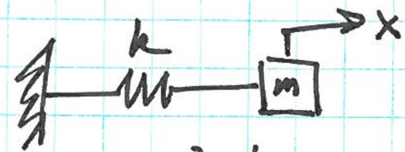
The Geometry of 2 DOF Systems by R. Rand

Start with 1 DOF system

$$\ddot{x} + \omega^2 x = 0$$



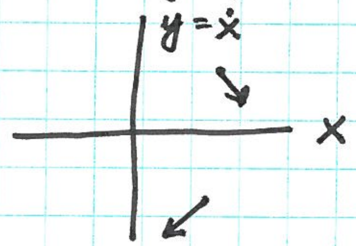
$$\omega^2 = \frac{k}{m}$$

Define $y = \dot{x} \Rightarrow \dot{y} = \ddot{x} = -\omega^2 x$

So we have a system of two first order ODEs:

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 x$$

View this in the x-y plane ("the phase plane")



At each point (x, y) there is a vector $(\dot{x}, \dot{y}) = (y, -\omega^2 x)$

"a vector field"

The motion flows along curves in the x-y plane which must satisfy the CONSERVATION OF ENERGY:

$$T + V = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 = h = \text{total energy}$$

Note: 1) h is determined by the initial conditions

2) $T+V=h$ follows from multiplying by $\dot{x}$:

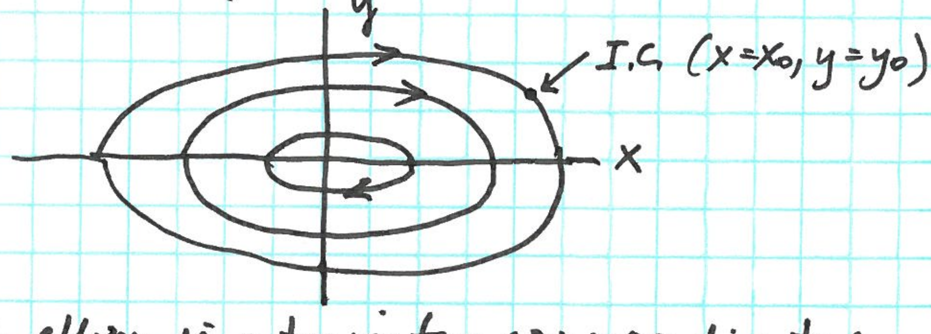
$$\dot{x}\left(\ddot{x} + \omega^2 x\right) = 0$$

$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) + \frac{d}{dt}\frac{\omega^2 x^2}{2} = 0$$

$$\frac{d}{dt}(T+V) = 0 \Rightarrow T+V = h$$

$$T+V=h \Rightarrow \underbrace{\frac{1}{2}y^2+\frac{1}{2}\omega^2x^2=h}_{\text{an ellipse}}$$

The x-y plane is filled with ellipses:



Each ellipse is a trajectory corresponding to a specific value of h.

There is particular ellipse and a specific value of h corresponding to each initial condition.

We may also think of $y(x)$ instead of $x(t)$ and $y(t)$. These are related by the chain rule:

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

Here $\frac{dy}{dt}=-\omega^2x$ and $\frac{dx}{dt}=y$, so we have

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{-\omega^2x}{y}$$

Separating variables,

$$y\,dy=-\omega^2x\,dx$$

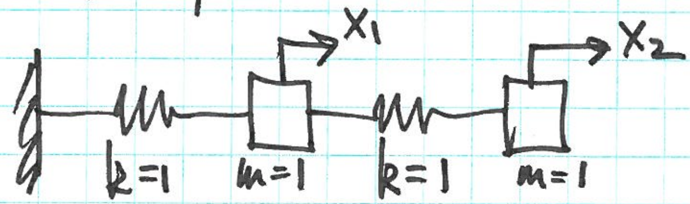
Integrating both sides,

$$\frac{y^2}{2}=-\omega^2\frac{x^2}{2}+\overset{h}{\overset{\shortparallel}{\text{constant}}}$$

$$\Rightarrow \frac{y^2}{2}+\frac{\omega^2x^2}{2}=h \qquad \text{(once again)}$$

Now let us go on to think about the geometry of phase space for a 2 DOF system

To make things easier to see, let's think about a specific example:



