2.3 Vector Spaces

MATH 294 FALL 1982 PRELIM 1 # 3a

2.3.1 Let $C[0,1]$ denote the space of continuous functions defined on the interval [0,1] (i.e. $f(x)$ is a member of $C[0,1]$ if $f(x)$ is continuous for $0 \leq x \leq 1$). Which one of the following subsets of $C[0,1]$ does **not** form a vector space? Find it and explain why it does not.

MATH 294 SPRING 1982 PRELIM 1 # 3

2.3.2 **a**)

- **i**) The subset of functions f which belongs to $C[0,1]$ for which $\int_0^1 f(s)ds = 0$.
- **ii**) The set of functions f in $C[0,1]$ which vanish at exactly one point (i.e. $f(x) = 0$ for only one x with $0 \leq x \leq 1$). Note different functions may vanish at different points within the interval.
- **iii**) The subset of functions f in $C[0,1]$ for which $f(0) = f(1)$.

b) Let $f(x) = x^3 + 2x + 5$. Consider the four vectors $\vec{v}_1 = f(x)$, $\vec{v}_2 = f'(x)$,$\vec{v}_3 = f''(x)$, $\vec{v}_4 = f'''(x)$, f' means $\frac{df}{dx}$.

- **i**) What is the dimension of the space spanned by the vectors? Justify your answer.
- **ii**) Express $x^2 + 1$ as a linear combination of the $\vec{v}_i$'s.

MATH 294 FALL 1984 FINAL # 1

2.3.3 **a**) Determine which of the following subsets are subspaces of the indicated vector spaces, and for each subspace determine the dimension of th- space. Explain your answer, giving proofs or counterexamples.

- **i**) The set of all vectors in $\Re^2$ with first component equal to 2.
- **ii**) The set of all vectors $\vec{x} = (x_1, x_2, x_3)$ in $\Re^3$ for which $x_1 + x_2 + x_3 = 0$.
- **iii**) The set of all vectors in $\Re^3$ satisfying $x_1^2 + x_2^2 - x_3^2 = 0$.
- **iv**) The set of all functions $f(x)$ in $C[0,1]$ such that $\int_0^1 f(x)dx = 0$. Recall that $C[0,1]$ denotes the space of all real valued continuous functions defined on the closed interval [0,1].

b) Find the equation of the plane passing through the points (0,1,0), (1,1,0) and (1,0,1), and find a unit vector normal to this plane.

MATH 294 SPRING 1985 FINAL # 5

2.3.4 Vectors $\vec{f}$ and $\vec{g}$ both lie in $\Re^n$. The vector $\vec{h} = \vec{f} + \vec{g}$

- **a**) Also lies in $\Re^n$.
- **b**) May or may not lie in $\Re^n$.
- **c**) Lies in $\Re^{\frac{n}{2}}$.
- **d**) Does not lie in $\Re^n$.

MATH 294 SPRING 1985 FINAL # 6

2.3.5 The vector space $\Re^n$

- **a**) Contains the zero vector.
- **b**) May or may not contain the zero vector.
- **c**) Never contains the zero vector.
- **d**) Is a complex vector space.

MATH 294 SPRING 1985 FINAL # 7

2.3.6 Any set of vectors which span a vector space

a) Always contains a subset of vectors which form a basis for that space.

b) May or may not contain a subset of vectors which form a basis for that space.

c) Is a linearly independent set.

d) Form an orthonormal basis for the space.

MATH 294 SPRING 1987 PRELIM 3 # 7

2.3.7 For what values of the constant a are the functions $\{\sin t \text{ and } \sin(t+a)\}$ in C^1_∞ linearly independent?

MATH 294 SPRING 1987 PRELIM 3 # 9

2.3.8 For problems (a) - (c) use the bases B and B' below:

$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

a) Given that $[\vec{v}]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ what is $[\vec{v}]_{B'}$?

b) Using the standard relation between $\Re^2$ and points on the plane make a sketch with the point $\vec{v}$ clearly marked. Also mark the point $\vec{w}$, where $[\vec{w}]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

c) Draw the line defined by the points $\vec{v}$ and $\vec{w}$. Do the points on this line represent a subspace of $\Re^2$?

MATH 294 FALL 1987 PRELIM 3 # 2

2.3.9 In parts (a) - (g) answer "true" if V is a vector space and "false" if it is not (no partial credit):

a) $V =$ set of all $x(t)$ in C_∞ such that $x(0) = 0$.

b) $V =$ set of all $x(t)$ in C_∞ such that $x(0) = 1$.

c) $V =$ set of all $x(t)$ in C_∞ such that $(D+1)x(t) = 0$.

d) $V =$ set of all $x(t)$ in C_∞ such that $(D+1)x(t) = e^t$.

e) $V =$ set of all polynomials of degree less than or equal to one with real coefficients.

f) $V =$ set of all rational numbers (a rational number can be written as the ratio of two integers, e.g., $\frac{4}{17}$ is a rational number while $\pi = 3.14\ldots$ is not)

g) $V =$ set of all rational numbers with the added restriction that scalars must also be rational numbers.

