M294 P3 SP87 #3

#1

$$x(t)=\begin{bmatrix} a\sin t + b\cos t \\ c\sin t + d\cos t\end{bmatrix} = a\begin{bmatrix}\sin t\\ 0\end{bmatrix} + b\begin{bmatrix}\cos t\\ 0\end{bmatrix} + c\begin{bmatrix}0\\ \sin t\end{bmatrix} + d\begin{bmatrix}0\\ \cos t\end{bmatrix}$$

$$= a\,v_1(t) + b\,v_2(t) + c\,v_3(t) + d\,v_4(t)$$

$\mathcal{B}=\{v_1, v_2, v_3, v_4\}$ is a basis and a,b,c,d the coordinates of $x(t)$.

Note $\dot v_1 = v_2$, $\dot v_2 = -v_1$, $\dot v_3 = v_4$, $\dot v_4 = -v_3$

so $T(av_1+bv_2+cv_3+dv_4) = -bv_1 + av_2 - dv_3 + cv_4$

which can be written

$$T\left(\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}_{\mathcal{B}}\right) = \boxed{\underbrace{\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}}_{{}_{\mathcal{B}}[T]_{\mathcal{B}}}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}}$$

Note: if you picked a different basis, your matrix will look different.

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#2

Finding $Tv=\lambda v$ amounts to solving $\begin{cases}\ddot x = \lambda x\\ x(0)=0\\ x(1)=0\\ x \text{ not identically } 0\end{cases}$

One such solution is

$$\boxed{\begin{aligned} x(t) &= \text{[redacted]} \sin \pi t\\ \lambda &= -\pi^2\end{aligned}}$$

Some other solns: $\begin{cases} x(t) = 17\sin 23\pi t\\ \lambda = -(23)^2\pi^2\end{cases}$ $\begin{cases} x(t) = -\sin 2\pi t\\ \lambda = -4\pi^2\end{cases}$

M294 F SP87 #2

3) DEFINITION: KERNEL IS THE SET OF (FUNCTIONS) VECTORS SUCH THAT $T(\underline{x}) = \underline{0}$

THIS IS THE CASE WHEN

$\dot{\underline{x}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underline{x}$, BUT THE SOLNS. OF THIS EQTN. CAN BE READ OFF

$\lambda = 2, 3$, $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

SO ALL FUNCTIONS OF THE FORM $\boxed{c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ IS THE KERNEL

$\boxed{\text{comb}\left(e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$ " " "

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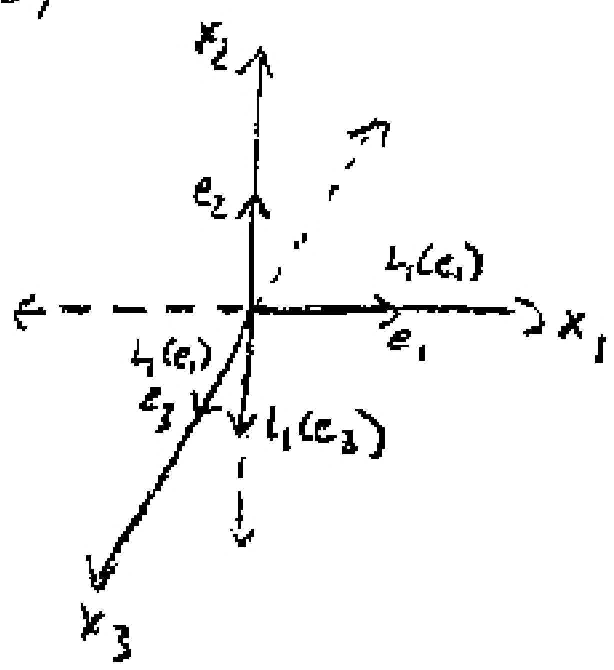
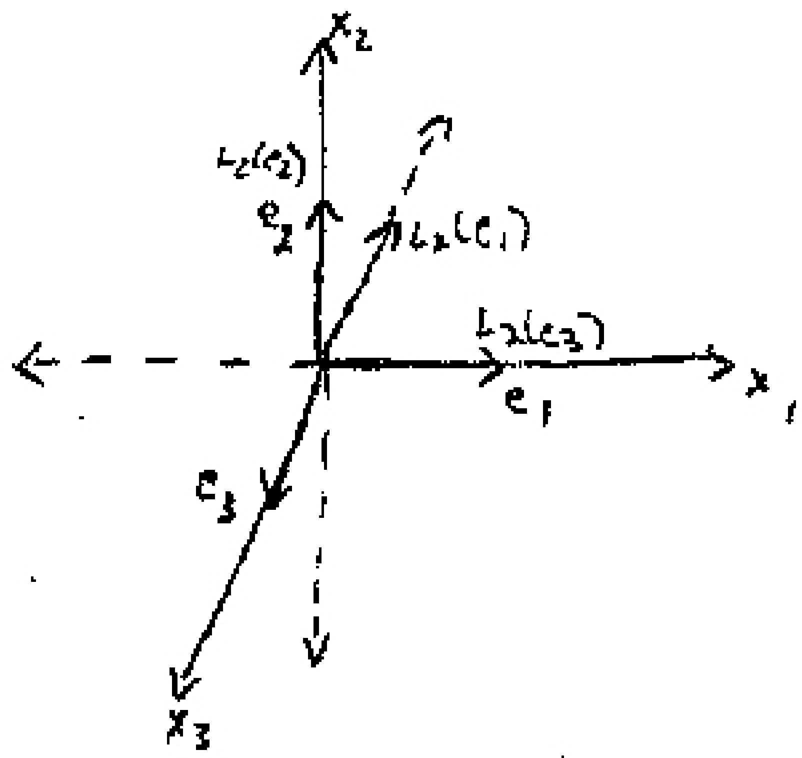
23) a)