To begin with, let's find the eqs. of motion and the principal coordinates:

$$T = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2, \quad V = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2$$

Lagrange's equations:

$$\ddot{x}_1 + 2x_1 - x_2 = 0$$

$$\ddot{x}_2 - x_1 + x_2 = 0$$

Ansatz: $x_1 = X_1 \cos\omega t, \quad x_2 = X_2 \cos\omega t$

$$-\omega^2 X_1 + 2X_1 - X_2 = 0$$

$$-\omega^2 X_2 - X_1 + X_2 = 0$$

$$\begin{bmatrix} -\omega^2 + 2 & -1 \\ -1 & -\omega^2 + 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution, det $= 0 \Rightarrow$

$$(-\omega^2 + 2)(-\omega^2 + 1) - 1 = 0$$

$$\omega^4 - 3\omega^2 + 1 = 0 \Rightarrow \omega^2 = \frac{3 \pm \sqrt{5}}{2}$$

$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}} = .618, \quad \omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}} = 1.618$$

Modal vectors:

$$(-\omega^2+2)\,X_1 - X_2 = 0 \Rightarrow X_2 = (-\omega^2+2)\,X_1$$

$$\omega_1 = .618 \Rightarrow {}_1X = \begin{pmatrix} 1 \\ 1.618 \end{pmatrix} \quad \text{in-phase mode}$$

$$\omega_2 = 1.618 \Rightarrow {}_2X = \begin{pmatrix} 1 \\ -.618 \end{pmatrix} \quad \text{out-of-phase mode}$$

Transform to principal coordinates p_1, p_2:

$$\text{Define } R = ({}_1X, {}_2X) = \begin{pmatrix} 1 & 1 \\ 1.618 & -.618 \end{pmatrix}$$

Set

$$x = Rp, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

i.e.

$$x_1 = p_1 + p_2$$
$$x_2 = 1.618\,p_1 - .618\,p_2$$

We have

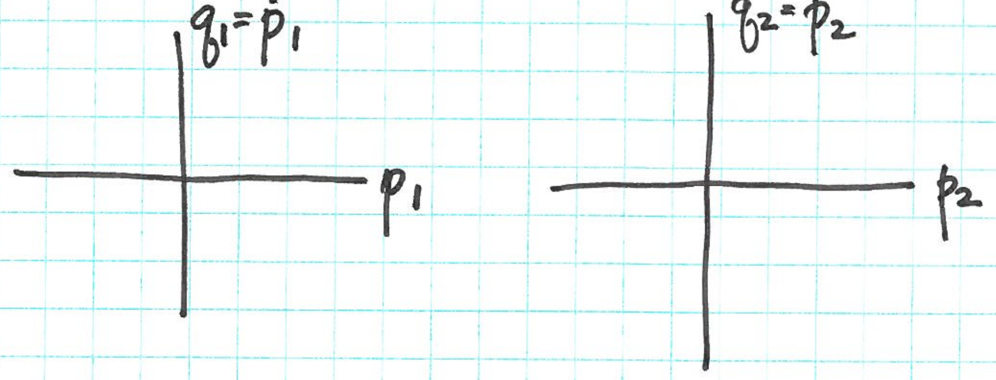
$$M\ddot{x} + Kx = 0$$
$$MR\ddot{p} + KRp = 0$$
$$\underbrace{R^t M R}_{D_1}\,\ddot{p} + \underbrace{R^t K R}_{D_2}\,p = 0$$

