>positive definite</u> iff

$$\vec{x} \neq 0 \Rightarrow \vec{x}^T M \vec{x} > 0 \qquad \forall \vec{x} \iff E_k > 0$$

for moving objects

Def: ~~K~~ is <u>positive semi-definite</u> iff

$$\vec{x} \neq 0 \Rightarrow \vec{x}^T \cancel{K} \vec{x} \geq 0 \qquad \forall \vec{x} \iff E_p \geq 0$$

$(1\times n)(n\times n)(n\times 1)$

$(1\times n)(n\times 1)$

1×1

NB: Directions that have $\omega_i = 0$ have no E_p associated with displacements in that direction. In this case (zero eigenvalue), we need to write our general solution $\vec{x}(t)$ as

$$\boxed{\omega_i = 0 \Rightarrow \vec{x}(t) = A_i \vec{u}_i + B_i t \vec{u}_i} \qquad (E_p = 0)$$

10/24 VIBRATIONS

Today
- Normal modes
- Linearization
- $M^{1/2}$

Normal Modes Summary

$$\boxed{\ddot{\vec{x}} = -M^{-1}K\vec{x} \Rightarrow \vec{x}_i(t) = \vec{u}_i e^{i\omega_i t}}$$

where we find the ω_i & $\vec{u}_i$ by

$$[\text{eigvecs, eigvals}] = \text{eig}(M^{-1}K)$$

$$\text{eigvecs} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \quad \text{eigvals} = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \ \omega_n^2 \end{bmatrix}$$

* IF $\omega_i = 0$, then use this general sol'n:

$$\boxed{\omega_i = 0 \Rightarrow \vec{x}(t) = (A_i + B_i t)\vec{u}_i}$$

Generally, there are <u>three ways</u> to solve the system

$$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$$

for some ICs

$$\vec{x}(0) = \vec{x}_0, \quad \dot{\vec{x}}(0) = \vec{v}_0$$

I. ode23

II. eig$(M^{-1}K)$ — see above

III. "The Official Method"

III. "The Official Method" for solving

$$\begin{cases} M\ddot{\vec{x}} + K\vec{x} = \vec{0} \\ \vec{x}(0) = \vec{x}_0 \quad , \quad \vec{v}(0) = \vec{v}_0 \end{cases}$$

This involves a change of coordinates to $\vec{q}$

$$\vec{q} = M^{1/2}\vec{x} \iff \vec{x} = M^{-1/2}\vec{q}$$

* We find $M^{1/2}$ (which is generally not unique) by diagonalizing M, then taking $\sqrt{\ }$ of the main diagonal, then un-diagonalizing. That assumes all M_{ij} are real and positive.

Then our system becomes

$$M(M^{-1/2}\ddot{\vec{q}}) + K(M^{-1/2}\vec{q}) = \vec{0}$$

$$\Rightarrow M^{-1/2}(M^{1/2}M^{1/2})M^{-1/2}\ddot{\vec{q}} + M^{-1/2}KM^{-1/2}\vec{q} = 0$$

($\uparrow$ mult. through by $M^{-1/2}$)

$$\Rightarrow \quad \ddot{\vec{q}} + M^{-1/2}KM^{-1/2}\vec{q} = 0$$

Define $\tilde{K} \equiv M^{-1/2}KM^{-1/2}$

Then our system is

$$\ddot{\vec{q}} + \tilde{K}\vec{q} = \vec{0}$$

for which we guess the solution

$$\vec{q} = \vec{v}e^{i\omega t}$$

$$\Rightarrow \quad -\omega^2\vec{v} + \tilde{K}\vec{v} = 0 \iff \tilde{K}\vec{v} = \omega^2\vec{v}$$

Which is the familiar eigen problem.

$$[\tilde{K} - \lambda I]\vec{v} = \vec{0}$$

NB: $\tilde{K}$ is SYMMETRIC (by construction)

Theorem. An $n \times n$ symmetric matrix has n linearly independent, mutually orthogonal eigenvectors and n distinct, real eigenvalues.

Corollary. If the above (symmetric) matrix is positive definite, its eigenvalues will also be positive.

So, $\tilde{K}$ has an orthonormal set of eigenvectors. Hence "normal modes."

And
$$\bar{q}(t) = \sum_{i=1}^{n} \left(A_i \cos \omega_i t + B_i \sin \omega_i t \right) \bar{v}_i$$

unless $\omega_i = 0 \Rightarrow (A_i + B_i t)\, \bar{v}_i$

Then
$$\bar{x}(t) = M^{-1/2}\, \bar{q}(t)$$

10/26 LECTURE: more normal modes | Emmie Nöther

1. Where from M, K?
   - Linearization
   - Quadratic L.E.
   - Linearization "on the fly"
2. Effective K/M
3. Symmetry

Symmetry in q (continuous symmetry, e.g. rotation or translation)
$\Leftrightarrow$ q "ignorable coordinate"
$\Leftrightarrow$ no q in EoM
$\Leftrightarrow$ Conservation law? only in conservative systems

Where from M, K ?

$$\boxed{M\ddot{\bar{x}} + K\bar{x} = \bar{0}}$$

We can get to this, our canonical MDoF vibration equation, in a number of ways:

I. (A) Start with full, nonlinear EoM (from AMB, LMB, COE, etc.)

(B) Find equilibrium where $\bar{x}^{*} = \text{CONST}$ solves the EoM

(C) Change variables to $\bar{x}$ that are $\bar{0}$ at equilibrium ($\bar{x}^{*} \longrightarrow \bar{x}$)

(D) Linearize about $\bar{x} = \bar{0}$ (Write the Taylor series, keep the first n terms $\Rightarrow$ $\mathcal{O}(n)$ approx.)

e.g. $$\dot{\theta} = \cos\theta \cdot \sin\theta = \left(1 - \frac{\theta^2}{2} + \frac{\theta^2}{4} - \dots\right)\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$\Rightarrow \dot{\theta} \simeq (1)\cdot(\theta) = \theta + \mathcal{O}(\theta^2)$$

or, e.g. $$\dot{\theta} = \frac{1}{1+\theta^2} \simeq 1 - \theta^2 + \dots$$

$$\Rightarrow \dot{\theta} \approx 1$$

(E) Write in matrix form.

II. Lagrange equations

(A) Find equilibrium ~~where~~ & change variables $\bar{x}^{*} \longrightarrow \bar{x} = \bar{0}$ at equil.

(B) Write Taylor series for E_p & E_k in the equilibrium neighborhood.

$$E_p = \cancel{E_{po}}^{\,0} + \underbrace{\cancel{\sum B_i x_i}}_{0 \text{ since } \frac{\partial E_p}{\partial \bar{x}} = 0 \text{ at } \bar{x} = 0 \dots} + \underbrace{\frac{1}{2}\sum_i \sum_j A_{ij} x_i x_j}_{= \frac{1}{2}\bar{x}' A \bar{x}} + \dots$$

$$E_k = \frac{1}{2}\dot{\bar{x}}' M \dot{\bar{x}}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

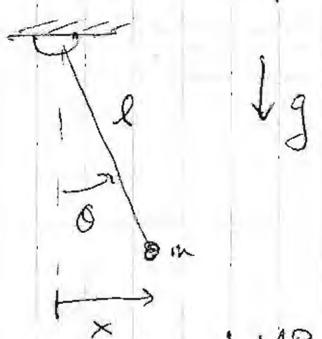
$\vdots$ (lots of deriv's, subscripts & algebra)

$$\Rightarrow M\ddot{\vec{x}} + K\vec{x} = 0$$

III. "Linearization on the fly" (ad hoc)

Look ahead to the goal of linear EoM and linearize as you go, keeping ~~only as many~~ just enough terms to arrive at that goal.

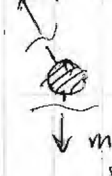
Ex: pendulum



FBD

"higher order terms"

$T = mg + \text{h.o.t.}$

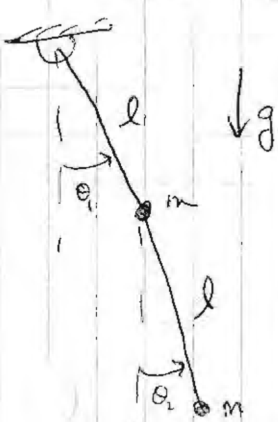


LMB in $\hat{x}$:

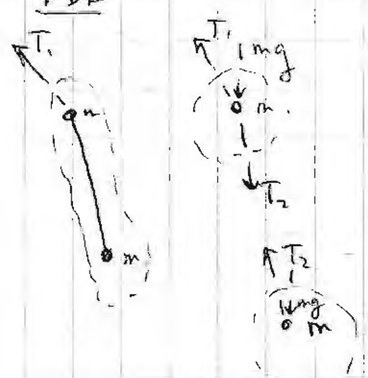
$$-T \cancel{\sin\theta} = m\ddot{x} \qquad \sin\theta \approx x/\ell$$

$$+ mg\frac{x}{\ell} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} \simeq -\frac{g}{\ell}x \iff \ddot{\theta} = -\frac{g}{\ell}\theta$$

Ex: simplest double pendulum



FBD



$T_1 \simeq 2mg$

$T_2 \simeq mg$

LMB, $\hat{x}$

$$-T_1\frac{x_1}{\ell} + T_2\frac{(x_2-x_1)}{\ell} = m\ddot{x}_1$$

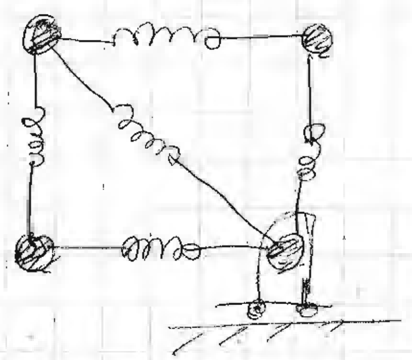
$$\Rightarrow -2mg\frac{x_1}{\ell} + \frac{mg}{\ell}(x_2-x_1) = m\ddot{x}_1$$

$$-T_2(x_2-x_1)/\ell = m\ddot{x}_2$$

$$\Rightarrow -\frac{mg}{\ell}(x_2-x_1) = m\ddot{x}_2$$

$$\Rightarrow \begin{cases} \ddot{x}_1 = \frac{g}{\ell}\cancel{(x_2-3x_1)}(x_2-3x_1) \\ \ddot{x}_2 = \frac{g}{\ell}(x_1-x_2) \end{cases}$$

ex **Lattice Vibration**



$\boxed{10/29}$:

NORMAL MODES ctd.

