Wave M294 P II SP87 #2

5) $\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x^2}$ $\qquad 0 = u(0,t) = u(3\pi, t)$

Guess $u = XT$

$$\frac{T''}{2T} = \frac{X''}{X}$$

$X(0) = X(3\pi) = 0 \Rightarrow X_n(x) = A_n \sin\frac{n\pi x}{3\pi} = A_n \sin\frac{nx}{3}$

$\frac{T''}{2T} = -\left(\frac{n}{3}\right)^2 \Rightarrow T_n(t) = B_n \sin\frac{\sqrt{2}n}{3}t + C_n \cos\frac{\sqrt{2}n}{3}t$

$u(x,t) = \sum_{n=1}^{\infty}\left(\tilde{A}_n \sin\frac{\sqrt{2}n}{3}t + \tilde{B}_n \cos\frac{\sqrt{2}n}{3}t\right)\sin\frac{nx}{3}$

$u(x,0) = \sum_{n=1}^{\infty}\tilde{B}_n \sin\frac{nx}{3} = \sin 5x \Rightarrow \tilde{B}_n = \begin{cases}1 & n=15\\ 0 & n\neq 15\end{cases}$

$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\frac{\sqrt{2}n}{3}\left(\tilde{A}_n \cos\frac{\sqrt{2}n}{3}t + \tilde{B}_n \sin\frac{\sqrt{2}n}{3}t\right)\sin\frac{nx}{3}$

$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\frac{\sqrt{2}n}{3}\tilde{A}_n \sin\frac{nx}{3} = \sin x \to \frac{\sqrt{2}n}{3}\tilde{A}_n = \begin{cases}1 & n=3\\ 0 & n\neq 3\end{cases}$

$\Rightarrow \tilde{A}_3 = \frac{\sqrt{2}}{2}$ and $\tilde{A}_n = 0$ $n\neq 3$

$\therefore u(x,t) = \frac{\sqrt{2}}{2}\sin\sqrt{2}t \sin x + \cos 5\sqrt{2}t \sin 5x$

$\therefore \boxed{u(1,1) = \frac{\sqrt{2}}{2}\sin\sqrt{2}\sin 1 + \cos 5\sqrt{2}\sin 5}$

M294 F SP87 #7

5) $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$
and $u(0,0) \neq u(1,0)$.

Let $u = XT$. $\Rightarrow \frac{X''}{X} = \frac{T''}{4T} = \alpha$

Since $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$,
$X'(0) = X'(1) = 0.$

So $\alpha = -\lambda^2$.

$\Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$

$X'(x) = \lambda(-A\sin\lambda x + B\cos\lambda x)$

$X'(0) = B\lambda = 0 \Rightarrow B = 0$.

$X'(1) = -A\lambda\sin(\lambda 1) = 0 \Rightarrow \lambda = n\pi$.

So $X_n(x) = A_n\cos n\pi x$

Then $T'' = -(2n\pi)^2 T$.

$\Rightarrow T_n(t) = B_n\cos 2\pi n t + C_n\sin 2\pi n t$.

Let $n = 1$. (Any odd n would work; B_1, C_1 also arbitrary.)

Take $u(x,t) = \cos\pi x\,(\cos 2\pi t)$ [$B_1 = 1$ all other Bs & Cs $= 0$]

Then $u(0,0) = 1$

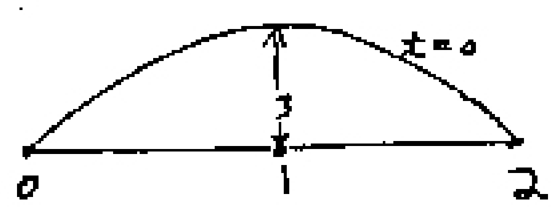
$u(1,0) = -1$.