gives, as usual,

$$\ddot{p}_1 + \omega_1^2 p_1 = 0$$
$$\ddot{p}_2 + \omega_2^2 p_2 = 0$$

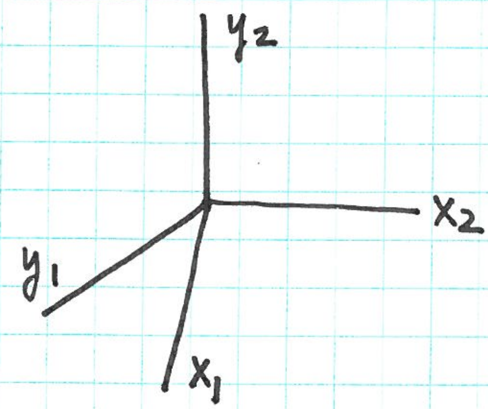
Let $q_1 = \dot{p}_1$, $q_2 = \dot{p}_2$

Then each of the principal modes lives in a 1 DOF p_i-q_i phase space:



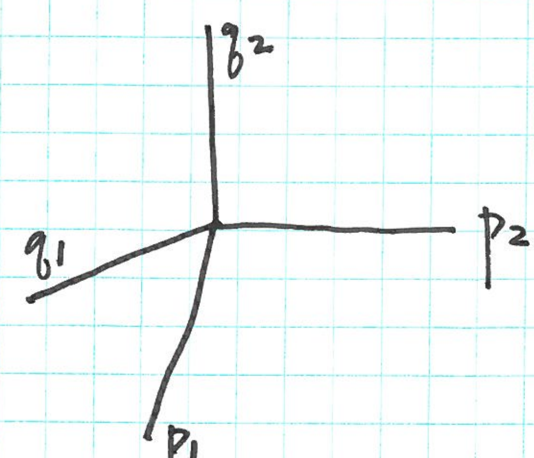
The phase space for the original 2 DOF system can be thought of in two ways: In physical coords,

$$\dot{x}_1 = y_1$$
$$\dot{y}_1 = -2x_1 + x_2$$
$$\dot{x}_2 = y_2$$
$$\dot{y}_2 = x_1 - x_2$$

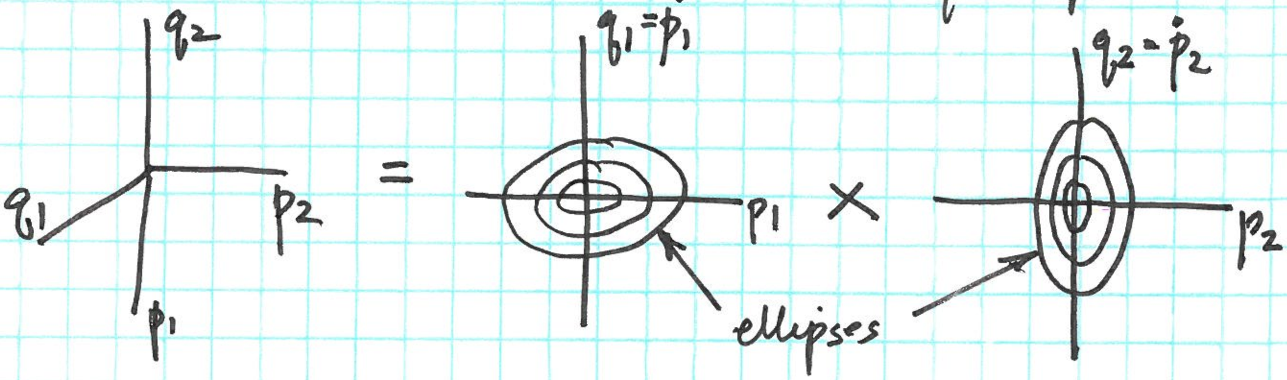


or in modal coordinates,

$$\dot{p}_1 = q_1$$
$$\dot{q}_1 = -\omega_1^2 p_1$$
$$\dot{p}_2 = q_2$$
$$\dot{q}_2 = -\omega_2^2 p_2$$

