TAM 2030 Dynamics Notes, Spring 2013, Cornell University, Ithaca, NY

Andy Ruina Lectures, Kevin Kircher recitations

DYNAMICS NOTES : Ruina @ cornell

Spring 2013

Kevin Kircher
OLH
Mon 10-12
Tues 1-2:30
Thur 102

①

TAM 2030 — HW Tues in lecture

Lecture 1

google Ruina   register i-clickers!

Date: No.

Mechanics	vocab		Equations (typical)
I. · material properties / constitutive laws / Force laws	eg. spring (k) dashpot (c) rigid	gravity (g/G) drag law μ (friction)	$T=k\Delta l$ "$F=kx$" $T=c\dot{l}$, $F=mg$, $F=-\frac{mMG}{r^2}$
II. · geometry / time kinematics	strain, velocity, angular velocity, acceleration, etc.		$\vec{a}=\dot{\vec{v}}$, $\vec{v}=\dot{\vec{r}}$, $v=\dot{x}$ $\epsilon=\frac{\Delta l}{l}$
III. · laws of mechanics / kinetics	momentum, kinetic energy, angular momentum.		$\sum\vec{F}=m_t\vec{a}_G$, $\dot{E}_k=P_{net}$ $\sum\vec{M}_{/c}=\dot{\vec{H}}_{/c}$

The 3 Pillars

STATICS vs DYNAMICS

- useful ☺   - not so useful ☹
- linear momentum balance $0=\sum_{\text{external}}\vec{F}$ $= m_t a_G$ (total mass × acceleration of center of gravity)

$$\vec{0} = \sum_{\text{external}}\vec{M}_{/c} = \sum\vec{r}_{i/c}\times m_i\vec{a}_i$$

(angular momentum balance)

GOAL FOR MECHANICS

- solve a general mechanics problem : given info about geometry, properties and forces, find other info about them.

SKILLS

- vectors : dot, cross, unit vectors ($\hat{n}$)
- ODEs : $x''+x=0$.
- Matlab : Plotting, animation, solve diff eq.

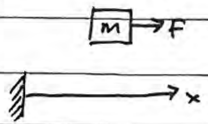
COMMON HW MISTAKES (things to learn to get right)

1) FBD always required with forces / angular momentum.
   large and clear? Did I check with someone?
2) Units must always be included and check out.
3) Differentiate btw $\overrightarrow{\text{vectors}}$ and scalars

1D DYNAMICS

- informal about vector notation for now.

look at a single "particle"



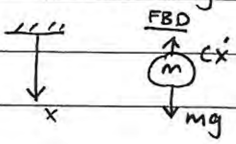
Basic Eqns: $\dot{v}$ or $v' = \frac{dv}{dt}$

$F = ma = mv' = m\ddot{x}$

$a = v'$

$v = x'$

eg. Particle falling in honey



(linear viscous honey)

linear momentum balance:

$\sum_{ext.} F = m_t a_G$

$= m\ddot{x}$

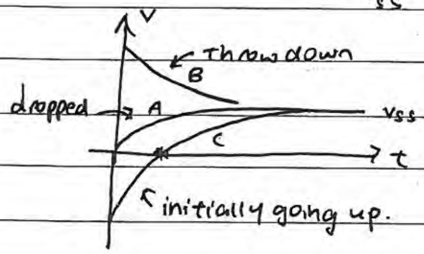
$m\ddot{x} + c\dot{x} = mg$

$m\dot{v} + cv = mg$. - 1st order

$\dot{v} = \frac{mg - cv}{m}$

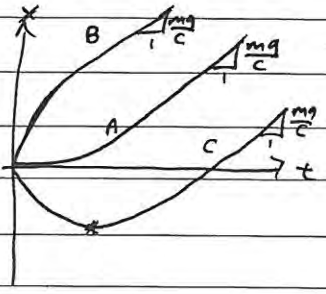
Find steady state speed: $mg - cv_{ss} = 0$

