Fourier Series

M294 PI SP87#2

16 a) $$\sin 6\pi x = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{3} + b_k \sin\frac{k\pi x}{3}\right)$$

There's no need to do integrals – your only possibility is $b_{18} = 1$, all other $a_k = b_k = 0$

In other words $\boxed{\sin 6\pi x = \sin 6\pi x}$.

b) $\sin 6\pi x$ is odd, so the Fourier series is equal to the sin series.

Again $\boxed{\sin 6\pi x = \sin 6\pi x}$

c) $$\sin 6\pi x = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{3}$$

and you have to either evaluate

$$a_k = \frac{2}{3}\int_0^3 \sin 6\pi x \cos\frac{k\pi x}{3}\,dx = \boxed{\begin{cases} 0, & k \text{ even} \\ \dfrac{8}{36\pi - \frac{k^2\pi}{9}}, & k \text{ odd}\end{cases}}$$

or issue the MuMath command

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2*DEFINT(SIN(2*#PI*X)*COS(K*#PI*X/3),
    X,0,3)/3 ;
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d) $$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + \cdots$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_0^3 2\,dx = 2$$

$$a_k = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{k\pi x}{3}\,dx = \frac{1}{3}\int_0^3 2\cos\frac{k\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\frac{\sin\frac{k\pi x}{3}}{\frac{k\pi}{3}}\right]_0^3 = \frac{2}{\pi}\sin k\pi = 0$$

$$b_k = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{k\pi x}{3}\,dx = \frac{1}{3}\int_0^3 2\sin\frac{k\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\frac{\cos\frac{k\pi x}{3}}{-\frac{k\pi}{3}}\right]_0^3 = \frac{-2}{k\pi}(\cos k\pi - 1)$$

So $b_1 = \frac{4}{\pi}$, $b_2 = 0$, $b_3 = \frac{4}{3\pi}$, $b_4 = 0$, $b_5 = \frac{4}{5\pi}$

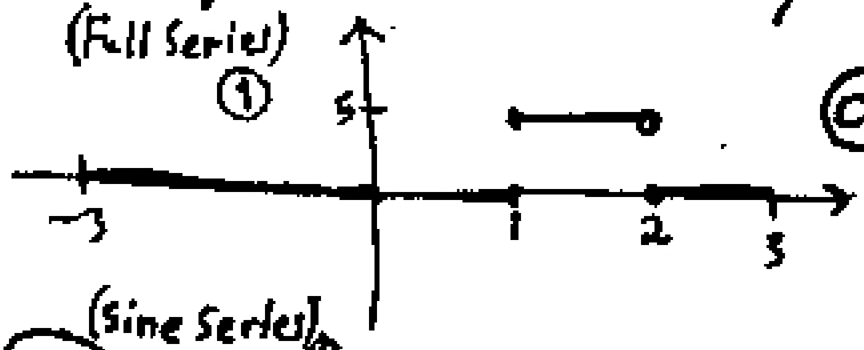
and

$$\boxed{f(x) = 1 + \frac{4}{\pi}\sin\frac{\pi x}{3} + \frac{4}{3\pi}\sin\frac{3\pi x}{3} + \frac{4}{5\pi}\sin\frac{5\pi x}{3} + \cdots}$$

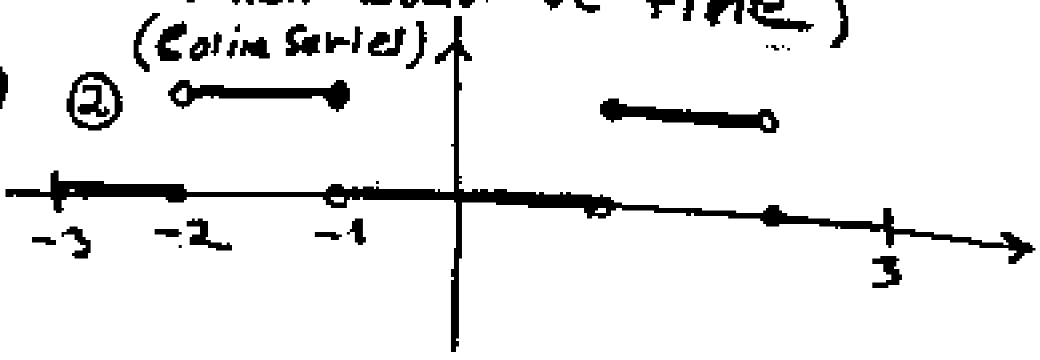
M294 F SP87 #8

17)

a) 3 choices (any one of which would be fine)