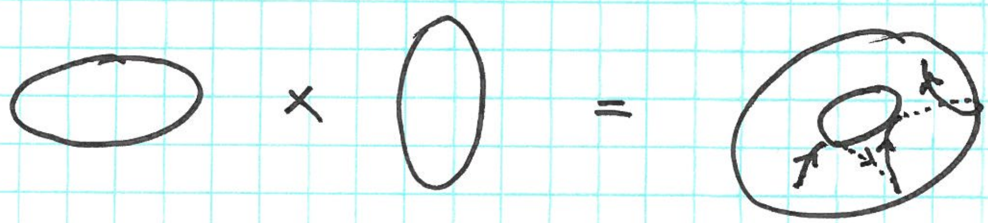


The system in modal coordinates is really two 1 DOF phase planes

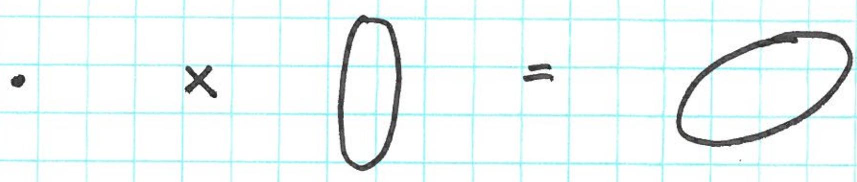


Initial conditions will single out a specific ellipse in the p_1-q_1 plane, and another in the p_2-q_2 plane.

Taken together, motion on the two ellipses can be thought of as occurring on a single <u>torus</u>



A special case occurs if the energy in one of the principal modes is zero: then the torus reduces to a circle:



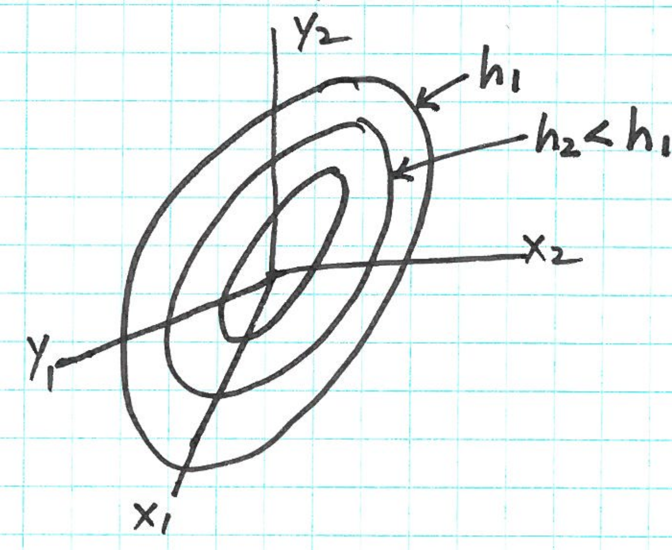
The 4 dimensional phase space is filled with non-intersecting tori. This is the case whether we use principal coordinates, as above, or physical coordinates x_i-y_i, in which case the tori are moved around by the transformation $x = Rp$.

In order to simplify the picture, we restrict attention to motions corresponding to the same total energy h.

$$T + V = h$$

$$\frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + \frac{1}{2} x_1^2 + \frac{1}{2}(x_1 - x_2)^2 = h$$

This object is a kind of 4 dimensional ellipsoid sitting in x_1-x_2-y_1-y_2 phase space. The whole space is composed of a continuum of these objects, all nested one within the next:



We will refer to one of these as an "energy manifold". The word "manifold" stands for a surface in n dimensions.