$v_{ss} = \frac{mg}{c}$ (terminal velocity)



velocity vs t

*



position vs t

DISCUSSION - 1D mechanics $^*\dot{(\,)} = \frac{d}{dt}(\,)$

Position: $\vec{r} = x\hat{\imath}$

velocity: $\vec{v} \equiv \dot{\vec{r}} = \frac{d}{dt}\vec{r} = \dot{x}\hat{\imath} = v\hat{\imath}$

acceleration: $\vec{a} \equiv \ddot{\vec{r}} = \frac{d^2}{dt^2}\vec{r} = \ddot{x}\hat{\imath} = \dot{v}\hat{\imath} = a\hat{\imath}$

$$\boxed{\begin{array}{l} v = \dot{x} \Leftrightarrow x(t) = x_0 + \int_{t_0}^{t} v(\tau)\,d\tau \\ \\ a = \dot{v} \Leftrightarrow v(t) = v_0 + \int_{t_0}^{t} a(\tau)\,d\tau \end{array}}$$

Recitation

QUESTIONS

Date: No.

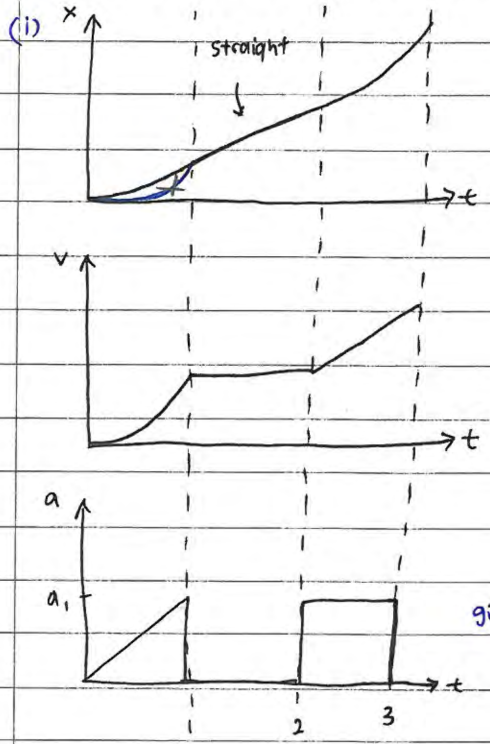
- Given the following a(t), with $a_1 = 6\ m/s^2$ and $v_0 = 0\ m/s$, (i) plot v(t), x(t)

(ii) Find v(t) for 0s-1s, 1s-2s, 2s-3s

(iv) Find displacement from t=0s to t=3s.

(iii) Find speed at t=3s.

(i)



* cubic - slower at first, then ~~steep~~.

given

$$\text{(ii)}\ a(t) = \begin{cases} \cancel{y=6x}\ \frac{6\,m/s^2}{1\,s}\,t & 0 \le t \le 1 \\ 0 & 1 \le t \le 2 \\ 6\,m/s^2 & 2 \le t \le 3 \end{cases}$$

$$v(t) = \begin{cases} 3t^2 & 0 \le t \le 1 \\ 3\,m/s & 1 < t \le 2 \\ \cancel{3+6t} & 2 < t \le 3 \end{cases}$$

$$v_0 + \int_2^t 6\,dt = 3 + 6(t-2) = 6t - 9$$

(iii) 9 m/s

(iv)

Lecture 2

viscosity

$F_D = C_D \nu A V$ viscous friction $\Rightarrow cv$ viscal low Reynold's no.

$= C_D \rho_f A V|V|$ $\Rightarrow cv|v|$ inertial quadratic, high R no.

two choices

where C_D = coefficient of drag (depends on shape)

ν = viscosity

ρ_f = density of fluid.

