Prelim 1 2012

1

	KE (T)	PE (V)
Wheel I	$\frac{1}{2} I \dot{\theta}^2$	0
Torsional Spring	0	$\frac{1}{2} k_1 \theta^2$
Linear Spring	0	$\frac{1}{2} k_2 (\theta r - x)^2$
Mass M	$\frac{1}{2} M \dot{x}^2$	$\cancel{M g x \sin 45} \to 0$

The potential energy change of mass M due to gravity balances with the potential energy change due to static deflections at equilibrium, so $V_M = 0$.

Displacement of $k_2 = \theta \cdot r - x = \Delta X_{k2}$

y-displacement of $M = X \sin 45 = \Delta y_m$

$L = T - V$

$u_i = \{x, \theta\}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_i}\right) - \frac{\partial L}{\partial u_i} = 0$$

$$L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M \dot{x}^2 - \frac{1}{2} k_1 \theta^2 - \frac{1}{2} k_2 (\theta r - x)^2$$

1) $M\ddot{x} + k_2(\theta r - x)(-1) = 0$

2) $I\ddot{\theta} + k_1 \theta + k_2 (\theta r - x) r = 0$

$$\boxed{\begin{aligned} &1)\ M\ddot{x} - k_2(\theta r - x) = 0 \\ &2)\ I\ddot{\theta} + k_1\theta + k_2 r^2 \theta - k_2 r x = 0 \end{aligned}}$$

② $L = T - V$ $\qquad F = \begin{bmatrix} 0 \\ F\cos\Omega t \end{bmatrix}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = F_i$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \qquad = \frac{1}{2}\dot{x}_1^2 + \dot{x}_2^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 \quad = \frac{1}{2}x_1^2 + (x_2^2 - 2x_2x_1 + x_1^2)$$

$$= x_2^2 - 2x_2x_1 + \frac{3}{2}x_1^2$$

$$L = T - V = \frac{1}{2}\dot{x}_1^2 + \dot{x}_2^2 - x_2^2 + 2x_1x_2 - \frac{3}{2}x_1^2$$

1) $\ddot{x}_1 - 2x_2 + 3x_1 = 0$

2) $2\ddot{x}_2 + 2x_2 - 2x_1 = F\cos\Omega t$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{M}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}}_{K}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ F\cos\Omega t \end{bmatrix}}_{F}$$

Take solution of form (at steady state):

$x_1 = AF\cos\Omega t \qquad x_2 = BF\cos\Omega t \qquad$ Want to minimize $|x_2|$

1) $-A\Omega^2 - 2B + 3A = 0 \quad \longrightarrow \quad A(3 - \Omega^2) = 2B$

2) $-2B\Omega^2 - 2A + 2B = 1 \quad \longrightarrow 2B(1 - \Omega^2) - 2A = 1$

$$2B(1 - \Omega^2) - \frac{4B}{3 - \Omega^2} = 1$$

$$B\left(2(1 - \Omega^2) - \frac{4}{3 - \Omega^2}\right) = 1 \quad \longrightarrow$$

2 cont'd

$$B = \frac{3-\Omega^2}{(2)(1-\Omega^2)(3-\Omega^2)-4}$$

Min $|X_2|$ when $B \to 0$

$B \to 0$ when $(3-\Omega^2) \to 0$

$\Omega^2 = 3$

$$B_{\Omega^2=3} = \frac{0}{-4} = 0$$

$\boxed{\Omega = \sqrt{3} \text{ gives smallest steady state response of } X_2}$

Alternate Solution:

Assume smallest amplitude of X_2 is when X_2 is not moving

In this mode of vibration, would be driving the simplified system at its natural frequency, as motion of m_1 will be $180°$ out of phase with the forcing.

Simplified system:

$$W_n = \Omega = \sqrt{\frac{K_{eff}}{m}}$$

$K_{eff} = 2+1$

$$\boxed{\Omega = \sqrt{3}}$$

③ (a) $\ddot{x} + 2n\dot{x} + x = 0$ Derivation of coefficients if you didn't know them!