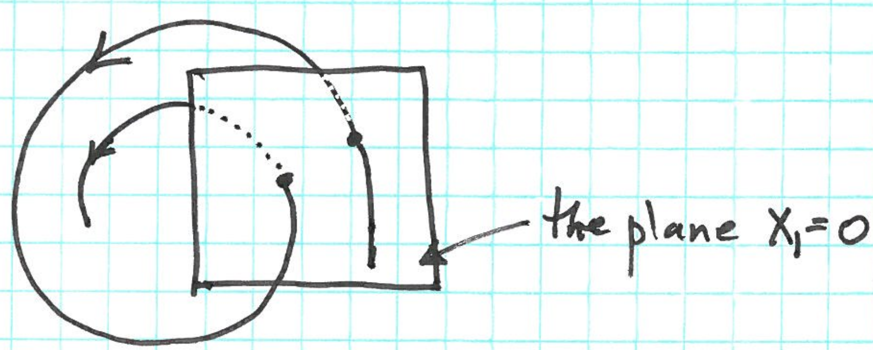
Each energy manifold is composed of a continuum of tori, and each torus represents a particular motion of the original system.

In order to understand how the tori are packed inside the energy manifold, we proceed as follows.

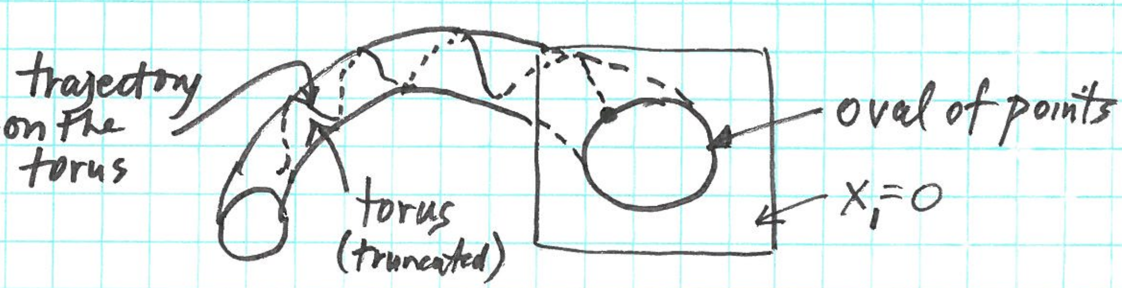
First note that the energy manifold is 3 dimensional. That is, if you choose values for x_1, x_2 and y_1, you can solve $T+V=h$ for y_2 (at least locally).

Since it is harder to picture 3 dimensional objects that 2 dimensional objects, we use a trick invented by Poincare to get a 2 dimensional look at the 3 dimensional energy manifold.

The trick is called a <u>Poincare map</u> and involves replacing an entire trajectory by a set of points, these being the places where the trajectory pierces the plane $X_1=0$ (going in the positive $\dot{X}_1$ direction)



So for example, a torus would look like a collection of points in an oval, in the Poincare map:



Now suppose we choose an initial condition which starts out on the plane $x_1=0$:

$$x_1=0,\quad x_2=x_{20},\quad y_2=y_{20}$$

where (x_{20}, y_{20}) is a point in the x_2-y_2 plane

and then we <u>COMPUTE</u> $y_1 (=\dot{x}_1)$ at $t=0$

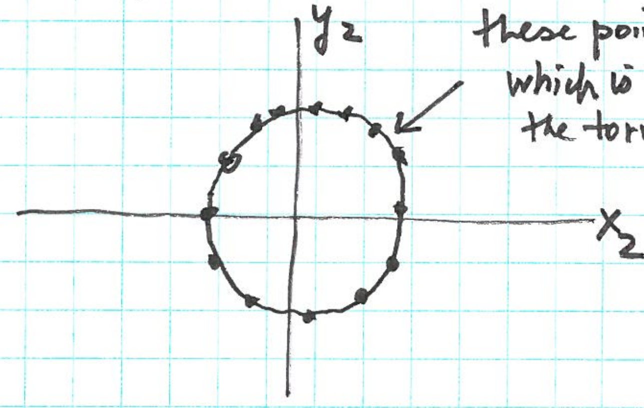
from $T+V=h$:

$$\frac{1}{2}y_1^2+\frac{1}{2}y_2^2+\frac{1}{2}x_1^2+\frac{1}{2}(x_1-x_2)^2=h$$

$$y_{10}=\sqrt{2h-y_{20}^2-x_{20}^2}$$

Now the trajectory with this initial condition takes off & travels through the energy manifold until it hits the plane $x_1=0$ with $\dot{x}_1>0$ again.