OR



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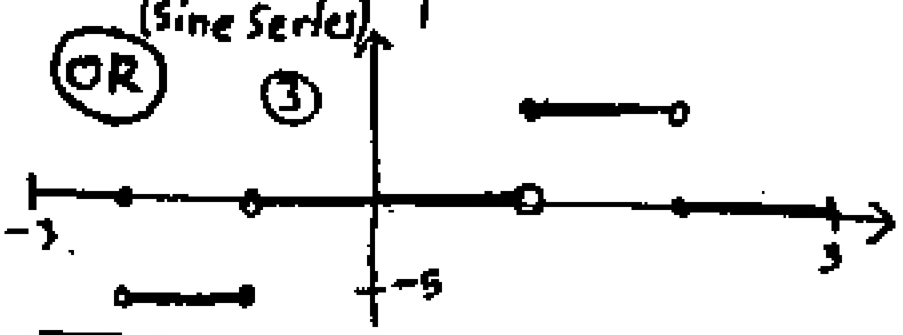


8 b) $b_{17} = \frac{1}{3}\int_{-3}^{3} f(t)\sin\frac{17\pi t}{3}\,dt =$

$\frac{1}{3}\int_1^2 5\sin\frac{17\pi t}{3}\,dt = \frac{5}{3}\left[\frac{-\cos\frac{17\pi t}{3}}{\frac{17\pi}{3}}\right]_1^2$

$= \frac{5}{17\pi}\left(\cos\frac{17\pi}{3} - \cos\frac{34\pi}{3}\right)$ for ①

$b_{17} = \frac{5}{17\pi}\left(+\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \boxed{\frac{5}{17\pi}}$

OR

$\boxed{0}$, odd function for ②

OR

$\frac{2}{3}\int_1^2 5\sin\frac{17\pi t}{3}\,dt = 2\cdot\frac{5}{17\pi} = \boxed{\frac{10}{17\pi}}$ for ③

(see ① above)

8 c) $S(1) = \frac{f(1-) + f(1+)}{2} = \boxed{\frac{1}{2}}$

8 d) $S(7.75) = S(1.75 + 6)$

$= S(1.75)$ (period 6)

$= \boxed{5}$

M294 PI SP88 #1

21) a) $\frac{1}{7} = a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx \quad \therefore \boxed{\int_{-\pi}^{\pi} f(x)\cos 2x\,dx = \frac{\pi}{7}}$

(could also integrate full series, all terms but 1 give zero.)

b) [F. Series at $x=0$] = [value of function if it is continuous and it is]

$= \boxed{1}$

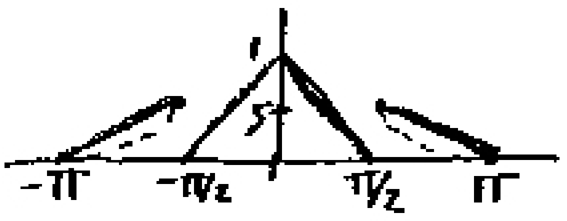
c) [F. Series at $x=\frac{\pi}{2}$] = average of left and right-hand limits

$(\frac{1}{2}+0)/2 = \boxed{\frac{1}{4}}$

d) $\boxed{b_1 = 0}$ since the function is even, no sine terms.