$L_1(\underline{x}) = A_1\underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\underline{x}$



$L_2(\underline{x}) = A_2\underline{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\underline{x}$

b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} = B_1$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = B_2$

d) AS SHOWN IN PARTS (B) AND (C) IT DOES.

Lin Trans. II

section 2.8

Math 294 - Fall '98 - Prelim 3 #1

1) (25 pt) Consider the following three vectors in $\mathcal{R}^3$:

$$\mathbf{y} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

[Note: $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.].

⇐ *Please put scrap work for problem 1 on the page to the left* ⇐.

⇓ *Put neat work to be graded for problem 1 below.* ⇓

(*If you need the space, clearly mark work to be graded on the scrap page.)* .

$$\langle u_1, u_1\rangle = [1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 3$$

$$\langle u_1, u_2\rangle = [1\ 1\ 1]\begin{bmatrix}1\\-1\\0\end{bmatrix} = 0$$

$$\langle u_2, u_2\rangle = [1\ {-1}\ 0]\begin{bmatrix}1\\-1\\0\end{bmatrix} = 2$$

$$\langle y, u_1\rangle = [1\ 0\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 2$$

$$\langle y, u_2\rangle = [1\ 0\ 1]\begin{bmatrix}1\\-1\\0\end{bmatrix} = 1$$

(§)

a) Find the orthogonal projection of y onto the subspace of $\mathcal{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\operatorname{Proj}_{\operatorname{span}\{u_1, u_2\}} y = \frac{\langle y, u_1\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle y, u_2\rangle}{\langle u_2, u_2\rangle} u_2$$

$$= \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \boxed{\frac{1}{6}\begin{bmatrix}7\\1\\4\end{bmatrix}}$$

plug into (§) to get

b) What is the distance between y and span$\{\mathbf{u}_1, \mathbf{u}_2\}$?

$$d = \left\| y - \frac{1}{6}\begin{bmatrix}7\\1\\4\end{bmatrix} \right\| = \frac{1}{6}\left\| \begin{bmatrix}6\\0\\6\end{bmatrix} - \begin{bmatrix}7\\1\\4\end{bmatrix} \right\| = \frac{1}{6}\left\| \begin{bmatrix}-1\\-1\\2\end{bmatrix} \right\|$$

$$= \frac{1}{6}(1+1+4)^{\frac{1}{2}} = \frac{\sqrt{6}}{6} = \boxed{\frac{1}{\sqrt{6}}}$$

c) In terms of the standard basis for $\mathcal{R}^3$, find the matrix of the linear transformation that orthogonally projects vectors onto span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

We find the matrix of a linear transformation by

$$A = [\ T(e_1)\ \ T(e_2)\ \ T(e_3)\ ]$$

projections of basis vectors of $\mathbb{R}^3$.

Replace y in eq (§) above to get

$$\boxed{A = \frac{1}{6}\begin{bmatrix}5 & -1 & 2\\ -1 & 5 & 2\\ 2 & 2 & 2\end{bmatrix}}$$

Can check this by verifying that $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $A\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}7\\1\\4\end{bmatrix}$