- Normal coordinates
- Simple interpretation
- Damping

Recall: $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$

~~We~~ To clean this up, i.e. get rid of M and get a nice, symmetric version of K, we change coordinates to

$$\vec{q} = M^{1/2}\vec{x}$$

whence

$$\ddot{\vec{q}} + \tilde{K}\vec{q} = \vec{0}$$

where

$$\tilde{K} \equiv M^{-1/2} K M^{1/2}$$

is real & symmetric.

Therefore $\tilde{K}$ has eigenvalues in the OLHP and eigenvectors $\vec{v}_i$ that are mutually orthonormal.

Our general solution is therefore

$$\begin{cases} \vec{q}_{gen.} = \sum (A_i \cos\omega_i t + B_i \sin\omega_i t)\vec{v}_i, & \omega_i \neq 0 \\ \text{OR} \\ \vec{q}_{gen} = \sum (A_i + B_i t)\vec{v}_i, & \omega_i = 0 \end{cases}$$

CHANGE TO "MODAL COORDINATES", r_i

Let P be the matrix whose columns are the eigenvectors of $\bar{K} = M^{-1/2} K M^{1/2}$.

or, $$P = \left[\vec{v}_1 \,\vdots\, \vec{v}_2 \,\vdots\, \cdots \,\vdots\, \vec{v}_n \right]$$

We now do another change of coordinates:

$$\vec{q} = P \vec{r}$$

Then $$\vec{x} = M^{-1/2} \vec{q} = M^{-1/2} P \vec{r}$$

$$\Rightarrow \vec{r} = P^{-1} M^{1/2} \vec{x}$$

But the columns of P are mutually orthogonal (since they're the eigenvectors of the symmetric matrix $\bar{K}$) and normalized by construction. So ...

$$\begin{bmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \leftarrow \vec{v}_2 \rightarrow \\ \leftarrow \vdots \rightarrow \\ \leftarrow \vec{v}_n \rightarrow \end{bmatrix} \left[\begin{matrix} \uparrow \\ \vec{v}_1 \\ \downarrow \end{matrix} \,\vdots\, \begin{matrix} \uparrow \\ \vec{v}_2 \\ \downarrow \end{matrix} \,\vdots\, \cdots \,\vdots\, \begin{matrix} \uparrow \\ \vec{v}_3 \\ \downarrow \end{matrix} \right] = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$ since $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$

$$\Rightarrow P^T \quad P = I$$

$$\Rightarrow \boxed{P^T = P^{-1}}$$

~~With the substitutio~~

With the transformation $\vec{q} = P\vec{r}$, our governing equation becomes

$$\ddot{\vec{q}} + \tilde{K}\vec{q} = \vec{0}$$

$$\Leftrightarrow P\ddot{\vec{r}} + \tilde{K}P\vec{r} = \vec{0}$$

$$\Rightarrow \ddot{\vec{r}} + P^T\tilde{K}P = \vec{0} \qquad \text{since } P^{-1} = P^T$$

Let's look at $P^T\tilde{K}P$. We know that

$$\tilde{K}P = \left[\lambda_1\vec{v}_1 \,\middle|\, \lambda_2\vec{v}_2 \,\middle|\, \cdots \,\middle|\, \lambda_n\vec{v}_n\right]$$

so

$$P^T\tilde{K}P = \begin{bmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \leftarrow \vec{v}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{v}_n \rightarrow \end{bmatrix} \left[\lambda_1\vec{v}_1 \,\middle|\, \lambda_2\vec{v}_2 \,\middle|\, \cdots \,\middle|\, \lambda_n\vec{v}_n\right] = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

again since $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$

So our governing equation is

$$\boxed{\ddot{\vec{r}} + \Lambda\vec{r} = \vec{0}} \Leftrightarrow \begin{cases} \ddot{r}_1 + \lambda_1 r_1 = 0 \\ \ddot{r}_2 + \lambda_2 r_2 = 0 \\ \vdots \\ \ddot{r}_n + \lambda_n r_n = 0 \end{cases}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 \ddots & \\ & & \lambda_n \end{bmatrix}$$

and the λ_i solve

$$\tilde{K}\vec{v}_i = \lambda_i\vec{v}_i$$

where $\tilde{K} = M^{-1/2} K M^{1/2}$

and $\vec{x} = M^{-1/2} P\vec{r}$

where $P = \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_n\right]$

So in the $\vec{r}$ basis we have n decoupled & vibration equations of the form

$$\ddot{r}_i(t) + \lambda_i \vec{r}_i(t) = \vec{0}$$

$$\Rightarrow \quad r_i(t) = \left(A_i \cos\omega_i t + B_i \sin\omega_i t\right)$$

To get the mode shape in physical coordinates, write

$$\vec{u}_1 = M^{-1/2} P \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

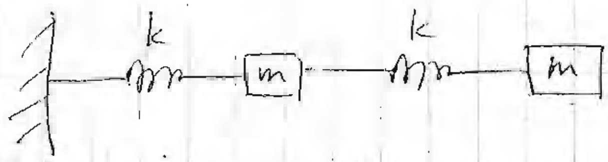
where the columns of $S =$ are the eigenvectors of $M^{-1}K$.

Physical interpretation of normal modes

- Each mass has simple harmonic motion,

$\Rightarrow$ same ~~k/m~~ effective k/m

ex)



- Calculate mode shape: $\bar{u} = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}$
- Calculate effective k's:

$$k_1^{eff} = k + (1 - x_2)$$

$$k_2^{eff} = k\,\frac{(x_2 - 1)}{x_2}$$

$$k_1^{eff} = k_2^{eff} \Rightarrow \frac{x_2 - 1}{x_2} = 1 + 1 - x_2$$

$$x_2 - 1 = 2x_2 - x_2^2$$

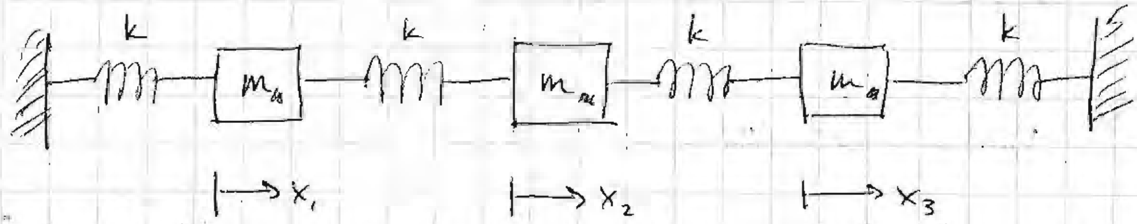
$$x_2^2 - x_2 - 1 = 0 \Rightarrow x_2 = \frac{1 \pm \sqrt{1+4}}{2}$$

"each bit of mass feels like it's in a harmonic oscillator."

Also, more obviously, the whole system feels like it's in a harmonic oscillator...

$\boxed{10/31}$ NORMAL MODES (cont'd).

QUIZ.



By intuition & plausibility arguments, find as many normal modes as you can (there are 3 total). Also find ω_i for each $\vec{v}_i$.

One is easy: the middle mass is at rest and the end masses oscillate in opposite directions.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad \omega_i = \sqrt{\frac{2k}{m}}$$

* Recall (& NB): All masses must "feel" the same effective spring stiffness, K^{eff}

Okay, what about all three moving in phase, at ~~their rest lengths~~ const. separation? Well, then the middle mass sees no net force (spring forces on either side cancel), and yet it's moving? That violates $\vec{F} = m\vec{a}$, so it's not possible.

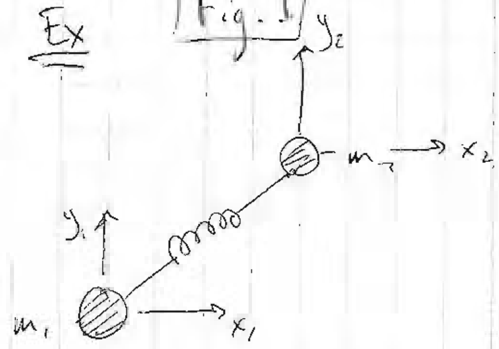
How about the outside masses moving ~~in phase~~ together, but the middle one doing something crazy/unknown. Guess this. If this were true, then our normal mode i would be

$$\vec{v}_2 = \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix}$$. Equate K_{eff}, solve for x.

MATLAB: $n \times n$ identity = eye(n)

eig($M^{-1}K$) = eig(K, M)

Ex | Fig. 1 |



Three normal modes are easy:

x-translation, y-translation, & circular motion, all with constant $\vec{x}_2 - \vec{x}_1$. $\omega = 0$ for all these.

The fourth is oscillation along $\vec{x}_2 - \vec{x}_1$.