General form (not knowing coefficients B or ω_D)

$$x(t) = Ae^{-Bt}\cos(\omega_D t + \phi)$$

$$-ABe^{-Bt}\cos(\omega_D t + \phi) - A\omega_D e^{-Bt}\sin(\omega_D t + \phi) = \dot{x}$$

$$+AB^2 e^{-Bt}\cos(\omega_D t + \phi) + AB\omega_D e^{-Bt}\sin(\omega_D t + \phi)$$
$$+ AB\omega_D e^{-Bt}\sin(\omega_D t + \phi)$$
$$- A\omega_D^2 e^{-Bt}\cos(\omega_D t + \phi) = \ddot{x}$$

To save space, $S = \sin(\omega_D t + \phi) \cdot e^{-Bt}$
$C = \cos(\omega_D t + \phi) \cdot e^{-Bt}$

Plug into Diff Eqn

$$AB^2C + 2AB\omega_D S - A\omega_D^2 C + 2n(-ABC - A\omega_D S) + AC = 0$$

$$C\left(AB^2 - A\omega_D^2 - 2nAB + A\right) = 0$$

$$CA(B^2 - \omega_D^2 - 2nB + 1) = 0 \Rightarrow B^2 - \omega_D^2 - 2nB + 1 = 0$$

$A \neq 0$, uninteresting solution

$$S(2AB\omega_D - A\omega_D \cdot 2n) = 0 \Rightarrow \cancel{2}B\cancel{\omega_D} - \cancel{2}n\cancel{\omega_D} = 0 \Rightarrow B = n$$

$$n^2 - \omega_D^2 - 2n^2 + 1 = 0$$
$$\omega_D = \sqrt{1-n^2} \Rightarrow x(t) = A \cdot e^{-nt}\cos(\sqrt{1-n^2}\,t + \phi)$$

3 (a) $\ddot{x} + 2n\dot{x} + x = 0$

General form for damped harmonic motion:

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x = Ae^{-\xi\omega_n t}\cos\left(\sqrt{1-\xi^2}\,\omega_n t + \phi\right)$$

$$\xi\omega_n = n \quad \omega_n^2 = 1 \Rightarrow \xi = n$$

$$x(t) = Ae^{-nt}\cos\left(\sqrt{1-n^2}\,t + \phi\right)$$ ← Same as obtained if you didn't know general form coeffs

$$x(t=0) = 0$$

$$A\cancel{e^0}\cos\phi = 0$$

$$\cos\phi = 0 \quad \phi = \frac{\pi}{2}$$

$$\boxed{x(t) = Ae^{-nt}\cos\left(\sqrt{1-n^2}\,t + \frac{\pi}{2}\right)}$$

(b) $\frac{x_1}{x_3} = 2$ ← over two periods

logarithmic decrement

$$\frac{x_1}{x_3} = \frac{e^{-n\cdot t_0}}{e^{-n\cdot(t_0+2T)}} \qquad T = \frac{2\pi}{\omega_d} \qquad \omega_d = \sqrt{1-n^2}$$

$$2 = e^{-n\cdot t_0 + n\cdot\left(t_0 + \frac{4\pi}{\sqrt{1-n^2}}\right)}$$

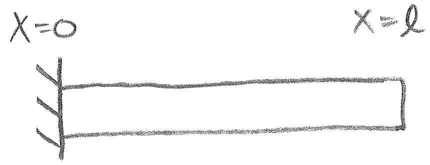
$$\ln(2) = \frac{n\cdot 4\pi}{\sqrt{1-n^2}} \qquad n^2\cdot\frac{16\pi^2}{(\ln(2))^2} - 1 + n^2 = 0$$

$$n^2\left(\frac{16\pi^2}{(\ln 2)^2} + 1\right) = 1$$

$$\boxed{n = \sqrt{\frac{1}{\frac{16\pi^2}{(\ln 2)^2} + 1}}}$$

MAE4770 Prelim 2 Solution

1



BC: longitudinal vibrations of a bar $\quad u_{tt} = c^2 u_{xx}$ where $c^2 = \frac{EA}{\rho}$

1) $u_x \big|_{x=l} = 0$ [free] $\qquad$ where $\rho = \frac{\text{mass}}{\text{length}}$

2) $u \big|_{x=0} = 0$ [clamped]

Assume a solution of the form

$u(x,t) = U_n(x) \cos(\omega_n t)$ $\qquad$ plug into $u_{tt} = c^2 u_{xx}$

$$-\omega_n^2 U_n(x) \cancel{\cos(\omega_n t)} = c^2 U_n''(x) \cancel{\cos(\omega_n t)}$$

$$U_n''(x) + \frac{\omega_n^2}{c^2} U_n(x) = 0$$

$$U_n(x) = a\cos\left(\frac{\omega_n}{c} x\right) + b\sin\left(\frac{\omega_n}{c} x\right)$$