MATH 294 FALL 1987 FINAL # 7

2.3.10 Consider the boundary-value problem

$X'' + \lambda X = 0 \; 0 < x < \pi$, $X(0) = X(\pi) = 0$, where λ is a given real number.

a) Is the set of all solutions of this problem a subspace of $C_\infty[0, \pi]$? Why?

b) Let $W =$ set of all functions $X(x)$ in $C_\infty[0, \pi]$ such that $X(0) = X(\pi) = 0$. Is $T \equiv D^2 - \lambda$ linear as a transformation of W into $C_\infty[0, \pi]$? Why?

c) For what values of λ is $Ker(T)$ nontrivial?

d) Choose one of those values of λ and determine $Ker(T)$

MATH 293 SPRING 1990? PRELIM 2 # 3

2.3.11 Is the set of vectors $\{\vec{v}_1 = e^{-t}, \vec{v}_2 = e^t\}$ in C^∞ linearly independent or dependent? (Justify your answer.)

MATH 293 SPRING 1992 PRELIM 2 # 3

2.3.12 W is the subspace of V_4 spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Find the dimension of W and give a basis.

MATH 293 SPRING 1992 PRELIM 2 # 4

2.3.13 V is the vector space consisting of vector-valued functions $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ where $x_1(t)$ and $x_2(t)$ are continuous functions of t in $0 \leq t \leq 1$. W is the subset of V where the functions satisfy the differential equations
$\frac{dx_1}{dt} = x_1 + x_2$ and $\frac{dx_2}{dt} = x_1 - x_2$
Is W a subspace of V?

MATH 293 SPRING 1992 PRELIM 2 # 6

2.3.14 V is the vector space consisting of all 2×2 matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Here the a_{ij} are arbitrary real numbers and the addition and scalar multiplication are defined by
$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$ and $cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$

a) Is $W_1 = \left\{ all \begin{bmatrix} a_{11} & a_{12} \\ 0 & 1 \end{bmatrix} \right\}$ a subspace? If so give a basis for W_1.

b) Same as part (a) for $W_2 = \left\{ \begin{bmatrix} a_{11} & -a_{12} \\ a_{12} & a_{11} \end{bmatrix} \right\}$.

c) Show that $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ are linearly independent.

d) What is the largest possible number of linearly independent vectors in V?

MATH 293 SPRING 1992 FINAL # 7

2.3.15 A "plane" in V_4 means, by definition, the set of all points of the form $\vec{u} + \vec{x}$ where $\vec{u}$ is a constant (fixed) vector and $\vec{x}$ varies over a fixed two-dimensional subspace of V_4. Two planes are "parallel" if their subspaces are the same. It is claimed that the two planes:

1^{st} plane:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}$$

2^{nd} plane:

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{pmatrix}$$

(where x_2, x_3 and x_4 can assume any scalar values) do not intersect and are not parallel. Do you agree or disagree with this claim? You have to give very clear reasons for your answer in order to get credit for this problem.

MATH 293 SUMMER 1992 FINAL # 3

2.3.16 **a**) Let V be the vector space of all 2 matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where a_{ij} , $i, j = 1, 2$, are real scalars.

Consider the set S of all $2 times 2$ matrices of the form $\begin{pmatrix} a+b & a-b \\ b & a \end{pmatrix}$ where a and b are real scalars.

i) Show that S is a subspace. Call it W.

ii) Find a basis for W and the dimension of W.

b) Consider the vector space V $\{f(t) = a + b\sin t + c\cos t\}$, for all real scalars a, b and c and $0 \le t \le 1$

Now consider a subspace W of V in which $\frac{df(t)}{dt} + f(t) = 0$ at $t = 0$

Find a basis for the subspace W.

MATH 293 FALL 1992 PRELIM 3 # 3

2.3.17 Let $C(-\pi, \pi)$ be the vector space of continuous functions on the interval $-\pi \le x \le \pi$. Which of the following subsets S of $C(-\pi, \pi)$ are subspaces? If it is not a subspace say why. If it is, then say why and find a basis.