4 deg. of freedom

$\Rightarrow$ 4 normal modes

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \text{x-trans} & \text{y-trans} & \text{circles} & \text{vibes} \\ \end{array}$$

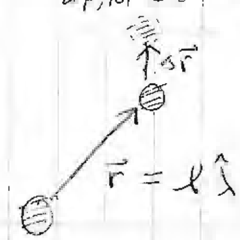
$$P = \left[\begin{array}{c|c|c|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$\omega = 0 \;,\; 0 \;,\; 0 \;,\; \sqrt{\dfrac{2k}{m_1+m_2}}$

$F_1 = -kx$

$E_{p1} = \frac{1}{2} k x^2$

$E_{p,TOT} = \sum \frac{1}{2} k \left((\Delta r \cdot \hat{\lambda})\right)^2$



* Writing this system up as $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$ is annoying. Guessing normal modes can save you tons of time.

11/2 : TRUSS VIBRATIONS

See above Figure 1.

Define $\hat{\lambda} = \vec{r}/|\vec{r}|$. Then $\vec{F} = -k(\hat{\lambda} \cdot \Delta\vec{r})\hat{\lambda} \Rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -k \underbrace{\begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix}}_{\bar{k}}$

then we can assemble our big $\tilde{K}$ matrix by

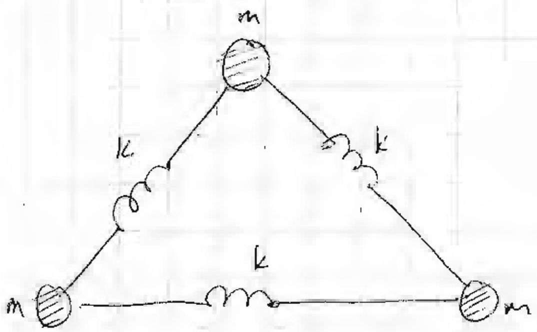
$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{bmatrix} = -\left[\begin{array}{c|c} \bar{k}_{2\times2} & -\bar{k} \\ \hline -\bar{k} & \bar{k}_{2\times2} \end{array}\right] \begin{bmatrix} \Delta r_{1x} \\ \Delta r_{1y} \\ \Delta r_{2x} \\ \Delta r_{2y} \end{bmatrix}$$

11/5

6, 6,
6, 3,

- Intuitive linear modes (cont'd)
- Computation (cont'd)
- Damping

<u>Quiz</u> (2D)

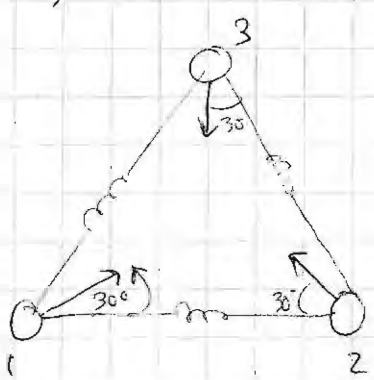


- 6 DoF
- 6 rows & cols in K & M
- 6 normal modes
- 3 $\omega_i = 0$ (2 trans, one circle)

equal masses,
equal springs.

- <u>Find one vibrating* mode:</u> $(\omega_i \neq 0)$
  - * If the masses are equal, then they all feel the same K_{eff}. Even if they're not equal, though, for any <u>normal mode</u> all the masses oscillate with the same frequency ω_i *.

So, one normal mode is the so-called "breathing mode:"



$$\vec{u}_i = \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ 0 \\ -1 \end{bmatrix}$$

To find ω_i, we use the fact that $K_{eff} = \text{CONST}$ for all three masses, when $K_{eff} = F/d$, d being the displacement.

$$\omega_i^2 = \left(\frac{K}{m}\right)_{eff} = \frac{F/d}{m} = \frac{2k\left(2\frac{\sqrt{3}}{2}\cancel{d}\right)\left(\frac{\sqrt{3}}{2}\right)/\cancel{d}}{m} = 3\frac{k}{m}$$

$$\Rightarrow \quad \omega_i = \sqrt{\frac{3k}{m}}$$

* What force are we talking about here?

Intuitive concept for normal modes:

Higher frequency ~~mo~~ normal modes occur when masses tend to move opposite one another.

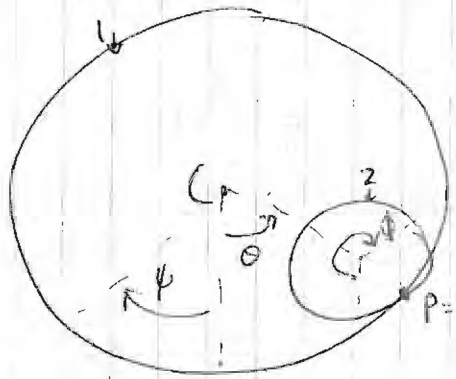
When masses move "more together", i.e. in similar spatial directions, then the spring forces between masses are small, because the springs don't stretch much. Therefore, the frequency of oscillation tends to be lower.

Both of these cases have synchronous motion, i.e. all masses move with the same ~~sy~~ temporal frequency.

So "higher spatial frequency $\Rightarrow$ higher temporal frequency"

& "lower spatial frequency $\Rightarrow$ lower temporal frequency"

LECTURE WED. 11/7/12



Roll w/o slip $\Rightarrow \vec{v}_{P/c} = \vec{V}_{P'/c}$

$\Leftrightarrow \vec{V}_{G/c} + \vec{\omega}_c \times \vec{r}_{P/G_2} = \vec{\omega}_1 \times \vec{r}_{P/c}$

$\vec{\omega}_1 = -\dot{\psi}\,\hat{k}$
$\vec{\omega}_2 = -\dot{\phi}\,\hat{k}$

$$\boxed{\dot{\vec{H}}_{/c} = \int \vec{r}_{/c} \times \vec{a}_{/\mathcal{F}}\, dm}$$

"The most fundamental quantity in mechanics"

DAMPING

3

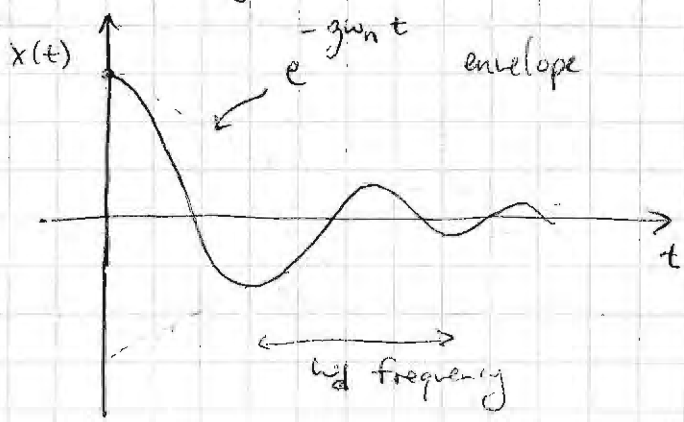
1 DoF case

Undamped : $\ddot{x} + \omega_n^2 x = 0 \Rightarrow x(t) = A\cos\omega_n t + B\sin\omega_n t$

Damped : $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$

$s = -\zeta\omega_n \pm i\omega_d$

$\omega_d = \sqrt{1-\zeta^2}$

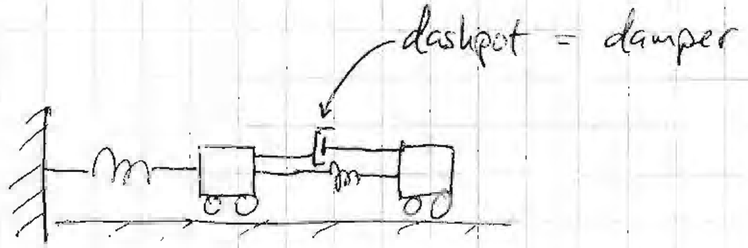


Normal Modes (recap)

$$M\ddot{\vec{x}} + K\vec{x} = 0 \iff \begin{aligned} \ddot{r}_1 + \omega_{n1} r_1 &= 0 \\ \ddot{r}_2 + \omega_{n2} r_2 &= 0 \\ &\vdots \\ \ddot{r}_n + \omega_{nn} r_n &= 0 \end{aligned}$$

"By diagonalization (change of basis to eigenvectors of) we reduce this n^{th} order system to n decoupled harmonic oscillators.

Damping

e.g. 1-D $\longrightarrow$ 

① $$\boxed{M\ddot{\vec x} + \underbrace{C\dot{\vec x}}_{\text{damping term}} + k\vec x = 0}$$

① is a ~~2nd order system~~ system of n 2nd order ODEs. Can we reduce it to a system of 1st order ODEs?

$$\vec z = \begin{bmatrix} \leftarrow \vec x \rightarrow \\ \leftarrow \dot{\vec x} \rightarrow \end{bmatrix} \qquad \vec x \ \& \ \dot{\vec x} : n \times 1 \quad (n \text{ masses})$$

$$\vec z : 2n \times 1$$

Let $\vec v = \dot{\vec x}$, then ① becomes

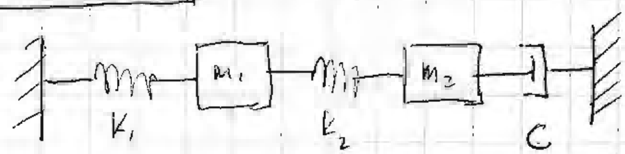
$$M\dot{\vec v} + C\vec v + k\vec x = 0 \iff M\dot{\vec v} = -C\vec v - k\vec x$$

so
$$\dot{\vec x} = \vec v$$
$$\dot{\vec v} = -M^{-1}C\vec v - M^{-1}k\vec x$$

$$\iff \dot{\vec z} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{\equiv A} \begin{bmatrix} \vec x \\ \vec v \end{bmatrix}$$

$$\iff \boxed{\dot{\vec z} = A\vec z}$$ guess soln: $\vec z = e^{\lambda t}\vec V$

DAMPING 11/9/12

ex. 