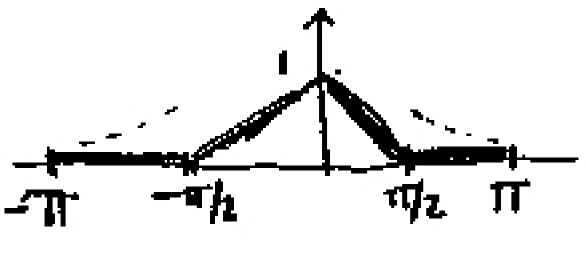
e) $\frac{a_0}{2}$ = average of the function $\boxed{\doteq \frac{5}{16}}$

I got $\frac{5}{16}$ because the average of f must be between the average of



which is $\frac{3}{8}$

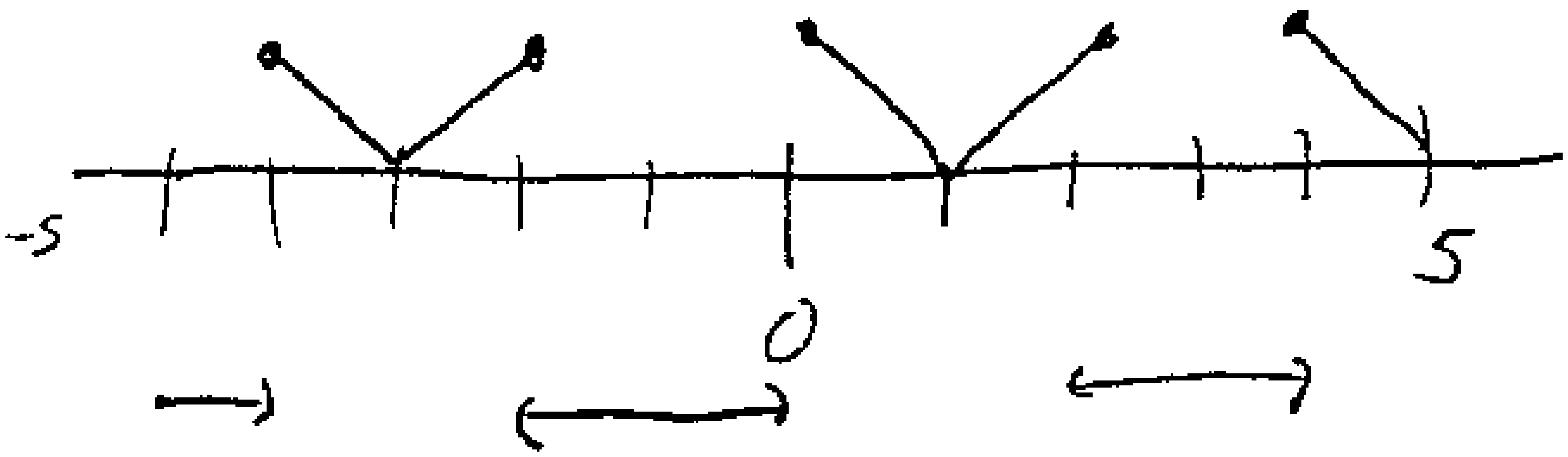
and the average of



which is $\frac{1}{4}$.

44) (a)

M294
P III
SP95
#1



$g(x)$

(b) One answer is

$$g(x) = \begin{cases} |x-(4k+1)|, & \text{if } 4k \le x \le 4k+2 \\ -1, & \text{if } 4k+2 < x < 4k+4 \end{cases}$$

M294 P III FA95 #2

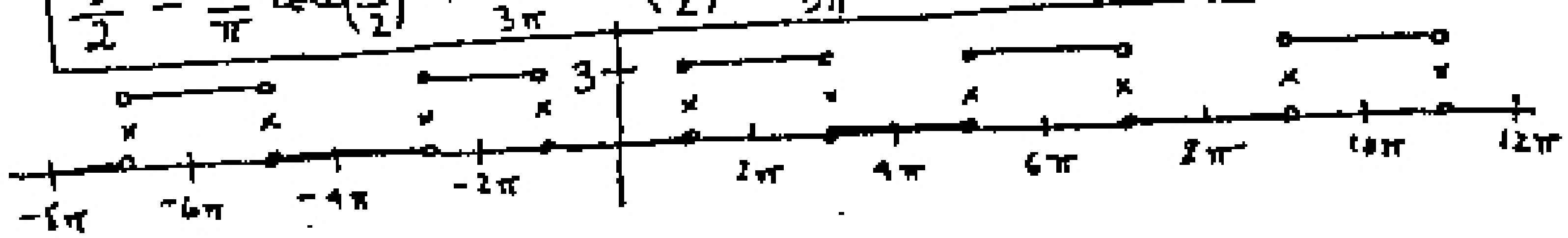
45) (a) f is even so $b_n = 0$, $a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\frac{n\pi x}{2\pi}\,dx$