$\boxed{\text{One solution is } u = \cos(\pi x)\cos(2\pi t)}$

M294 P II SP88 #5

8) $u(x,0) = 3\sin\frac{\pi x}{2}$

As x goes from 0 to 2, $\frac{\pi x}{2}$ goes from 0 to π which is a half cycle of the sin which looks like



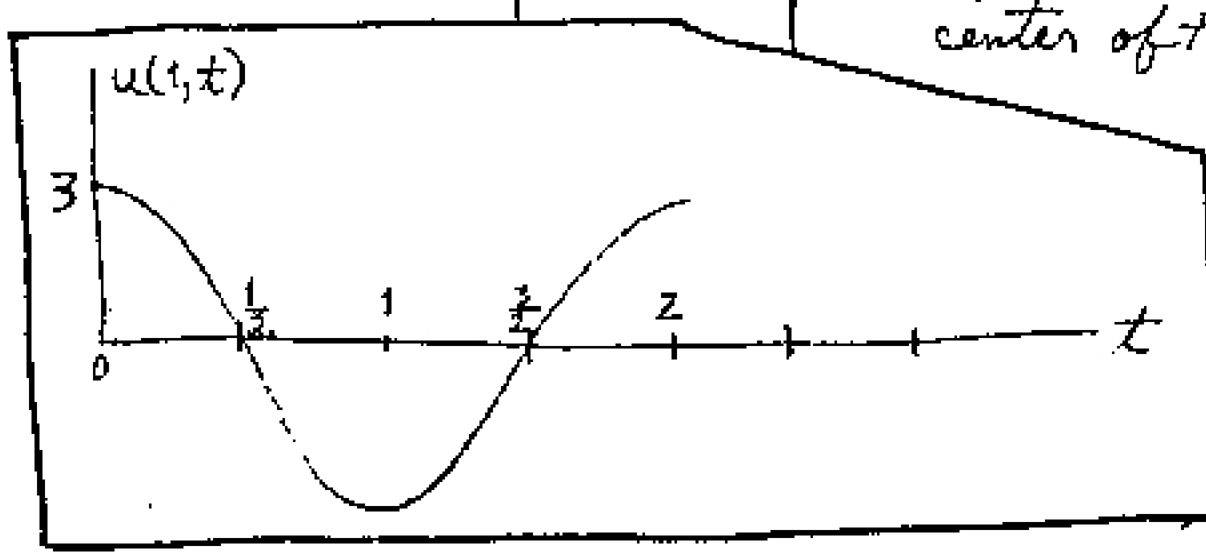
$\ell = 2$

The soln to $\begin{cases} u_{tt} = \overset{\alpha^2}{4} u_{xx} \\ u_t(x,0) = 0 \end{cases}$ $\alpha = 2$

with this $u(x,0)$ is $\left\{\text{picking term of form } b_n \sin\left(\frac{n\pi x}{\ell}\right)\cdot\cos(n\pi\alpha t/\ell)\right\}$

$3\sin\frac{\pi x}{2}\cos\pi t$

So $u(1,t) = \boxed{3\cos\pi t}$ = displacement of the center of the string



11) $u(x,t) = 8\sin 13\pi x\cos 13\pi t - 2\sin 31\pi x\cos 31\pi t$
$- 8\dfrac{\sin 8\pi x\sin 8\pi t}{8\pi} + 12\dfrac{\sin 88\pi x\sin 88\pi t}{88\pi}$

because (i) each $\sin\lambda x\cos\lambda t$ or $\sin\lambda x\sin\lambda t$ solves $u_{tt} = u$ and linearity, (ii) $u(0,t) = 0 = u(1,t)$ because $\sin n\pi = 0$, and (iii) $u(x,0) = 8\sin 13\pi x - 2\sin 31\pi x$ ✓
$u_t(x,0) = -8\sin 8\pi x + 12\sin 88\pi x$ ✓

M294 F FA95 #5

15) $u(x,t) = F(x+t) + G(x-t)$

(a) $u(x,0) = F(x) + G(x)$, $u_t(x,0) = F'(x) - G'(x)$

(b) $u_t(x,0) = 0$ gives $F'(x) = G'(x)$, so $F(x) = G(x) + C$

Then $u(x,0) = e^{-x^2}$ gives $G(x) + C + G(x) = e^{-x^2}$ so you may take $G(x) = \frac{1}{2}e^{-x^2}$, $C = 0$, $F(x) = \frac{1}{2}e^{-x^2}$, and get

$u(x,t) = \frac{1}{2}e^{-(x+t)^2} + \frac{1}{2}e^{-(x-t)^2}$