LINEAR VISCOUS CASE

$C = C_D \nu A$

LMB : $\Sigma F = ma \Rightarrow mg - cv = m\dot{v}$

$\dot{v} = g - \frac{c}{m} v$ (to obtain number solution)

$\dot{v} + \frac{c}{m} v = g$ (for mathematical ODE)

2 ways to write ODE

$V = \dot{x}$

homogeneous

means any particular solution

$v(t) = V_h + V_p$

$\frac{dv}{dt} + \frac{c}{m} v = g.$

$\mu = e^{\frac{c}{m}t}$

$e^{\frac{c}{m}t} v = \int e^{\frac{c}{m}t} g \, dt + C =$ ~~$e^{\frac{c}{m}t} gt + C$~~

~~$v = gt + Ce^{-\frac{c}{m}t}$~~

$\frac{mg}{c} e^{\frac{c}{m}t} + C$, $v = \frac{mg}{c} + Ce^{-\frac{c}{m}t}$

Try to do w/out lecturer help.

$V_p = \frac{mg}{c}$ particular soln.

$V_h = C_1 e^{-\frac{c}{m}t}$ homo. soln.

$v(t) = C_1 e^{-\frac{c}{m}t} + \frac{mg}{c}$, since $v(0) = 0$,

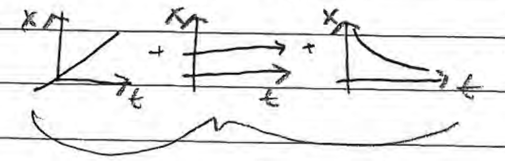
$v(t) = \frac{mg}{c}(1 - e^{-\frac{c}{m}t})$ soln. of ODE w/ I.C.

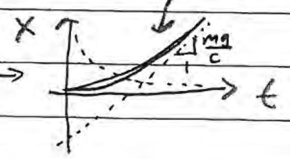
$x = x(0) + \int_0^t v(t')\, dt'$, $x(0) = 0$ Integrate to find position

$x(t) = \int_0^t \frac{mg}{c}(1 - e^{-\frac{c}{m}t'})\, dt'$

$= \frac{mg}{c} t' \Big|_0^t + \Big[e^{-\frac{c}{m}t'} (\frac{m}{c})(\frac{mg}{c})\Big]_0^t$

$= \frac{mg}{c} t - \frac{m^2 g}{c^2} + \frac{m^2 g}{c^2} e^{-\frac{c}{m}t}$





soln. for pos vs time

A

(see * on pg. 2)

the 3 terms in the ODE soln. for x(t)

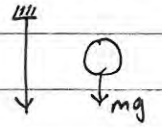
coeff of viscous drag.

$[C_D] = [\frac{F}{v}] = [\frac{\text{mass}}{\text{time}}]$

$\underbrace{\text{arguments of sines and exponentials have to be dimensionless}}$ → CHECK.

← if not ⇒ mistake

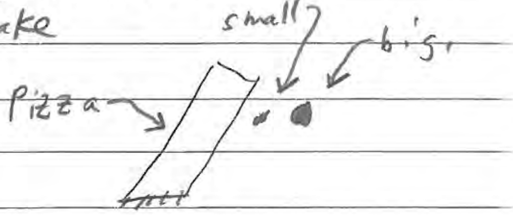
eg. Galileo's falling balls (neglect air friction)



$\sum F = ma$

$mg = m\dot{v}$

$\dot{v} = g$ → no mass in eqn, so all balls fall the same



eg. with air friction, quadratic drag.

$m = \frac{4}{3}\pi r^3 \rho_b$ Area $= \pi r^2$

LMB $mg - F_D = ma$

$$\frac{4}{3}\pi r^3 \rho_b\, g - C_D \rho_f \pi r^2 v|v| = m\dot{v}$$

$$\dot{v} = \frac{1}{\frac{4}{3}\pi r^3 \rho_b}\left[\frac{4}{3}\pi r^3 \rho_b\, g - C_D \rho_f \pi r^2 v|v|\right]$$

$$= g - C_D\left(\frac{3}{4}\right)\frac{\rho_f}{\rho_b}\left(\frac{1}{r}\right) v|v|$$

↳ smaller for big balls. ∴ for same material, bigger balls fall faster. but for cones, mass $\propto r^2$ ∴ drag $\propto$ weight

see demo on www site)