$$= \frac{1}{\pi}\int_{\pi}^{2\pi} 3\cos\frac{n\pi x}{2\pi}\,dx = \begin{cases} 3 & , \text{if } n=0 \\ \frac{3}{\pi}\frac{2}{n}\left[\sin\frac{nx}{2}\right]_{\pi}^{2\pi}, & n>0 \end{cases}$$

$$= \begin{cases} 3 & , \text{if } n=0 \\ \frac{6}{n\pi}\left(\underbrace{\sin n\pi}_{0} - \sin\frac{n\pi}{2}\right) & n>0 \end{cases}$$

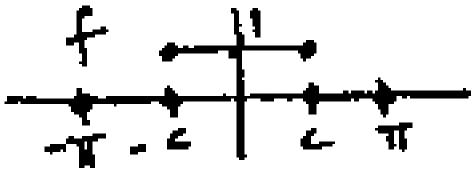
$$\boxed{\frac{3}{2} - \frac{6}{\pi}\cos\left(\frac{x}{2}\right) + \frac{6}{3\pi}\cos\left(\frac{3x}{2}\right) - \frac{6}{5\pi}\cos\left(\frac{5x}{2}\right) + \cdots}$$

(b)



M294F FA95 #4

#5)



(a) f is even so all $b_n = 0$; $a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{2c}{\pi}$,

$n>0$: $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^c \cos nx\,dx = \frac{2}{n\pi}\sin nx\Big|_0^c = \frac{2\sin nc}{n\pi}$

(b) when $c=\pi/4$ and $x=\pi/2$, f is continuous at x, so

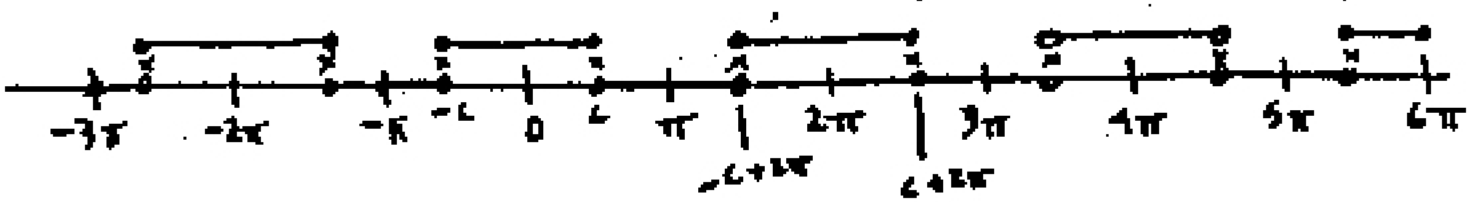
$$f(x) = \frac{c}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin nc}{n\pi}\cos nx$$ holds, so

$$0 = \frac{\pi/4}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin\frac{n\pi}{4}}{n\pi}\cos\frac{n\pi}{2}$$

$$= \frac{1}{4} + \frac{2}{\pi}\left(-\frac{1}{2} + \frac{1}{6} - \frac{1}{10} + \frac{1}{14} - \cdots\right)$$

rearranging, $\frac{\pi}{8} = \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \cdots$

(c)



(d) the insulated B.C. $u_x = 0$ at ends tells us to use the solutions $e^{-\left(\frac{n\pi k}{l}\right)^2 t}\cos\frac{n\pi x}{l}$ which come from separation of variables, and since $u(x,0)$ is the same as $f(x)$ in this problem, the answer is

$$\boxed{u(x,t) = \frac{c}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin nc}{n\pi} e^{-n^2 t}\cos nx}$$

This is the heat equation, which we have solved in class by separation of variables for these boundary conditions.