In this way we obtain a collection of points in the x_2-y_2 plane:



these points lie on a curve which is the intersection of the torus with $x_1=0$

Had we chosen a different initial condition, we'd have obtained a different curve.
By looking at all such curves we can see how the tori are arranged in the energy manifold.

We proceed now to find analytical expressions for the curves in the x_2–y_2 plane which are the intersections of the tori with the energy manifold.

From $\ddot{p}_1 + w_1^2 p_1 = 0$ we have

$$\frac{1}{2}\dot{p}_1^2 + \frac{1}{2} w_1^2 p_1^2 = C \qquad \text{(conservation of energy)}$$

Next we want to write this in terms of the physical coords.

$$x = Rp \Rightarrow p = R^{-1} x$$

For $R = \begin{pmatrix} 1 & 1 \\ 1.618 & -.618 \end{pmatrix}$, find $R^{-1} = \begin{pmatrix} .2764 & .4472 \\ .7236 & -.4472 \end{pmatrix}$

$$\Rightarrow \quad p_1 = .2764\, x_1 + .4472\, x_2$$

$$p_2 = .7236\, x_1 - .4472\, x_2$$

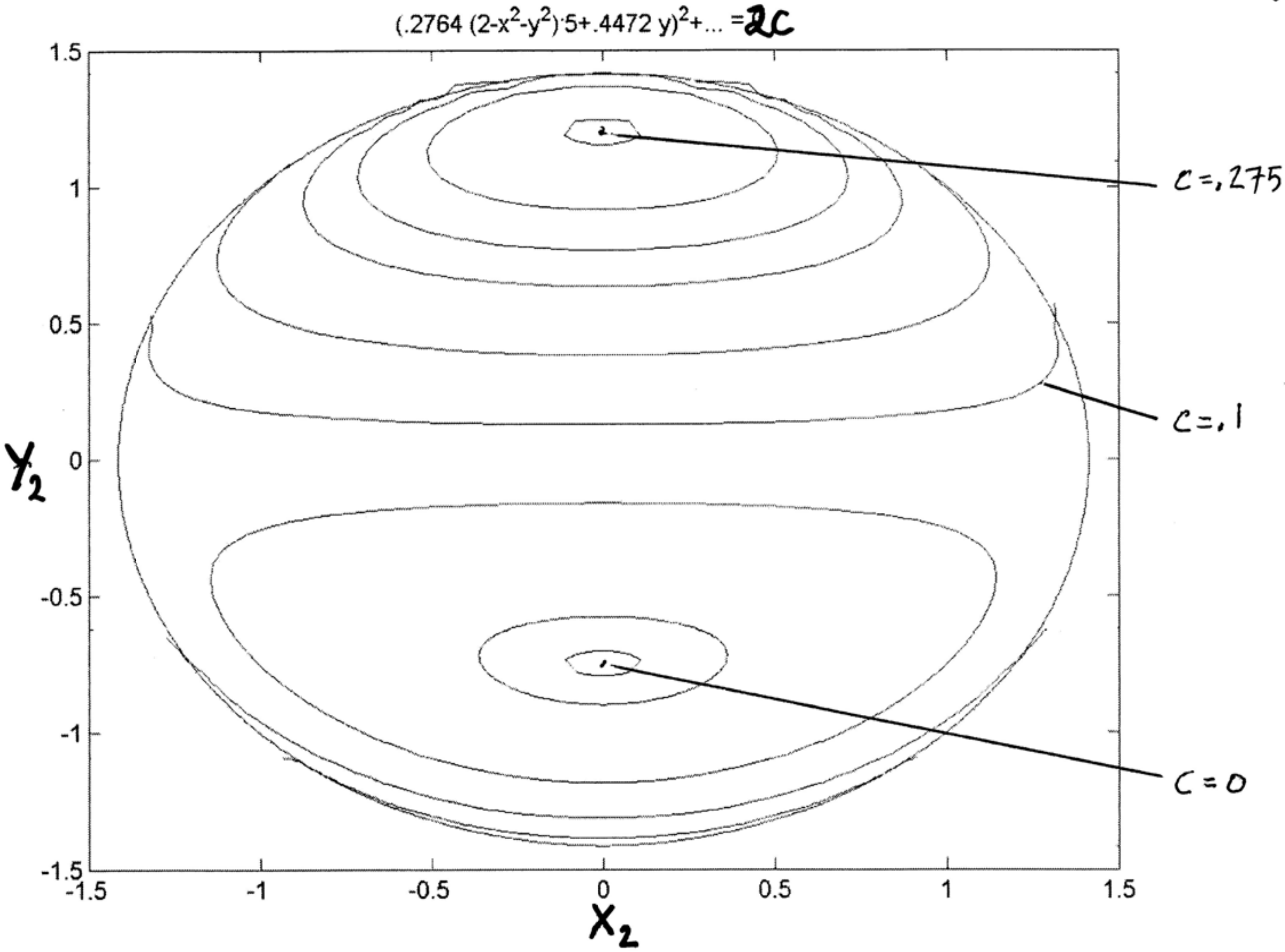
$$\frac{1}{2}\dot{p}_1^2 + \frac{1}{2} w_1^2 p_1^2 = C \Rightarrow \frac{1}{2}(.2764\, \overset{y_1}{\overset{\shortparallel}{\dot{x}_1}} + .4472\, \overset{y_2}{\overset{\shortparallel}{\dot{x}_2}})^2$$