Note: You must show that the basis you choose consists of linearly independent vectors. In what follows a_0, a_1 and a_2 are arbitrary scalars unless otherwise stated.

a) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$

b) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$, subject to the condition $\int_{-\pi}^{\pi} f(x)dx = 2\pi$

c) S is the set of functions of the form $f(x) = 1 + a_1 \sin x + a_2 \cos x$, subject to the condition $\int_{-\pi}^{\pi} f(x)dx = 0$

MATH 293 FALL 1992 PRELIM 2 # 5

2.3.18 Consider all polynomials of degree ≤ 3

$$P_3 = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3\}, -\infty < t < \infty$$

They Form a vector space. Now consider the subset S of P_3 consisting of polynomials of degree ≤ 3 with the conditions

$$p(0) = 0, \frac{dp}{dt}(0) = 0$$

Is S a subspace W of P_3? Carefully explain your answer.

MATH 293 FALL 1992 PRELIM 2 # 6

2.3.19 Given a vector space V_4 which is the space of all vectors of the form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ for all real x_1, x_2, x_3, x_4,
consider the set S of vectors in V_4 of the form

$$S = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 3 \\ -2 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

for all values of scalars a, b and c.
Is the set S a subspace W of V_4? Explain your answer carefully.

MATH 293 FALL 1992 FINAL # 3d

2.3.20 Let S be the set of all vectors of the form $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the usual mutually perpendicular unit vectors. Let W be the set of all vectors that are perpendicular to the vector $\vec{v}_1 = \vec{i} + \vec{j} + \vec{k}$. Is W a vector subspace of V_3? Explain your answer.

MATH 293 FALL 1994 PRELIM 2 # 5

2.3.21 In each of the following, you are given a vector space V and a subset W. Decide whether W is a subspace of V, and prove that your answer is correct.

a) V is the space $M_{2,2}$ of all 2×2 matrices, and W is the set of 2×2 matrices A such that $A^2 = A$

b) V is the space of differentiable functions, and W is the set of those differentiable functions that satisfy $f'(3) = 0$.

MATH 293 FALL 1994 PRELIM 2 # 4

2.3.22 **a**) Let M denote the set of ordered triples (x, y, z) of real numbers with the operations of addition and multiplication v=by scalars c defined by

$$(x, y, z) \oplus (x', y', z') = (x + z', y + y', z + z')$$

$$c \odot (x, y, z) = (2c, cy, cz).$$

Is M a vector space? Why?

b) Consider the vector space $\Re^4$. Is the subset S of vectors of the form (x_1, x_2, x_3, x_4) where x_1, x_2, and x_3 are arbitrary and $x_4 \leq 0$ a subspace? Why?

c) Consider the vector space P_2 of polynomials of degree ≤ 2. Is the subset S of polynomials of the form $p(t) = a_0 + a_1 t + (a_0 + a_1)t^2$ a subspace? Why?

MATH 293 FALL 1994 PRELIM 3 # 5

2.3.23 Answer each of the following as True or False. If false, explain, by an example.

a) Every spanning set of $\Re^3$ contains at least three vectors.

b) Every orthonormal set of vectors in $\Re^5$ is a basis for $\Re^5$.

c) Let A be a 3 by 5 matrix. Nullity A is at most 3.

d) Let W be a subspace of $\Re^4$. Every basis of W contains at least 4 vectors.

e) In $\Re^n, ||cX|| = |c|\,||X||$

f) If A is an $n \times n$ symmetric matrix, then rank $A = n$.

MATH 293 FALL 1994 FINAL # 4

2.3.24 **a**) Find a basis for the space spanned by: {(1,0,1),(1,1,0),(-1,-4,3)}.

b) Show that the functions $e^{2x}\cos(x)$ and $e^{2x}\sin(x)$ are linearly independent.

MATH 293 FALL 1994 PRELIM 3 # 2

2.3.25 Which of the following sets of vectors is linearly independent? Show all work.

a) In P_2: $S = \{1, t, t^2\}$

b) In $\Re^3$: $S = \{(1, 2, -1), (6, 3, 0), (4, -1, 2)\}$

MATH 293 SPRING 1995 PRELIM 3 # 3

2.3.26 Let P_3 be the space of polynomials $p(t)$ of degree ≤ 3. Consider the subspace $S \subset P_3$ of polynomials that satisfy

$$p(0) + \left.\frac{dp}{dt}\right|_{t=0} = 0$$

a) Show that S is a subspace of P_3.

b) Find a basis for S.

c) What is the dimension of S?

MATH 293 FALL 1995 PRELIM 3 # 3

2.3.27 Let P_3 be the space of polynomials $p(t) = a_0 + a_1 t + a_2 t + a_3 t^3$ of degree ≤ 3. Consider the subset S of polynomials that satisfy

$$p''(0) + 4p(0) = 0$$

Here $p''(0)$ means, as usual, $\left.\frac{d^2p}{dt^2}\right|_{t=0}$.

a) Show that S is a subspace of P_3. Give reasons.

b) Find a basis for S.

c) What is the dimension of S? Give reasons for your answer.