We found $u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2}kt}$

The heat eqn is $k\,u_{xx} = u_t$,

Thus We have $k=1$, $L=\pi$ (given in eqn & seen in the B.C.)

$\therefore \boxed{u(x,t) = \sum a_n \sin nx\, e^{-n^2 t}}$

The a_n come from the Fourier expansion of the initial condition. Fortunately

From problem 2, we have $x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$ for the Fourier sine series.

Thus, this is the expression of the initial condition $\left\{ u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx \right. \Rightarrow a_n = \frac{2(-1)^{n+1}}{n}$

$\therefore \boxed{u(x,t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\, e^{-n^2 t}}$

The first 2 terms are $u = \frac{2}{1}(-1)^2 \sin x\, e^{-t} + \frac{2}{2}(-1)^3 \sin 2x\, e^{-4t}$

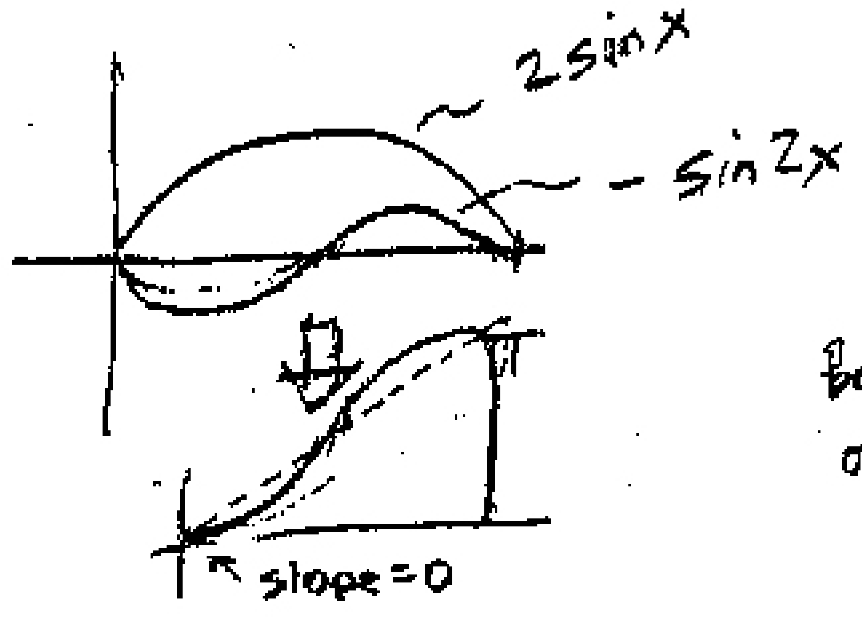
In general, $u(x,t) = 2\sin x\, e^{-t} - \sin 2x\, e^{-4t} + \ldots\ldots$

at $t=0$ $\quad u(x,0) = 2\sin x - \sin 2x + \ldots\ldots$

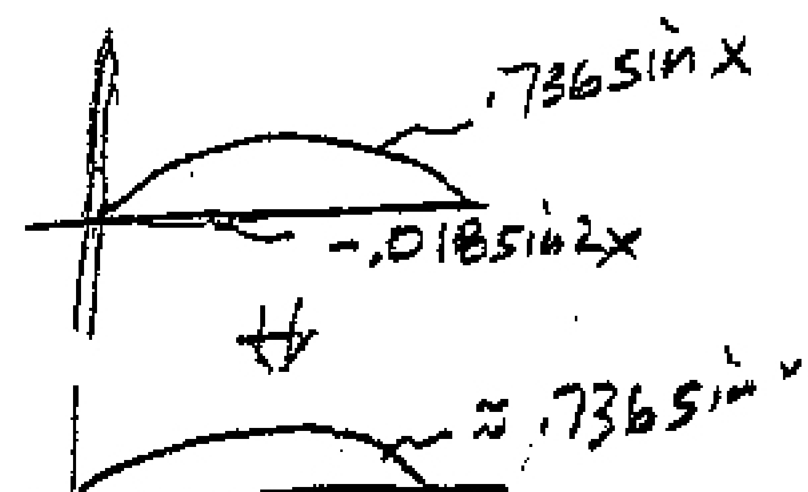
$t=1$ $\quad u(x,1) = 2e^{-1}\sin x - e^{-4}\sin 2x + \ldots\ldots$

$= .736 \sin x - .018 \sin 2x + \ldots$

$\underline{t=0}$



$\underline{t=1}$



Because of rapid decay of 2nd term, soln even @ t=1 is essentially just 1st term.