FBDs

kx_1 $\qquad$ $k(x_2 - x_1)$ $\qquad$ $c\dot{x}_2$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$$

In the 1D analog, from ~~the~~ here we can get

$$m\ddot{x} + c\dot{x} + kx = 0$$

$c = 0 \Rightarrow \quad \ddot{r} + \omega^2 r = 0$

$c \neq 0 \Rightarrow \quad \boxed{\ddot{r} + 2\zeta\omega_n\dot{r} + \omega_n^2 r = 0}$

Can we produce a similar-looking vector equation for our MDOF system with damping?

Ans: NO, but we can cheat & do something similar.

Let $\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$, $\dot{\vec{x}} = \vec{v}$, $\dot{\vec{v}} = -M^{-1}K\vec{x} - M^{-1}C\dot{\vec{x}}$

$$\Rightarrow \quad \dot{\vec{z}} = \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]}_{\equiv A}\vec{z} \iff \dot{\vec{z}} = A\vec{z}$$

Aside : scalar equations

$$\dot{x} = ax \Rightarrow x(t) = x_0 e^{at} \quad \text{(guess)}.$$

where $e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$

We can check the ODE sol'n guess :

$$x(t) = x_0 e^{at} \Rightarrow a\,x(t) = a x_0 \left(1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots\right)$$

$$\Rightarrow a\,x(t) = x_0 \left(a + a^2 t + a\frac{(at^2)}{2!} + a\frac{(at)^3}{3!} + \dots\right)$$

That's the RHS. Let's check the LHS :

$$\dot{x}(t) = \frac{d}{dt}\left(x_0\left(1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots\right)\right)$$

$$= x_0\left(a + a^2 t + a\left(\frac{a^2t^2}{2!}\right) + \dots\right.$$

$$= a x_0 \left(1 + at + \frac{(at)^2}{2!} + \dots\right) = a\vec{x}_0 e^{at} \checkmark$$

So that works. Let's try the matrix analog:

Define $e^{At} \equiv I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$

where $A^2 = AA$, etc.

Then we can guess $\vec{z} = e^{At}\vec{z}_0$ and check it by the same process.

Summing up,

$$\boxed{\dot{\vec{z}} = A\vec{z} \Longrightarrow \vec{z}(t) = e^{At}\vec{z}_0}$$

We can solve this a few ways:

1. ode45
2. $\vec{z} = e^{At}\vec{z}_0$
3. Guess $\vec{z} = e^{\lambda_i t}\vec{u}_i$. Then $\dot{\vec{z}} = \lambda_i \cancel{e^{\lambda_i t}}\vec{u}_i = A\cancel{e^{\lambda_i t}}\vec{u}_i$
$\Rightarrow A\vec{u}_i = \lambda_i\vec{u}_i \Leftrightarrow \boxed{(A - \lambda_i I)\vec{u}_i = 0}$

$$\boxed{\dot{\vec{z}} = A\vec{z}} \; : \; \underline{\text{Solutions}}.$$

1. $\boxed{\text{ode 45}}$: <u>numerical solver</u>
2. <u>Matrix exponential</u> : $\boxed{\vec{z}(t) = e^{At}\vec{z}_0}$
3. <u>Eigen. decomposition</u> : Guess $\boxed{\vec{z}(t) = e^{\lambda_i t}\vec{u}_i}$

then $\lambda_i \vec{u}_i = A\vec{u}_i \Rightarrow$ ~~de~~ $|\lambda_i I - A| = 0$

solve for $\lambda_i, \vec{u}_i$.

If A has a full set ($n\times n$ $A \Rightarrow n$ eigenvectors) of linearly independent eigenvectors, then the general solution takes the form

$$\vec{z} = \sum_i C_i e^{\lambda_i t}\vec{u}_i \quad , \quad \text{if } \lambda_i \neq 0$$

$$\Leftrightarrow \quad \vec{z} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$\Leftrightarrow \quad \cancel{\underbrace{\vec{z}}_{n\times 1} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_n t} \end{bmatrix}}_{n\times n,\ \text{diag}} \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{n\times 1}}$$

$$\vec{z} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

If $\vec{z}(t=0) = \vec{z}_0$, then our matrix eqn is

$$\vec{z}(t) = \underbrace{\begin{bmatrix} \\ \text{eigvecs} \\ \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \text{exp. of} \\ \text{eigvals} \\ \text{(diag)} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_n \end{bmatrix}}_{\vec{c}}$$

$$\Leftrightarrow \quad \vec{z}(t) = PD\vec{c}$$

$$\vec{z}(0) = \vec{z}_0 = P \begin{bmatrix} e^0 & & 0 \\ & e^0 & \\ 0 & & e^0 \end{bmatrix} \vec{c} = PI\vec{c} = P\vec{c}$$

$$\Rightarrow \quad \vec{c} = P^{-1}\vec{z}_0 = P \backslash \vec{z}_0$$

(that's how we find the coefficient vector $\vec{c}$)

NB: Since ~~solutions~~ the general solution to an ODE is unique, the "matrix exponential" and "eigenstuff" must agree. And they do.

NB: Matlab $\longrightarrow$ a matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

in the system $\dot{\vec{z}} = A\vec{z}$, represents

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}\right\} \text{ sinusoids, or } \dot{x} = v,\ \dot{v} = \ddot{x},\ \ddot{x} = -x \ \left(\begin{matrix}\text{harmonic} \\ \text{oscillator}\end{matrix}\right)$$

$$\dot{x}_3 = 3x_3 \ \} \ \text{real exponential}$$

$\boxed{\text{11/12 : Damping}}$

Recall: springs, masses, dampers $\Rightarrow$

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$$

$$\Rightarrow \quad \dot{\vec{z}} = A\vec{z}, \quad \text{with} \quad \vec{z} = \begin{bmatrix} \vec{x} \\ \dot{\vec{x}} \end{bmatrix}$$

$$\text{and} \quad A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]$$

$\vec{x}: n\times 1$, $\vec{v} = \dot{\vec{x}}: n\times 1$, $\vec{z}: 2n\times 1$, $A: 2n\times 2n$

- Matlab note on the function <u>diag.m</u>: when fed a <u>matrix</u>, it returns the main diagonal elements in a column vector. When fed a <u>vector</u>, it returns a diagonal matrix with that vector as its main diagonal.

- Of our three solution techniques (ode45, expm(A*t)$\vec{z}_0$, and eigendecomposition), only the third has problems. This is because we're not guaranteed a full set of eigenvectors & nondegenerate eigenvalues.

<u>Lagrange with friction/air resistance:</u> e.g. $P_{out} = \frac{1}{2}(x_2 - x_1)^2$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{bi}} - \frac{\partial \mathcal{L}}{\partial q_{bi}} + \frac{\partial P_{out}}{\partial \dot{q}_{bi}} = \frac{\partial P_{in}}{\partial \dot{q}_{bi}}$$

* Check signs.

11/14 : MDoF DAMPING CONT'D

Recall: no damping (1) $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$

*2 equivalent methods (2) $\dot{\vec{z}} = A\vec{z}$, $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$

ex. $M = 1$, $K = 1$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$ "Phil and Sally"

Some notes on the equivalence of these methods:

- Method 1: $M\ddot{\vec{x}} + K\vec{x} = \vec{0}$ (Let's work on this eqn. w/o guessing a sol'n.)

Let $\boxed{\vec{x} = M^{-1/2}\vec{q}}$ (change of basis)

Then our ODE system is

$$MM^{-1/2}\ddot{\vec{q}} + KM^{-1/2}\vec{q} = \vec{0} \iff M^{1/2}\ddot{\vec{q}} + KM^{-1/2}\vec{q} = \vec{0}$$

$$\Rightarrow \ddot{\vec{q}} + M^{-1/2}KM^{-1/2}\vec{q} = 0$$

Define $\tilde{K} = M^{-1/2}KM^{-1/2}$. Then ODE sys is

$$\boxed{\ddot{\vec{q}} + \tilde{K}\vec{q} = \vec{0}}$$

Now let P be the matrix of eigenvectors of $\tilde{K}$

$$P = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$, $\vec{u}_i$, λ_i an eigenpair of $\tilde{K}$.