1D - 1 deg of freedom ← general case — **Different Cases**

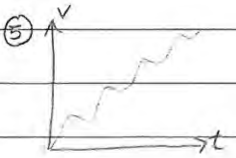
▨ → $F(x, v, t)$

$F = ma \Rightarrow a = \frac{F}{m} = \dot{v}$

$\dot{x} = v$.

$$\boxed{\begin{aligned} V_{new} &= V_{old} + a.\Delta t \\ x_{new} &= x_{old} + V.\Delta t \end{aligned}}$$

Euler's method



1) $F = 0.,\ a = 0$, v constant
2) $F =$ constant, $a =$ constant $= \frac{F}{m}$
$v = v_0 + a_0 t$, $x = x_0 + v_0 t + a_0 t^2/2$.
3) $F = -cv$ (viscous drag).
$\dot{v} = \frac{F}{m} = -\frac{c}{m}v \Rightarrow v = v_0 e^{\frac{c}{m}t}$
4) $F = -cv|v|$ (quadratic drag)
5) $F = F(t) = a \cos t$ (oscillating)
6) $F = -kx$, $x(t) = \overset{A}{\_}\cos(\underline{\ }t) + \overset{B}{\_}\sin(\underline{\ }t)$.
$\sqrt{\frac{k}{m}}$

Special cases

MATLAB

Date: No.

1 DOF motion in 1D. ODE solving

```
xnew = x+v*h ; % h is timestep (Δt)
vnew = v + a*h ;
tnew = t+h;
```

} basic idea of calculations.

- look at file posted online.

```
F = -k*x;
a = F/m;
```

DISCUSSION 2. (Recitation)

- ODE review

✓ ?

1) $$\begin{cases} \ddot{x} = \gamma \dot{x}^2 & [\gamma] = \frac{1}{m} \\ x(0)=0,\ v(0)=v_0 \end{cases}$$ (negative quadratic drag)

$$\dot{x} = v$$

$$\dot{v} = \gamma v^2 \Leftrightarrow \frac{dv}{dt} = \gamma v^2$$

$$\int_{v_0}^{v} \frac{dv}{v^2} = \int_0^t \gamma\, dt$$

$$\left[-\frac{1}{v}\right]_{v_0}^{v} = \gamma t' \Big|_0^t$$

$$-\frac{1}{v} + \frac{1}{v_0} = \gamma t$$

$$\frac{1}{v} = \frac{1}{v_0} - \gamma t$$

$$v(t) = \frac{1}{1/v_0 - \gamma t}$$

$$\frac{dx}{dt} = \frac{1}{1/v_0 - \gamma t} \cancel{= \frac{1}{\frac{1-v_0\gamma t}{v_0}}} \cancel{= \frac{v_0}{1-v_0\gamma t}}$$

$$\cancel{\int \frac{1}{1-v_0\gamma t} dt = \int \frac{1}{v_0} dx}$$

$$\int_0^t \frac{1}{1/v_0 - \gamma t} dt = \int_0^x 1\, dx$$

$$\left[-\frac{1}{\gamma} \ln\left|\frac{1}{v_0} - \gamma t\right|\right]_0^t = x - 0$$

$$-\frac{1}{\gamma}\left[\ln\left|\frac{1}{v_0} - \gamma t\right| - \ln\left|\frac{1}{v_0}\right|\right] = x$$

$$-\frac{1}{\gamma}\left[\ln\left|\frac{1}{v_0} - \gamma t\right| + \ln v_0\right] = x$$

$$-\frac{1}{\gamma} \ln|1 - v_0\gamma t| = x(t).$$

↳ check units $[v_0 \gamma t] = \frac{1}{m} \cdot \frac{m}{s} \cdot s = \overset{1}{\cancel{0}}$ dimensionless