To satisfy BC, $U_n(x)$ must satisfy BC's as BC's are true forall t

BC 1) $-a\frac{\omega_n}{c}\sin\left(\frac{\omega_n}{c} x\right) + b\frac{\omega_n}{c}\cos\left(\frac{\omega_n}{c} x\right) = 0$ $\quad$ at $x = l$

$$\rightarrow \quad -a\sin\left(\frac{\omega_n}{c} l\right) + b\cos\left(\frac{\omega_n}{c} l\right) = 0$$

BC 2) $a\,\underbrace{\cos(0)}_{1} + b\,\underbrace{\sin(0)}_{0} = 0$

$$a = 0$$

$\longrightarrow \quad b\cos\left(\frac{\omega_n}{c} l\right) = 0$ $\qquad$ $b \neq 0$ (otherwise trivial soln)

$\frac{\omega_n}{c} l$ must be $\frac{\pi}{2}, \frac{3\pi}{2}$, etc $\qquad \frac{\omega_n}{c} l = \frac{(2n+1)}{2}\pi$

$n = 0, 1, 2, \ldots$

1 continued

$$\omega_n = \frac{(2n+1)}{2}\pi\frac{c}{l}$$

$$\boxed{\omega_n = \frac{(2n+1)}{2}\frac{\pi}{l}\sqrt{\frac{EA}{\rho_{length}}} \qquad n=0,1,2\ldots}$$

(2) $$Q = \frac{EA}{\rho_{vol}A}\frac{\int_0^l V'(x)^2\,dx}{\int_0^l V(x)^2\,dx}$$ where $\rho_{vol} = \frac{\text{mass}}{\text{volume}}$

since A is constant (Note $\rho_{vol} = \frac{\rho_{length}}{A}$)

$$V(x) = a + bx + cx^2$$

BC's: longitudinal vibrations of a bar

$$V_x\Big|_l = 0 \rightarrow b + 2cx = 0 \rightarrow b + 2cl = 0 \rightarrow b = -2cl$$

$$V\Big|_0 = 0 \rightarrow a = 0$$

$$V(x) = c(-2lx + x^2)$$

$$V'(x) = c(-2l + 2x)$$

[Note: can take $c=1$ as it cancels out in Q anyway]

Num: $$\int_0^l V'(x)^2 dx = c^2\int_0^l 4x^2 - 8lx + 4l^2\,dx$$

$$= c^2\left(\frac{4}{3}x^3 - 4lx^2 + 4l^2x\right)\Big|_0^l$$

$$= c^2\left(\frac{4}{3}l^3 - 4l^3 + 4l^3\right) = c^2\left(\frac{4}{3}l^3\right)$$

2 continued

den: $\int_0^l V(x)^2 dx = c^2 \int_0^l x^4 - 4lx^3 + 4l^2x^2 \, dx$

$$= c^2 \left(\frac{x^5}{5} - l x^4 + \frac{4}{3} l^2 x^3 \right) \Big|_0^l$$

$$= c^2 l^5 \left(\frac{1}{5} - 1 + \frac{4}{3} \right)$$

$$\left(\frac{1}{5} + \frac{1}{3} \right)$$

$$= c^2 l^5 \left(\frac{8}{15} \right)$$

$$Q = \frac{E}{\rho_{vol}} \frac{\cancel{c^2} \frac{4}{3} l^3}{\cancel{c^2} \frac{8}{15} l^5} = \frac{5}{2} \frac{E}{l^2 \rho_{vol}}$$