Then $\vec{q} = r_1\vec{u}_1 + r_2\vec{u}_2 + \dots + r_n\vec{u}_n$

let

$$\iff \vec{q} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \iff \boxed{\vec{q} = P\vec{r}}$$ Second change of basis

Our ODE system now becomes, after the change of bases

$$\vec{q} = M^{1/2}\vec{x}, \quad \vec{r} = P^{-1}\vec{q}$$

$$\Rightarrow \ddot{\vec{q}} + \tilde{K}\vec{q} = \vec{0} \iff P\ddot{\vec{r}} + \tilde{K}P\vec{r} = \vec{0}$$

$$\Rightarrow \ddot{\vec{r}} + P^{-1}\tilde{K}P\vec{r} = \vec{0}$$

But $P^{-1} = P^{T}$ since P is orthogonal, so

$$\ddot{\vec{r}} + P^{T}\tilde{K}P\vec{r} = \vec{0}$$

But P was built from the (mutually orthogonal) eigenvectors of $\tilde{K}$, so

$$\tilde{K}\vec{u}_i = \lambda_i\vec{u}_i$$

$$\begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} \tilde{K} \left[\vec{u}_1 \,\middle|\, \vec{u}_2 \,\middle|\, \cdots \,\middle|\, \vec{u}_n\right]$$

rows of K

$$= \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \\ \vdots \\ \tilde{K}_n \end{bmatrix} \left[\vec{u}_1 \,\middle|\, \vec{u}_2 \,\middle|\, \cdots \,\middle|\, \vec{u}_n\right]$$

$$= \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} \left[\lambda_1\vec{u}_1 \,\middle|\, \lambda_2\vec{u}_2 \,\middle|\, \cdots \,\middle|\, \lambda_n\vec{u}_n\right]$$

$$= \begin{bmatrix} \lambda_1\vec{u}_1\cdot\vec{u}_1 & & \\ & \lambda_2\vec{u}_2\cdot\vec{u}_2 & \\ & & \ddots \\ & & & \lambda_n\vec{u}_n\cdot\vec{u}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \; \lambda_n \end{bmatrix} \equiv \Lambda$$

So the changes of bases

$$\boxed{\vec{x} = M^{-1/2}\vec{q} \quad , \quad \vec{q} = P\vec{r}}$$

gave us the ODE system

$$\boxed{\ddot{\vec{r}} + \Lambda \vec{r} = \vec{0}}$$

where $\boxed{[P, \Lambda] = \text{eig}(\tilde{K})}$

$$\Longleftrightarrow \quad \begin{aligned} \ddot{r}_1 + \lambda_1 r_1 &= 0 \\ \ddot{r}_2 + \lambda_2 r_2 &= 0 \\ &\vdots \\ \ddot{r}_n + \lambda_n r_n &= 0 \end{aligned}$$

n decoupled harmonic oscillators in r basis

Method 2 : $\dot{\vec{z}} = A\vec{z}$

$$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix} , \quad A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]$$

$\vec{v}, \vec{x} : n \times 1$; $\vec{z} : 2n \times 1$; $A : 2n \times 2n$; $C, M, K : n \times n$

Comparing to Method 1, let's set $C = 0$.
Then

$$\begin{bmatrix} \dot{\vec{x}} \\ \dot{\vec{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$$

Guess a sol'n $e^{\lambda_i t}\vec{w}_i$, where $\vec{w}_i, \lambda_i$ is an eigenpair of $M^{-1}K$.

Then our solution is

$$\vec{z} = \begin{bmatrix} \vec{w}_x \\ \lambda \vec{w}_x \end{bmatrix} \quad \text{where } \lambda \vec{w}_x = \vec{w}_r$$

Now let's add damping:

Method 1: $M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = \bar{0}$

"Trouble!"

Method 2: $\approx$ OK. (if we ignore the highly improbable cases where we have insufficient eigenvectors & repeated eigenvalues)

So, method 2 is mostly okay when we add damping. However, there's some subtlety, because now we have complex eigenstuff.

Look at one "mode":

$$\begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix} = e^{\lambda t}\, \overline{\omega} \qquad (\text{complex } \lambda \text{ and } \vec{\omega})$$

If we take the real part, we get... $(\lambda = \alpha + i\beta)$ $(\overline{\omega} = \vec{u} + i\vec{v})$

$$\overline{Z} = e^{(\alpha+i\beta)t}(\vec{u} + i\vec{v})$$

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{u} + i\vec{v})$$

Then $Re\{\vec{Z}\} = \underbrace{e^{\alpha t}\cos\beta t\,\vec{u} - e^{\alpha t}\sin\beta t\,\vec{v}}_{\text{One "complex" mode}}$

Now all the K_{eff} stuff we did goes out the window, since each "normal" mode is a sum of a sine ~~x~~ cosine term.

This sucks. So, following the experts, we WISH IT AWAY.

~~It~~ Wishing away the ugly complex normal modes gives us "modal damping"

Here we assume that using the basis $\vec{r} = P^{-1} M^{1/2} \vec{x}$ from our undamped modes will give us a bunch of decoupled ~~that~~ damped oscillators of the form

$$\begin{cases} \ddot{r}_1 + 2\omega_n \zeta \dot{r}_1 + \omega_n^2 r_1 = 0 \\ \ddot{r}_2 + 2\omega_n \zeta \dot{r}_2 + \omega_n^2 r_2 = 0 \\ \vdots \\ \ddot{r}_n + 2\omega_n \zeta \dot{r}_n + \omega_n^2 r_n = 0 \end{cases}$$

In general, this is JUST NOT TRUE.

We can, however, cook up ~~the~~ cases where this IS true, e.g.

$$\boxed{C = \alpha M + \beta K}$$

We can go through the matrix algebra to show that this leads us to decoupled damped oscillators.

11/16 Damping & Forcing $\boxed{M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)}$

$\boxed{\vec{x} = M^{-1/2}\vec{q}} \Rightarrow \ddot{\vec{q}} + M^{-1/2} C M^{-1/2} \dot{\vec{q}} + M^{-1/2} C M^{-1/2} \vec{q} = M^{-1/2} \vec{F}(t)$

3 approaches: ① ode45 (best), ② e^{At} (less useful; control theorists); ③ modes : valid if damping is SMALL.

$$\Rightarrow \boxed{\ddot{\vec{q}} + \tilde{C}\dot{\vec{q}} + \tilde{K}\vec{q} = M^{-1/2}\vec{F}(t)}$$

where $\tilde{K} = M^{-1/2} K M^{-1/2}$ and $\tilde{C} = M^{-1/2} C M^{-1/2}$

Let $V = [\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_n]$: eigenvectors of $\widetilde{K}$.

Change bases <u>again</u> via $\boxed{\vec{q} = V\vec{r}}$

Then we have

$$V\ddot{\vec{r}} + \widetilde{C}V\dot{\vec{r}} + \widetilde{K}V\vec{r} = M^{-1/2}\vec{F}(t)$$

But V is orthogonal, so $V^{-1} = V^T$ and we have

$$\ddot{\vec{r}} + V^T\widetilde{C}V\dot{\vec{r}} + V^T\widetilde{K}V\vec{r} = V^TM^{-1/2}\vec{F}(t)$$

But $V^T\widetilde{K}V$ is the diagonalization of $\widetilde{K}$, so it's just Λ.

We WISH for a bunch of decoupled, damped, driven oscillators. But $V^T\widetilde{C}V$ is NOT DIAGONAL in general, so we don't get that — except for a few <u>SPECIAL CASES</u>:

$$\boxed{\ddot{\vec{r}} + V^T\widetilde{C}V\dot{\vec{r}} + \Lambda\vec{r} = \widetilde{\vec{F}}(t)} \quad \widetilde{\vec{F}}(t) \equiv V^TM^{-1/2}\vec{F}(t)$$

<u>Special Case I</u>: $C = 0$. Then

$$\ddot{\vec{r}} + \Lambda\vec{r} = \widetilde{\vec{F}}(t) \iff \begin{aligned} \ddot{r}_1 + \lambda_1 r_1 &= F_1(t) \\ \ddot{r}_2 + \lambda_2 r_2 &= F_2(t) \\ &\vdots \\ \ddot{r}_n + \lambda_n r_n &= F_n(t) \end{aligned}$$

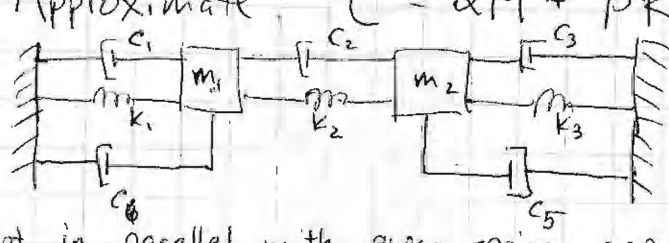
<u>Special Case II</u>: Just WISH that $V^T\widetilde{C}V$ is diagonal. ← "MODAL DAMPING"

* Okay for gray boxy experiments, but not for theory since we don't know HOW to wish $V^T\widetilde{C}V$ into diagonality.

<u>Special Case III</u>: Make an approximation of the system...

(a) replace $V^T\widetilde{C}V$ with diagonal elements of $V^T\widetilde{C}V$

(b) Approximate $C = \alpha M + \beta K$



$c_1 = \beta k_1$
$c_2 = \beta k_2$
$c_3 = \beta k_3$
$c_6 = \alpha m_1$
$c_5 = \alpha m_2$

"Put a dashpot in parallel with every spring, proportionally w/ β.
Put a dashpot on every mass, connecting it to the world, with damping α."

Special Case III cont'd

Writing $C = \alpha M + \beta K$ (α, β scalars) gives us

$$\ddot{r}_i + 2\zeta_i \omega_{ni} \dot{r}_i + \omega_{ni}^2 = \tilde{F}_i$$

where $\zeta_i = \frac{1}{2}\left(\beta \omega_i + \frac{\alpha}{\omega_i}\right)$

"For high frequency modes (ω_i large), the K-term β is important."

"For low frequency modes (ω_i small), the M-term α is important."

11/19 : VIBRATIONS OF CONTINUOUS MEDIA

Punchline: The (1-D linear non-dispersive) wave equation:

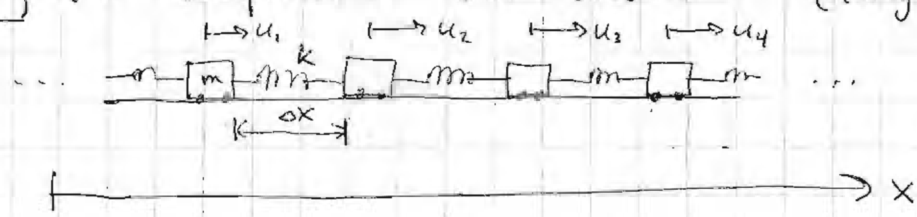
$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

e.g. $u(x,t) = \sin(\omega_1 t)\cos(\omega_2 x)$ where ω_1 & ω_2 are related.