2) MATLAB

$$\left.\begin{aligned}\dot{x} &= v\\ \dot{v} &= \gamma v^2\end{aligned}\right\}$$ problem setup for MATLAB.

```
% Define time span we're interested in, let's say 0 - 10sec.
tspan = linspace (0, 10, 1000);
% Define initial conditions.
   x0 = 0;   v0 = 400;
   z0 = [x0 ; v0] ;     % column vector.
% Solver
   [t  zarray] = ode45(@rhs , tspan, z0);          % zarray will be
```

$$\begin{bmatrix} x(0) & v(0)\\ x(0.001) & v(0.001)\\ \vdots & \vdots\\ x(10) & v(10)\end{bmatrix}$$

```
% Plot
   x = zarray(: , 1) ;
   plot(t, x)


 function  zdot = rhs(t, z)
        x = z(1) ;
        v = z(2) ;
        xdot = v;
        vdot = γv^2;        % depends on what acceleration function is.
        zdot = [xdot ; vdot ];
 end
```

ENERGY METHODS

Definitions : Power $P = \vec{F}\cdot\vec{v} = Fv \text{ (1D)}$, $[P] = W = 1\ J/s$

Work $\Delta W = \vec{F}\cdot\Delta\vec{r} = F\Delta x$, $[W] = J = 1\ N\cdot m$

$W = \int \vec{F}\cdot d\vec{r} = \int_{x_0}^{x} F\,dx$

Kinetic E $E_k = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}mv^2$, $[E_k] = J$.

Potential E $E_p = -\int_{x_0}^{x} F(x')\,dx'$

gravitational $E_{p,grav} = mgh$

spring $E_{p,spring} = \frac{1}{2}k(\Delta l)^2$ $[k] = \frac{N}{m}$

Recitation cont'd

Useful Formulae

1) $P = \dot{E}_k$
$\uparrow$ power

$$\dot{E}_k = \frac{d}{dt}(\frac{1}{2}mv^2) = m\frac{d}{dt}(\frac{1}{2}v^2) = m(\frac{1}{2})(2v)(\dot{v}) = mva = Fv$$

$$* \frac{d}{dt}(\frac{1}{2}v^2) = va.$$

2) $W = \Delta E_k$
$\uparrow$ work

3) No dissipation → KE + PE consistent
conservation of energy.

Lecture 4

QUIZ: $\ddot{x} = x$

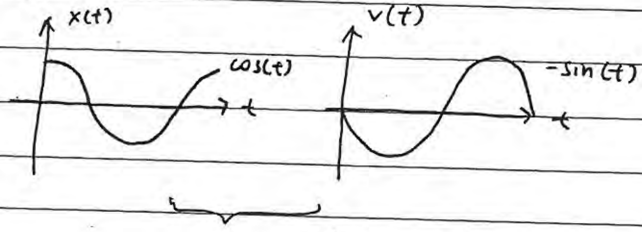
a) $\sin(t)$
b) $\cos(t)$ } for $\ddot{x} = -x$ (harmonic)
c) e^t
d) e^{-t}
e) More than one or none of above. ✓

>> dbquit Matlab hint (if stuck in debugger)

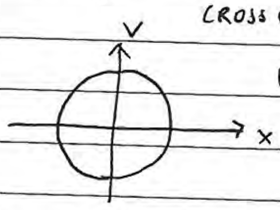
When using MATLAB (euler's method) for harmonic oscillator equation, why does total energy increased over time?

$$\ddot{x} = -x$$

$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -x\end{aligned}}$$

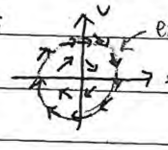


CROSS PLOT.