$$+ \frac{1}{2}(.618)^2 (.7236\, x_1 - .4472\, x_2)^2 = C$$

Now set $x_1 = 0$ for Poincare map and

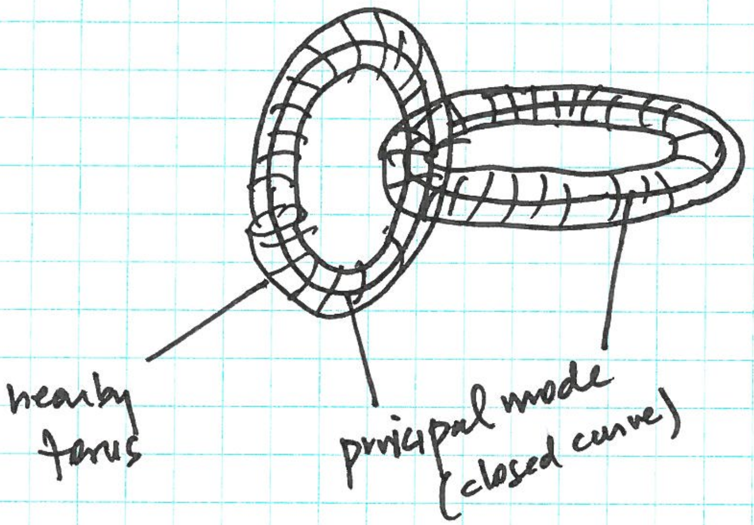
$$y_1 = \sqrt{2h - y_2^2 - x_2^2} \quad \text{from p. G9}$$

$$\left(.2764\sqrt{2h - y_2^2 - x_2^2} + .4472\, y_2\right)^2 + (.618)^2 (.4472\, x_2)^2 = 2C$$



h=1

So we see this arrangement:

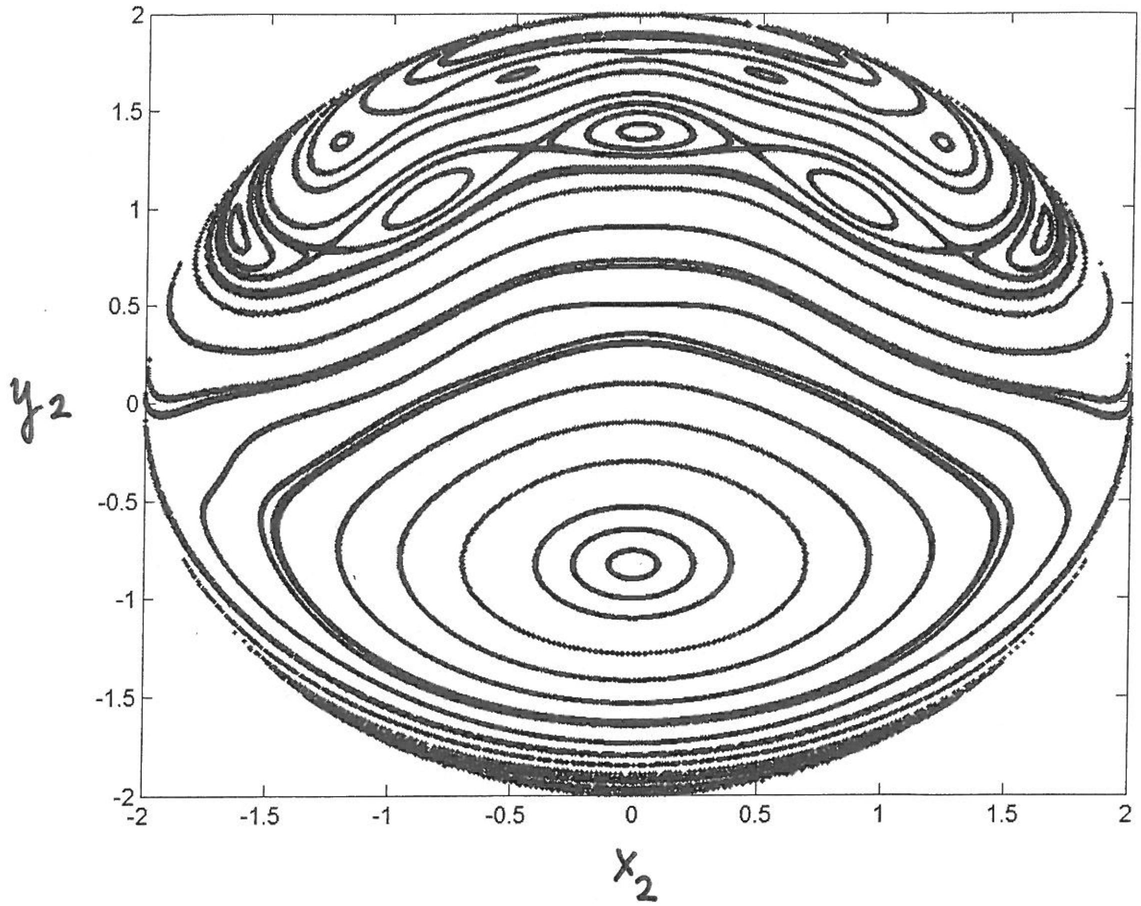


A single parameter family of torus flows go from the nbhd of one principal mode to another.

Diagrams like the foregoing Poincaré map are important when nonlinear terms are included in the system.

See next page where the comparable diagram is given for the same system, except a <u>nonlinear spring</u> is included.

The dynamics have become more complicated.

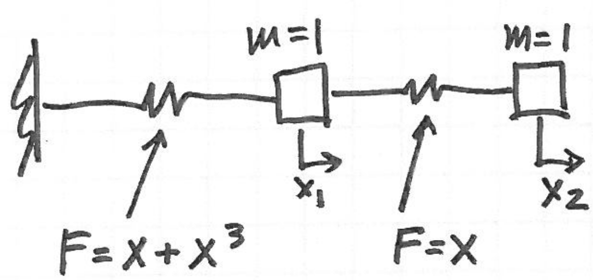


$h=2$

NONLINEAR SPRING

$$V = \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}(x_1 - x_2)^2$$

$$T = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2$$



$$\ddot{x}_1 = -x_1 - x_1^3 - (x_1 - x_2)$$

$$\ddot{x}_2 = -(x_2 - x_1)$$