Here the $\cos(\omega_2 x)$ term is the continuous analog of our normal mode $\vec{V}_i$ in the discrete case.

A Few Ways to Derive the Wave Equation

1. 1-D compressive "bar waves" (longitudinal)



Label points by initial position.

Mass i FBD

$-T_{i-1}\hat{i} \leftarrow \{ \square \} \rightarrow T_{i+1}\hat{i}$

$T_{i-1} = k(u_i - u_{i-1})$

$T_{i+1} = k(u_{i+1} - u_i)$

LMB: $\{\sum \vec{F}^{ext} = m\vec{a}\} \cdot \hat{i} \Rightarrow \cancel{m\ddot{u}_i =}$

$$\Rightarrow m\ddot{u}_i = k\left[(u_{i+1} - u_i) - (u_i - u_{i-1})\right]$$

$$m\ddot{u}_i = k\left[(u_{i+1}-u_i)-(u_i-u_{i+1})\right]$$

$$= k\left[\Delta u(x+\Delta x) - \Delta u(x)\right]$$

$$= k\Delta(\Delta u(x))$$

Let $u_{i+1} = u(x+\Delta x)$
$u_i = u(x)$
$u_{i-1} = u(x-\Delta x)$

Dividing by Δx twice gives

$$\frac{m\,\ddot{u}_i}{\Delta x\,\Delta x} = k\,\frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta x}\right)$$

Letting $\Delta x \longrightarrow 0$ (sloppy calculus), we have

$$\frac{m}{(\Delta x)^2}\frac{d^2u}{dt^2} = k\,\frac{\partial^2 u}{\partial x^2}$$

$$\left(\frac{m}{\Delta x}\right)\frac{d^2u}{dt^2} = k\Delta x \cdot \frac{\partial^2 u}{\partial x^2}$$

Define $\gamma \equiv \frac{m}{\Delta x}$: linear mass density

and note that $k = \frac{F}{\text{displacement}}$, so

$$\gamma\,\ddot{u} = \frac{F}{\text{disp}}\,\Delta x\,\frac{\partial^2 u}{\partial x^2} = \frac{F}{\left(\frac{\text{disp}}{\Delta x}\right)}\frac{\partial^2 u}{\partial x^2} = \frac{F}{\text{strain}}\,\frac{\partial^2 u}{\partial x^2}$$

$[Pa]$
$\uparrow$

Let $\gamma = \rho A$ and $\frac{F}{\text{strain}} = E\cdot A$. Then $[E] = \left[\frac{F}{A}\right] = \left[\frac{N}{m^2}\right]$

$$\rho \cancel{A}\,\ddot{u} = E\,\cancel{A}\,\frac{\partial^2 u}{\partial x^2}$$

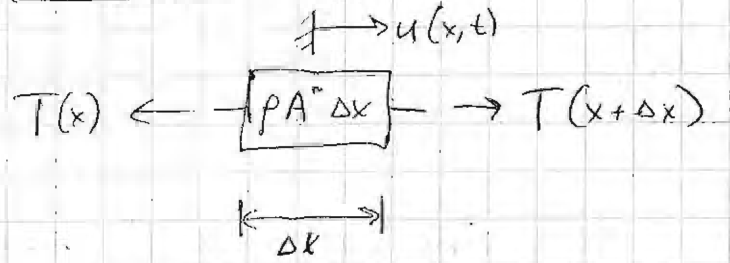
$$\ddot{u} = \frac{E}{\rho}\,\frac{\partial^2 u}{\partial x^2}$$

Define $c^2 \equiv \frac{E}{\rho}$, then

$$\boxed{\ddot{u} = c^2\,\frac{\partial^2 u}{\partial x^2}}$$

1-D bar waves
longitudinal

Derivation $\boxed{2}$



$m = \rho A$

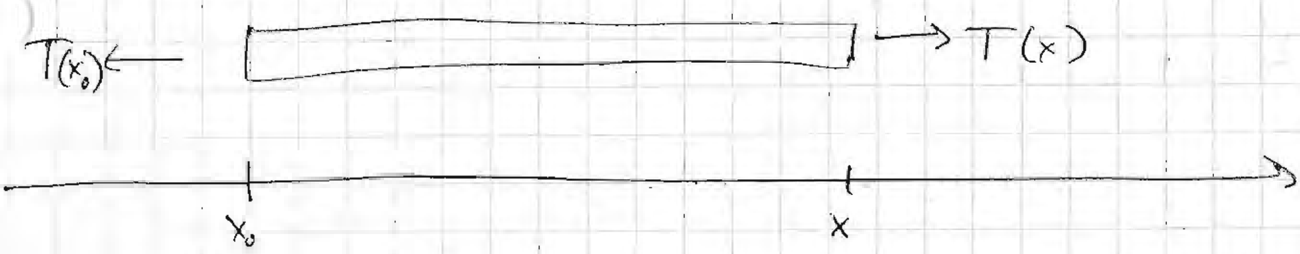
LMB: $\Sigma F = ma \quad : \quad T(x+\Delta x) - T(x) = \rho A \ddot{u}$

$$\cancel{TLT} \quad \Delta x \, \frac{\partial T}{\partial x} = \rho A \Delta x \, \ddot{u}$$

Let $\quad T = A \epsilon E \quad , \quad \epsilon = \frac{\partial u}{\partial x}$

$$\boxed{u_{xx} = \frac{\rho}{E} u_{tt}}$$

Derivation $\boxed{3}$



<u>F = ma</u> for a finite section:

$$\left\{ T(x) - T(x_0) = \int_{x_0}^{x} \rho A \ddot{u} \, x' dx' \right\}$$

$$\frac{d}{dx}\{\text{LMB}\} \Rightarrow \frac{\partial T}{\partial x} = \frac{d}{dx} \int_{x_0}^{x} \rho A \ddot{u} \, x' dx \qquad \cancel{\text{FTC}}$$

$$\text{FTC} \Rightarrow \frac{\partial T}{\partial x} = \rho A \ddot{u} \Big|_{x_0}^{x}$$

Define $T \equiv \sigma A$

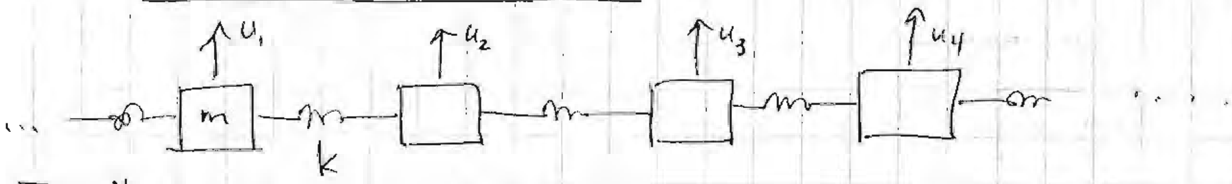
$\sigma \equiv E \epsilon$

$\epsilon \equiv \frac{du}{dx}$

Then $\frac{\partial}{\partial x}\left(AE \frac{du}{dx} \right) = \rho A \ddot{u}(x)$

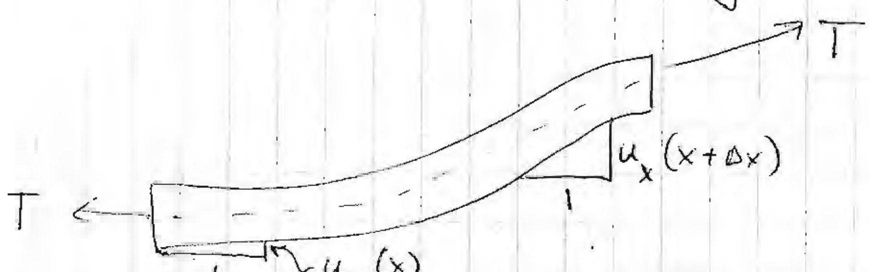
$$\cancel{A}E \frac{\partial^2 u}{\partial x^2} = \rho A \ddot{u} \quad \Rightarrow \quad \boxed{u_{xx} = \frac{\rho}{E} u_{tt}}$$

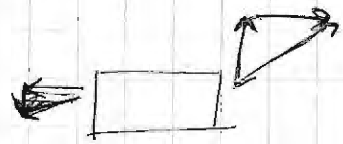
Transverse Waves



In the ~~For~~ linear vibration regime, ALL SPRING FORCES ARE HORIZONTAL. To first order, then, there is <u>no</u> vibration in the vertical direction. $\tilde{K}$ has eigenvectors with ZERO eigenvalues for any vertical motion.