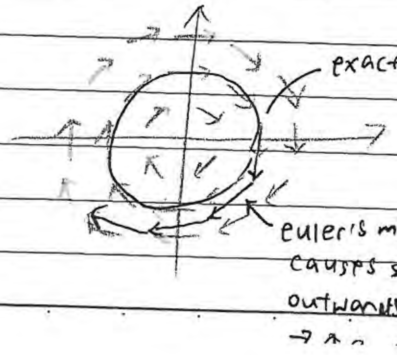


Phase plane

or look at ODE



exact solution



exact solution is circle

euler's method causes spiralling outwards

Harmonic Oscillator

From FBD & LMB,

ODE $m\ddot{x} = -kx$

$m\dot{x}\ddot{x} = -kx\dot{x}$

$m\frac{d}{dt}(\frac{1}{2}\dot{x}^2) = -k\frac{d}{dt}(\frac{1}{2}kx^2)$

$\frac{d}{dt}(Ek) = \frac{d}{dt}(E_p)$

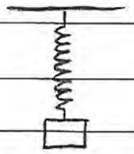
↳ kinetic energy ↳ potential energy

$\frac{d}{dt} E_{total} = 0 \Rightarrow E_{total} = constant$

Feb 5, 2013

Lecture 5

- 1 DOF oscillator; forcing and damping
- MATLAB for this.



DEMO

If you hold it completely still, movement will damp out (friction)

Resonance – at natural frequency, hand barely has to move up and down to produce huge movements.

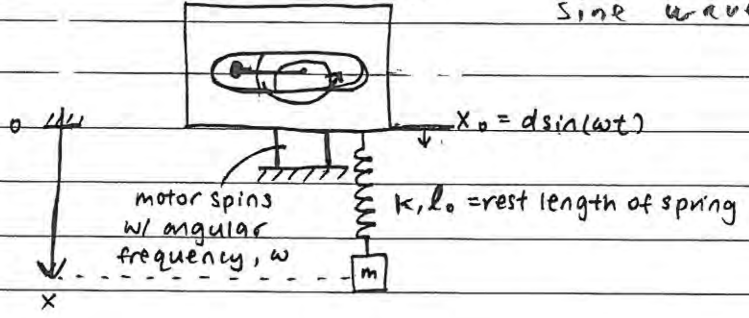
If you move your hand up and down very slowly, load moves up and down with hand.

At high frequencies, load moves up and down a bit. (opposite hand motion)

HANGING MASS

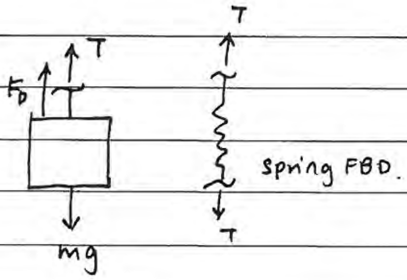
- with oscillating support
- with air friction

"Scotch Yoke" to generate sine wave



FBD

$$T = k(l - l_0)$$
$$= k\underbrace{((x - x_0) - l_0)}_{\Delta l = \cancel{\text{increase}}\ \text{stretch of spring}}$$



$$F_D = c\dot{x}$$

LMB: $\Sigma F = ma$

$$mg - T - F_D = m\ddot{x}$$

$$mg - k(x - x_0 - l_0) - c\dot{x} = m\ddot{x} \quad ①$$

2 standard ways to rewrite ① :

A) Standard linear ODE way

$$m\ddot{x} + c\dot{x} + kx = mg + k\underset{\uparrow\, d\sin\omega t}{x_0} + k\ell_0$$

For vibrations, interested in particular solution of

$$m\ddot{x} + c\dot{x} + kx = d\sin(\omega t)$$

$$\Rightarrow \quad x = A\cos(\omega t) + B\sin(\omega t)$$

Key result : $A^2 + B^2$ is big if ω is close to $\sqrt{\frac{k}{m}}$ → RESONANCE

B) Numerical solution

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= [mg - k(x - d\sin\omega t - \ell_0) - cv]/m \end{aligned}\right\} \text{2 first order ODEs.}$$

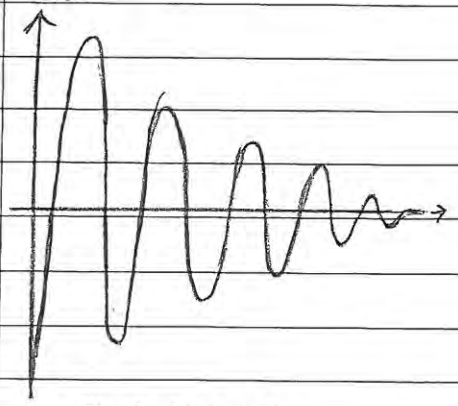
→ can be done with MATLAB. ☺

(1c)

lots of examples on course www site.