M294 SP96 F #5

49) T, T, F, F, F, T, T, T, T, T

M294 SP97 P1 #3

Problem 50)

It is an odd function $\Rightarrow$ $a_0 = 0$, $a_n = 0$ $\qquad p = 2\pi$

$$b_n = \frac{1}{\pi} \cdot 2 \int_0^{\pi} f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^{\pi} 3 \sin nt \, dt = \frac{-6}{n\pi} \left[\cos nt \right]_0^{\pi}$$

$$b_n = -\frac{6}{n\pi} \left[\cos n\pi - 1 \right]$$

i.e. $$b_n = -\frac{6}{n\pi} \left[(-1)^n - 1 \right]$$

b-) $n = 1$:

$$b_1 = -\frac{6}{n\pi} \left[-1 - 1 \right] = \frac{12}{\pi}$$

$n = 3$:

$$b_3 = -\frac{6}{3\pi}(-2) = \frac{4}{\pi}$$

$n = 5$:

$$b_5 = \frac{12}{5\pi}$$

$n = 2$:

$$b_2 = 0$$

$n = 4$:

$$b_4 = 0$$

5 a) $L = \pi$

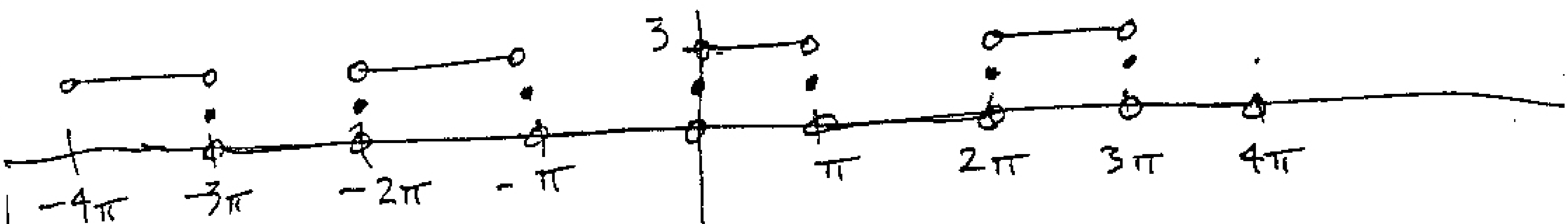
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi}\int_0^{\pi} 3 \, dx = 3$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 3\cos(nx)\,dx = \frac{3}{\pi}\left.\frac{\sin(nx)}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 3\sin(nx)\,dx = \frac{3}{\pi}\left.\frac{\cos(nx)}{-n}\right]_0^{\pi} = \frac{3}{\pi}\,\frac{\cos(n\pi)-1}{-n}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{6}{n\pi} & n \text{ odd} \end{cases}$$

$$f(x) \sim \frac{3}{2} + \frac{6}{\pi}\sin(x) + \frac{6}{3\pi}\sin(3x) + \cdots$$



series converges to $3/2$ at $x = 0$.

b) $u_t + u = 3u_x$ If $u(x,t) = X(x)T(t)$ then

$$XT' + XT = 3X'T$$

$\div XT$ $$\frac{T'}{T} + 1 = \frac{3X'}{X} = \text{constant } \lambda$$

$$\boxed{3X' = \lambda X} \qquad \boxed{T' + T = \lambda T}$$

M294 FA92 F #7 Section 5.1

56) (a) -1, $-\pi/3$, $\pi/3$; 0, 0, π ; 1, 0.2, 0.2
(jumps)

(b) $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x)$ $-\pi \le x \le \pi$ because