That's because our treatment so far has left out ~~depended on~~ PRE-STRESS. We need to add a vertical tension term to get a restoring force and hence vibrations.





$$F_y = m a_y \implies T\, u_{xx}(\Delta x) = \rho A \Delta x\, u_{tt}$$

$$\implies \boxed{u_{tt} = \frac{T}{\rho A} u_{xx}} \qquad c^2 \equiv \frac{T}{\rho A}$$

Solution of the Wave Equation : $\boxed{u_{tt} = c^2 u_{xx}}$

Guess: $\boxed{X(x)\, T(t) = u(x,t)}$ "separation of variables"

Then $u_{tt} = X(x)\ddot{T}(t)$ and $u_{xx} = X''(x)\, T(t)$

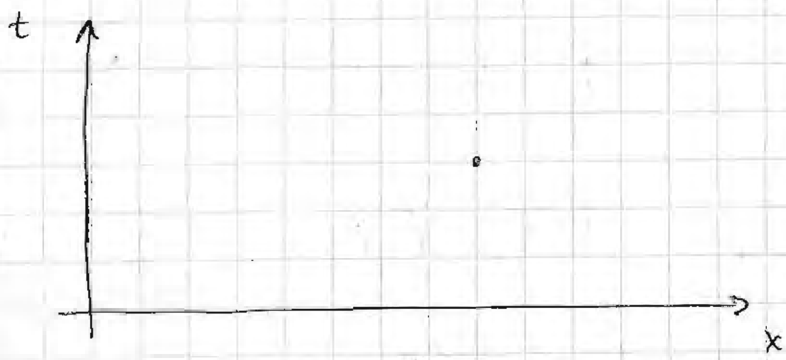
So $u_{tt} = c^2 u_{xx} \implies X(x)\ddot{T}(t) = c^2 X''(x) T(t)$

$$\implies \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} \quad \forall x, t$$

So we have $f(x) = \frac{1}{c^2} g(t)$

But this can only be true if both $f(x) = \text{CONST}/_x$

and $g(t) = \text{CONST}/_t$



Wave Eqn. $\boxed{\ddot{u} = \frac{T}{\rho A} u''}$ " Stiffness from pre-stress "

or Torsion Eqn: $\boxed{\ddot{\theta} = \frac{G}{\rho} \theta''}$

Compare to $\boxed{M\ddot{\bar{x}} = -K\bar{x}}$ ...

(mass)(accel) $\propto$ −(displacement from relaxed shape)

To solve the wave equation, we look for modal solutions in the continuum case.

Assume our solution is separable:

$$u(x,t) = X(x)\, T(t)$$

Then $u_{xx} = \frac{1}{c^2} u_{tt} \iff T(t) X''(x) = \frac{1}{c^2} X(x) \ddot{T}(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{\ddot{T}(t)}{T(t)}$$

$$f(x) = \frac{1}{c^2} g(t) \Rightarrow \begin{array}{l} f(x) = \text{CONST} \\ g(t) = \frac{1}{c^2} \cdot \text{CONST} \end{array} \quad \forall x,t \, !$$

Call $\frac{X''}{X} = -\lambda^2$, $\frac{\ddot{T}}{T} = -\lambda^2 c^2$

Then the wave equ. becomes

$$X'' + \lambda^2 X = 0 \quad , \quad \ddot{T} + (\lambda c)^2 T = 0$$

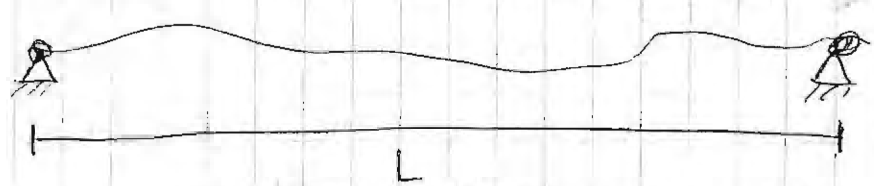
$$\Rightarrow \quad X(x) = A_x \cos \lambda x + B_x \sin \lambda_x \quad , \quad T(t) = A_t \cos \lambda ct + B_t \sin \lambda ct$$

$$\Rightarrow u(x,t) = X(x)T(t) = \left(A_x \cos \lambda x + B_x \sin \lambda x\right)\left(A_t \cos \lambda ct + B_t \sin \lambda ct\right)$$

* Big $\lambda \Rightarrow$ small $\nu = \frac{1}{\lambda} \Rightarrow$ FAST OSCILLATIONS

Let's look at the CANONICAL EX:

$\boxed{\text{String pinned at two ends}}$



In the general (infinite L) case we have an infinite number of particular solutions; in fact, an uncountably infinite #. We tame that infinity by our boundary conditions:

$$u(0,t) = 0 \quad \Rightarrow \quad C = 0$$

$$u(L,t) = 0 \quad \Rightarrow \quad D = 0 \text{ (boring) or } \sin \lambda L = 0$$

$$\Rightarrow \lambda L = n\pi, \quad n = 1,2,3,\ldots$$

$$\lambda = \frac{n\pi}{L}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(A \cos\left(\frac{n\pi ct}{L}\right) + B \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

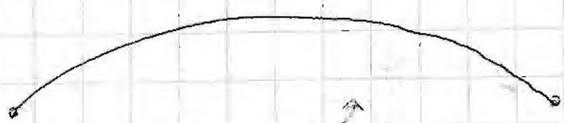
$$\text{So, } u(x,t) = \sum_{n=1}^{\infty} \underbrace{\left(A\cos\left(\frac{n\pi c}{L}t\right) + B\sin\left(\frac{n\pi c}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)}_{\text{"eigen-vectors"}}$$

wave numbers $\frac{n\pi}{L} = \lambda_n$

$\Updownarrow$

frequencies $\omega_n = \frac{n\pi c}{L} = c\lambda_n$

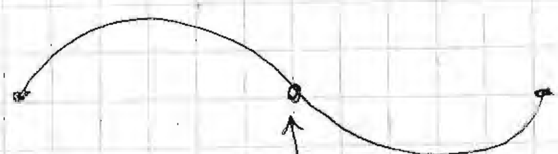
<u>1^{st} mode</u>, $n=1$:
"1^{st} harmonic"



"the fundamental"

$\omega_1 = \frac{\pi c}{L}$

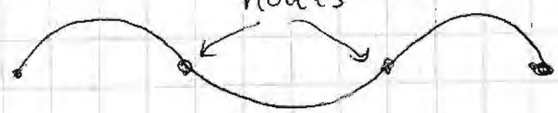
<u>2^{nd} mode</u>, $n=2$:
"octave", "2^{nd} harmonic"



$\omega_2 = \frac{2\pi c}{L} = 2\omega_1$

"nodes" = zeros at the modes/"eigenvectors"

<u>3^{rd} mode</u>, $n=3$
"3^{rd} harmonic"



$\omega_3 = \frac{3\pi c}{L} = 3\omega_1$

In music, every note with an integer multiple frequency of C is ALSO C. A piano keyboard spans 7 octaves, so high C has a frequency

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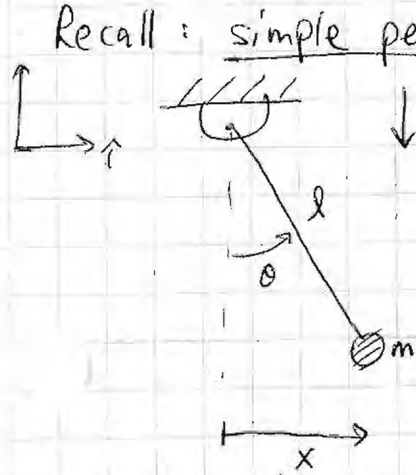
$\boxed{11/26}$ <u>Today: string vibes & energy</u>

Recall: <u>simple pendulum</u>



$\downarrow g$

$E_p = 0$
"datum"

<u>FBD</u>

<u>Energy</u>

$$E_k = \frac{1}{2}m(l\dot{\theta})^2$$

$$x = l\sin\theta \simeq l\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

To 2^{nd} order, $x = l\theta$

$$E_p = mgy = mg(-l\cos\theta) = -mgl\left(1 - \frac{\theta^2}{2} + \cdots\right)$$

$$E_p \approx -mgl\left(1 - \frac{\theta^2}{2}\right)$$

and $\theta \simeq \frac{x}{l}$

So, to 2^{nd} order,

$$E_p = -mgl + mg\frac{x^2}{l\,2}$$

To 2^{nd} order, we have our energy expressions:

$$E_k = \frac{1}{2} m \dot{x}^2 \qquad \dot{\theta} = \frac{\dot{x}}{\ell}, \quad (\ell\dot{\theta})^2 = \dot{x}^2$$

$$E_p = -mg\ell + \frac{mgx^2}{2\ell}$$

$$\mathcal{L} = T - V = E_k - E_p = \frac{1}{2} m\dot{x}^2 - mg\ell + \frac{mgx^2}{2\ell}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \Rightarrow \cdots$$

$$\Rightarrow \boxed{\begin{array}{l} \ddot{x} + \frac{g}{\ell} x = 0 \\ \text{OR } \ddot{\theta} + \frac{g}{\ell}\theta = 0 \end{array}}$$

NB: This ~~works~~ derivation is equally valid using x or θ as

1) our minimal coordinate, since — to 2^{nd} order — they are proportional.

2) Need to calculate E_k & E_p to 2^{nd} order

3) In linearized ODEs & sol'ns, $y \equiv 0$ (to 1^{st} order), yet we needed $y \neq 0$ to get our E_p expression.

4) There's no length scale in our ODEs. So we can't talk meaningfully about "large" or "small" oscillations

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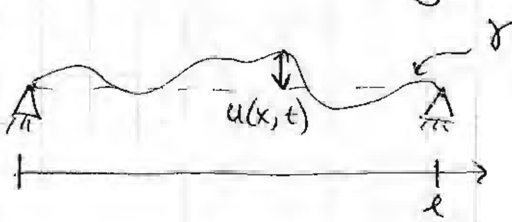
Recall: Vibrating Spring (it vibrates because of the tension in the string, not the

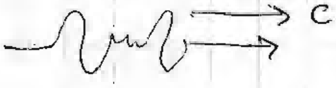


Wave eqn: $u_{tt} = c^2 u_{xx}$

Aside.

D'Alambert's soln: $u(x,t) = \begin{cases} f(x-ct) \\ \text{or } g(x+ct) \end{cases}$ for any functions f and g

Take any shape you want and it propagates down the length."