Basic Idea for Code (details posted online)

```
L0 =                  % rest length
d =                   % forcing amplitude
omega =               % forcing fequency
m =                   % mass
k =                   % spring constant
g =                   % gravity
c =                   % drag constant
h =                   % timestep size = Δt
t0 =      ; t1 =      % interval to solve ODE.

n = (t1-t0)/h         % number of timesteps.
n=floor(n)

% initialize  t, x, v arrays , 1st components to be t0, x0, v0 respectively.

% for loop. calulating x and v values for various t values. call Function.
%           using laws of mechanics and kinematics:
                a = F/m;
                xnew = x + v*h;
                vnew = v + a*h;

% Plot results

function F=rhs(x,v,t,k,c,d,L0,m,g,omega)
     F = -k*(x-d*sin(omega*t)-L0)-c*v ± m*g;
end
```

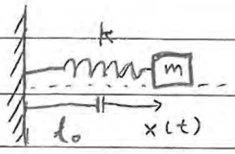


some damping.

Recitation

Discussion 3 - HARMONIC OSCILLATOR

- Derivation

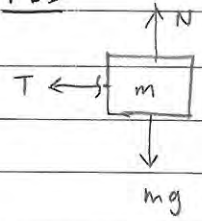


Experimental "fact" : spring force = (spring constant) * (stretch)

$$T = \underset{\uparrow \text{ linear}}{k}x$$

* Not all springs are like this. Some have quadratic / cubic terms.

FBD



LMB

$\Sigma F = ma$

$-T = m\ddot{x}$

EOM

$m\ddot{x} = -kx$

$\ddot{x} + \frac{k}{m}x = 0.$

Solution

$$x(t) = A\cos(\lambda t + \phi) \underset{\text{trig. identity}}{\Longleftrightarrow} x(t) = B\cos(\lambda t) + C\sin(\lambda t)$$

- DEFINITIONS & FORMULAE

Spring constant, k $[\frac{N}{m}]$

rest length , l_0 [m]

oscillation frequency, $\lambda = \sqrt{\frac{k}{m}}$ $[\frac{rad}{s}]$

Period , $T = \frac{2\pi}{\lambda}$ [s]

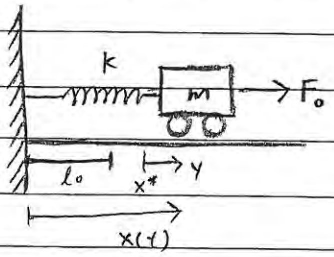
Amplitude , A [m] - determined from initial conditions

Phase angle , ϕ [rad] - determined from initial conditions.

Kinetic Energy, $E_k = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\lambda^2\sin^2(\lambda t + \phi)$ [J]

Potential Energy , $E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\lambda t + \phi)$

- EXAMPLE : CART & SPRING



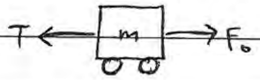
A cart of mass m, attached to spring (k, l_0), is driven by a constant force F_0.

I.C. : $x(0)=L$

$\dot{x}(0)=0$

a) FBD → LMB → EoM → Soln.

b) Find E_k, E_p, check E_{total} = constant (no friction)



$\Sigma F=ma$

$F_0-T=m\ddot{x}$

$m\ddot{x}=F_0-k(x-l_0)$

$\ddot{x}+\frac{k}{m}x=(F_0+kl_0)\frac{1}{m}$

SOLUTION: change of variables

(1) Find equilibrium position, x^*

(2) Change variables to $y=x-x^*$ ⇒ $\ddot{y}=\ddot{x}$, $\dot{y}=\dot{x}$

(3) Use ICs to resolve A, ϕ ⇒ $y(0)=L-x^*$, $\dot{y}(0)=0$

(4) Check E_k+E_p = constant.

⇒ Another way: x^* is a particular solution, just plug it into a homogeneous solution.