Hint: What constraint, if any, does the given formula impose on the constants a_0, a_1, a_2, and a_3 of a general $p(t)$?

MATH 293 FALL 1995 PRELIM 3 # 4

2.3.28 We define a new way of "adding" vectors by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 y_2 \end{pmatrix}$$

and use ordinary scalar multiplication.

a) Is the commutative axiom "$x + y = y + x$" satisfied?

b) Is the associative axiom "$x + (y + z) = (x + y) + z$" satisfied?

c) How about the distributive law "$a(x + y) = ax + ay$"?

d) Is this a vector space?

Give reasons.

MATH 293 SPRING 1996 PRELIM 3 # 1

2.3.29 The set W of vectors in $\Re^3$ of the form (a, b, c), where $a + b + c = 0$, is a subspace of $\Re^3$.

a) Verify that the sum of any two vectors in W is again in W.

b) The set of vectors

$$S = \{(1, -1, 0), (1, 1, -2), (-1, 1, 0), (1, 2, -3)\}$$

is in W. Show that S is linearly dependent.

c) Find a subset of S which is a basis for W.

d) If the condition $a + b + c = 0$ above is replaced with $a + b + c = 1$, is W still a subspace? Why/ why not

MATH 293 SPRING 1996 PRELIM 3 # 3

2.3.30 Which of the following subsets are bases for P_2, the vector space of polynomials of degree less than or equal to two? You do *not* need to show your work.

$S_1 = \{1, t, 1 - t, 1 + t\}$, $S_2 = \{t^2, t^2 + 2, t^2 + 2t\}$, $S_3 = \{1 + t + t^2, t, t^2\}$

MATH 293 SPRING 1996 FINAL # 4

2.3.31 Suppose $\vec{v}_1, \dots, \vec{v}_p$ are vectors in $\Re^n$. Then $Span\{\vec{v}_1, \dots, \vec{v}_p\}$ is always:

a) a linearly independent set of vectors
b) a linearly dependent set of vectors
c) a basis for a subspace of $\Re^n$
d) the set of all possible linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$,
e) none of the above.

MATH 294 SPRING 1997 FINAL # 7.2

2.3.32 Which of the following subsets are subspaces of the vector space P_2 of polynomials of degree ≤ 2? (No Justification is necessary.) Express your answer as e.g.: SUBSPACE: a,b,c,d; NOT: e

a) $\{\ p(t) \big| p'(t) = 0, \text{ all t}\}$
b) $\{\ p(t) \big| p'(t) - 1 = 0, \text{ all t}\}$
c) $\{\ p(t) \big| p(0) + p(1) = 0\}$
d) $\{\ p(t) \big| p(0) = 0 \text{ and } p(1) = 0\ \}$
e) $\{\ p(t) \big| p(0) = 0 \text{ and } p(1) = 1\ \}$

MATH 294 FALL 1997 PRELIM 2 # 3

2.3.33 Let W be the subspace of $\Re^4$ defined as

$$W = span\left(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -6 \\ 4 \end{pmatrix}\right)$$

a) Find a basis for W. What is the dimension of W?

b) It is claimed that W can be described as the intersection of two linear spaces S_1 and S_2 in $\Re^4$. The equation of S_1 and S_2 are

$$S_1 : x - y = 0$$

and

$$S_2 : ax + by + cz + dw = 0,$$

where a, b, c, d are real constants that must be determined. Find one possible set of values of a, b, c and d.

MATH 294 FALL 1997 PRELIM 2 # 6

2.3.34 Let V be the vector space of 2×2 matrices.

a) Find a basis for V.

b) Determine whether the following subsets of V are subspaces. If so, find a basis. If not, explain why not.

i) $\{A \text{ in } V \mid \det A = 0\}$

ii) $\left\{ A \text{ in } V \mid A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

c) Determine whether the following are linear transformations. Give a short justification for your answers.

i) $T : V \to V$, where $T(A) = A^T$,

ii) $T : V \to \Re^1$, where $T(A) = \det(A)$.

MATH 294 SPRING 1998 PRELIM 3 # 5

2.3.35 True or False? Justify each answer.

a) In general, if a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis of V.

b) A linearly independent set in a subspace H is a basis for H.

c) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d) If an $n \times n$ matrix A is diagonalizable, it is invertible.

MATH 293 SPRING ? FINAL # 6

2.3.36 Give a definition for addition and for scalar multiplication which will turn the set of all pairs $(\vec{u}, \vec{v})$ of vectors, for $\vec{u}, \vec{v}$ in V_2, into a vector space V.

a) What is the zero vector of V?

b) What is the dimension of V?

c) What is a basis for V?

MATH 293 FALL 1998 PRELIM 2 # 2

2.3.37 Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix},$$

a) Show by a calculation that its determinant is nonzero.

b) Calculate its inverse by any means.