$\frac{4}{3} \cdot \frac{15}{8} = \frac{5}{2}$

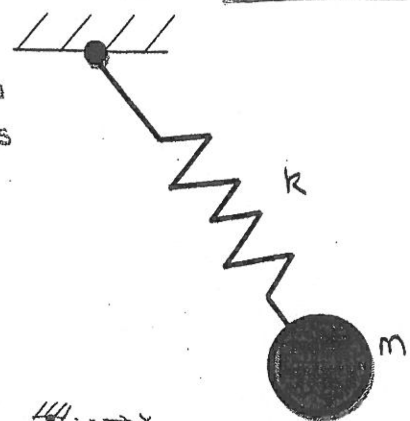
$$w_1 \leq \sqrt{Q}$$

$$\boxed{w_1 \leq \sqrt{\frac{5}{2} \frac{E}{\rho_{vol} \, l^2}}}$$

3

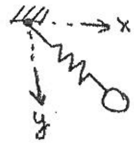
MAE 4770 - Mid Term 2 PROBLEM 3

★ "derive the model"
find the equations of motion, clearly label/explain all variables & coordinates



l_0 = unstretched length

• Cartesian Coordinates



$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$

$U = -mgy + \frac{1}{2} k \left(\sqrt{x^2+y^2} - l_0 \right)^2$

$L = T - U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + mgy - \frac{1}{2} k \left(\sqrt{x^2+y^2} - l_0 \right)^2$

$\frac{d}{dt} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$ & $\frac{d}{dt} \frac{\delta L}{\delta \dot{y}} - \frac{\delta L}{\delta y} = 0$

$\frac{\delta L}{\delta \dot{x}} = m\dot{x}$ $\qquad$ $\frac{\delta L}{\delta \dot{y}} = m\dot{y}$

$\frac{d}{dt} \frac{\delta L}{\delta \dot{x}} = m\ddot{x}$ $\qquad$ $\frac{d}{dt} \frac{\delta L}{\delta \dot{y}} = m\ddot{y}$

$\frac{\delta L}{\delta x} = -kx + \frac{k l_0 x}{\sqrt{x^2+y^2}}$ $\qquad$ $\frac{\delta L}{\delta y} = mg - ky + \frac{k l_0 y}{\sqrt{x^2+y^2}}$

so equations of motion in Cartesian coordinates are

$m\ddot{x} + kx - \frac{k l_0 x}{\sqrt{x^2+y^2}} = 0$ & $m\ddot{y} - mg + ky - \frac{k l_0 y}{\sqrt{x^2+y^2}}$

[redacted]

• Polar Coordinates  $x = l\sin\theta$ $\quad y = l\cos\theta$ $\quad \dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta$ $\quad \dot{y} = \dot{l}\cos\theta - l\dot{\theta}\sin\theta$

convert T & U from cartesian to polar coordinates:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$= \frac{1}{2}m\left[(\dot{l}\sin\theta + l\dot{\theta}\cos\theta)^2 + (\dot{l}\cos\theta - l\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{l}^2\sin^2\theta + 2\dot{l}l\dot{\theta}\cos\theta\sin\theta + l^2\dot{\theta}^2\cos^2\theta + \dot{l}^2\cos^2\theta - 2\dot{l}l\dot{\theta}\cos\theta\sin\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

$$T = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = -mgy + \frac{1}{2}k\left(\sqrt{x^2+y^2} - l_0\right)^2$$

$$= -mgl\cos\theta + \frac{1}{2}k\left(\sqrt{l^2\sin^2\theta + l^2\cos^2\theta} - l_0\right)^2$$

$$U = -mgl\cos\theta + \frac{1}{2}k(l - l_0)^2$$

$$L = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta - \frac{1}{2}k(l-l_0)^2$$

$$\frac{d}{dt}\frac{\delta L}{\delta\dot{\theta}} - \frac{\delta L}{\delta\theta} = 0 \quad \& \quad \frac{d}{dt}\frac{\delta L}{\delta\dot{l}} - \frac{\delta L}{\delta l} = 0$$

$$\frac{\delta L}{\delta\dot{\theta}} = ml^2\dot{\theta} \qquad \frac{\delta L}{\delta\dot{l}} = m\dot{l}$$

$$\frac{d}{dt}\frac{\delta L}{\delta\dot{\theta}} = ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} \qquad \frac{d}{dt}\frac{\delta L}{\delta\dot{l}} = m\ddot{l}$$

$$\frac{\delta L}{\delta\theta} = -mgl\sin\theta \qquad \frac{\delta L}{\delta l} = ml\dot{\theta}^2 + mg\cos\theta - k(l-l_0)$$

so equations of motion in polar coordinates are

$$l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\sin\theta = 0 \quad \& \quad m\ddot{l} - ml\dot{\theta}^2 + k(l-l_0) - mg\cos\theta = 0$$