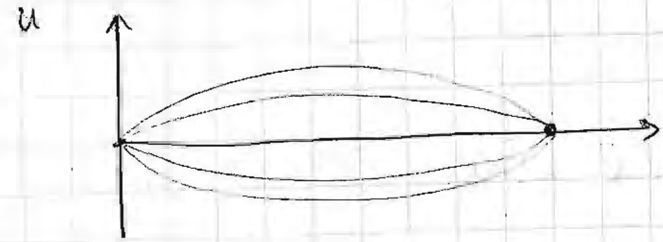
"Traveling Wave"

Our separable solution $u(x,t) = X(x)T(t)$ is a "Standing Wave" sol'n: the same shape staying in one place but oscillating up & down. (As opposed to D'Alambert's traveling wave sol'n.)

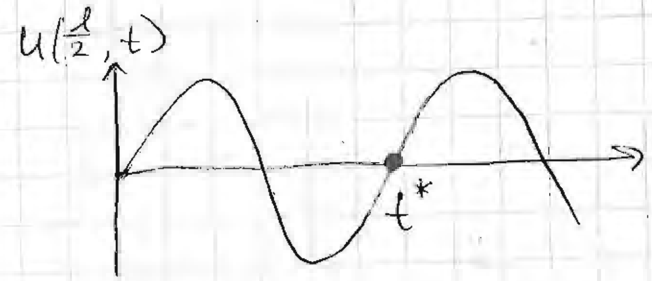
$$u(x,t) = X(x)T(t) \quad , \quad BC's \quad u(0,t) = u(l,t) = 0$$

$$\Rightarrow \quad u = \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \sin\left(\frac{n\pi x}{l}\right)$$

$$\omega_n \equiv \frac{n\pi c}{l}$$



$n=1$: "Fundamental"
"lowest mode"
"1^{ST} (0^{th}?) harmonic"



$$\omega_n t^* = 2\pi \quad \forall n$$

$$t^* = \frac{2\pi}{\omega_n} = \frac{2l}{L}$$

New idea: Use energy to find $\dot{E}_T = 0$ (or $\mathcal{L}$.)

Fundamental vibration frequency. (w/o our ODE approach)

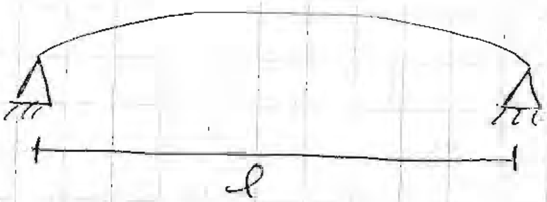
Method:
1. Guess mode shape
2. Calculate E_k & E_p
3. Use $\mathcal{L}$ or $\dot{E}_T = 0$

1. Guess mode shape: treat as 1 DoF system

Guess $u(x,t) = f(x)\,g(t)$

with an $f(x)$ that meets our BC's and gives rise to an ODE for $g(t)$.

Guess



Parabola, $f(x) = x(\ell - x)$

* This is a wrong, but still interesting guess.

2. Calculate E_k & E_p for our guess

K.E.

$$u(x,t) = B\,f(x)\,g(t) \Rightarrow \dot{u} = Bf\dot{g}$$

$E_k = \frac{1}{2}\int v^2 dm$

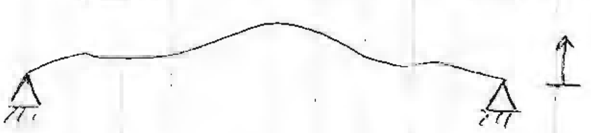
$$E_k = \frac{1}{2}\int_0^\ell \underbrace{\left(B\,f(x)\,\dot{g}(t)\right)^2}_{v}\underbrace{(\rho A\,dx)}_{dm}$$

$$= \frac{1}{2}\rho A B^2 \dot{g}(t)^2 \int_0^\ell f(x)^2 dx$$

$$= \frac{1}{2}\rho A B^2 \dot{g}(t)^2 \int_0^\ell x^2(\ell - x)^2 dx = \ldots$$

$$\boxed{E_k = \cancel{\tfrac{1}{2}}\rho A B^2 \dot{g}(t)^2 \frac{\ell^5}{60}}$$

PE (a) Assume inextensible string. Then...



$y = 0$ when string is straight (that's our datum)

$$y = s - s_0 = \underbrace{\int_0^\ell \sqrt{1 + (u')^2}\,dx}_{\text{arc length}} - \ell$$

$$E_p = mgy = \left[\int_0^\ell \left(1 + \frac{(u')^2}{2} + \ldots\right)dx - \ell\right]mg$$

$$= \frac{1}{2}mg\left[\int_0^\ell (u')^2 dx \cancel{\ell}\right] + O(x^3)$$

Good to 2^{nd} order

Alternately, (b) assume stretchy spring; calculate strain energy. $\boxed{\text{SAME ANS.}}$

$$E_p = \frac{T}{2} B^2 \int_0^{\ell} \left[\frac{d}{dx}\{f(x)\, g(t)\}\right]^2 dx$$

$$= \frac{TB^2 g(t)^2}{2} \int_0^{\ell} \left(\frac{d}{dx}\left[x(\ell - x)\right]\right)^2 dx = \ldots$$

$$= \frac{TB^2 g(t)^2 \ell^3}{6}$$

3. Use $\dot{E}_T = 0$ or $\mathcal{L}$ to find EoM

$$\dot{E}_T = \frac{d}{dt}(E_k + E_p) = 0$$

$$\frac{d}{dt}\left[\frac{\rho A B^2 \dot{g}^2 \ell^5}{60} + \frac{TB^2 g^2 \ell^3}{6}\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\rho A \ell^2 \dot{g}^2}{10} + T g^2\right] = 0$$

$$\Rightarrow \frac{\rho A \ell^2}{10} \frac{d}{dt} \dot{g}^2 + T \frac{d}{dt} g^2 = 0$$

$$\frac{\rho A \ell^2}{10}(2\dot{g}\ddot{g}) + T(2g\dot{g}) = 0$$

$\ldots$

$$\ddot{g} + \frac{T \cdot 10}{\rho A \ell^2} g = 0$$

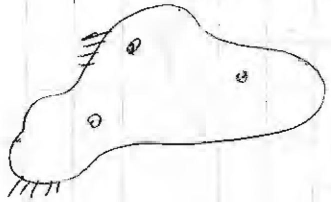
we guessed a parabolic mode shape, which is wrong. But...

Compare this to the actual solution, and the error is only

$$\frac{\sqrt{10}}{\pi} = 1.006 \ldots$$

Moral: If you guess a mode shape that is wrong but reasonable, you will always get an answer that's close to the real answer.

11/28 : (1) Raleigh Method (cont'd) to find freq. of lowest mode
(2) Vibration Absorber



Some structure, a mix of point masses & continuum

* Every elastic structure has normal modes (atoms, crystals, machines, ...)

Raleigh's Method for finding lowest mode frequency

1. Assume mode shape & that meets B.C.'s :

$$u(x,t) = f(x)\ g(t)$$

x := position and/or index (in discrete or mixed discrete/continuous prob's)

* Best if $f(x)$ is as smooth as possible

2. Calculate E_p & E_k
3. Write $\mathcal{L}$ & find EoM $\Rightarrow \ddot{g} + \omega^2 g = 0$

RESULT : $\omega \geq \omega_n$, the <u>actual</u> soln. of the system's ODE(s) or PDE(s)

experience : $\omega \approx \omega_n$ if your guess $u(x,t)$ was "good"

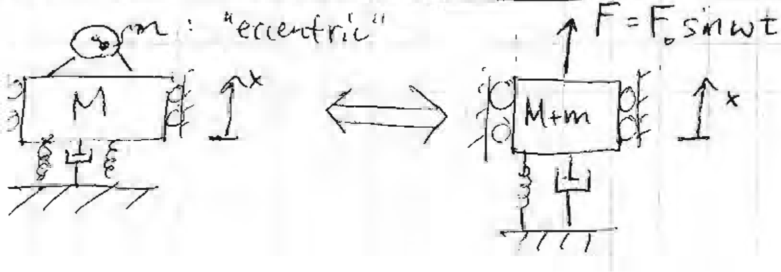
Aside :

$$\begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \text{forcing} \\ \dot{q}_1^2 \\ q \\ \vdots \end{bmatrix}$$

Using the Matlab solve.m requires inverting M, where $M^{-1} = \frac{1}{\det M} \mathrm{Cof}\, M$ which has $n!$ terms in $\det M$ for an $n \times n$ M.

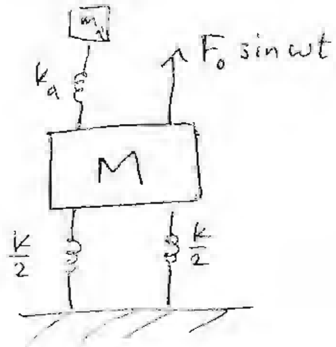
A better (more runtime efficient) command is jacobian.m

Vibration Absorbers (linked to the last HW problem)



FINAL EXAM : show the equivalence at these systems

TRICK : $\boxed{\text{VIBRATION ABSORBER}}$



If you add little mass m on spring w/ const. k_0 (m & k_0 chosen "well"), then the little mass will go nuts and the big mass will stay still.

To do this...

1. Look at steady state: $\begin{bmatrix} x \\ X \end{bmatrix} = \sin\theta \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ (undamped)

2. Pick m_a & k_a such that the position of mass M is constant. Which means the force on M from m exactly cancels the ~~forcing ten~~ driving force, i.e. $\sqrt{\frac{k_a}{m_a}} = \omega$ (from $F(t) = F_0 \sin\omega t$)