@ equilibrium, $\ddot{x}=0$

(1) $\frac{k}{m}x^*=\frac{F_0+kl_0}{m}$

$x^*=\frac{F_0}{k}+l_0$

(2) $\ddot{y}+\frac{k}{m}\left[y+\frac{F_0}{k}+l_0\right]=\frac{F_0+kl_0}{m}$

$\ddot{y}+\frac{k}{m}y+\frac{F_0}{m}+\frac{kl_0}{m}=\frac{F_0+kl_0}{m}$

$\ddot{y}+\frac{k}{m}y=0$

$\Rightarrow y(t)=A\cos(\lambda t+\phi)$

$\dot{y}=-\lambda A\sin(\lambda t+\phi)$

$\dot{y}(0)=-\lambda A\sin(\lambda t+\phi)=0 \Rightarrow \phi=0$

$y(0)=A\cos(\lambda t)=A=L-\frac{F_0}{m}-l_0$

Lecture 6

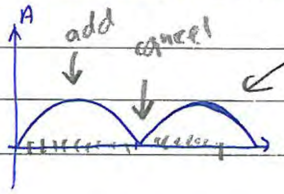
$k=1$, $m=1$, $\omega=1.01$ (very close to resonance frequency)

$m\ddot{x}+kx=\sin\omega t$

$x(0)=0$

$\dot{x}(0)=0.$

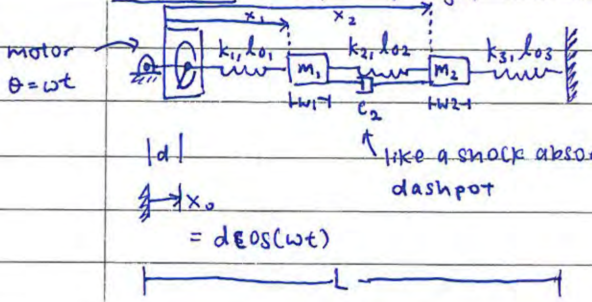
⇒ what is amplitude of oscillation? "beating"



"beat" phenomenon: homog. soln & part. soln. cancel then add then cancel etc

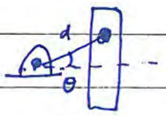
↑ every 100 cycles?

EXAMPLE – 1D, 3 springs, 2 masses.

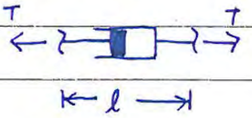


$= d\cos(\omega t)$

↑ like a shock absorber, dashpot



Dashpot

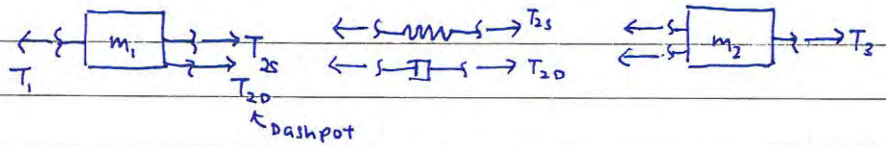


$T=c\dot{l}$

FBDs

mass 1

mass 2



* sign convention
- pulling away : +ve
- getting longer : +ve

pretending that we know x and v at time t,

$$T_1 = k_1(\Delta l_1) = k_1(l_1 - l_{01}) = k_1(x_1 - x_0 - l_{01})$$

$$= k_1(x_1 - d\cos\omega t - l_{01})$$

$$T_{2S} = k_2(l_2 - l_{02}) = k_2(\underbrace{x_2 - (x_1 + w_1)}_{l_2} - l_{02})$$

$$T_{2D} = c_2(\dot{l}_2) = c_2(\dot{x}_2 - \dot{x}_1)$$

$$T_3 = k_3(\Delta l_3) = k_3(x_3 - (x_2 + w_2) - l_{03})$$

LMB

mass 1 $\Sigma F = ma$

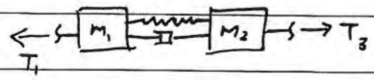
$$T_{2S} + T_{2D} - T_1 = m_1\ddot{x}_1$$

$$T_3 - T_{2S} - T_{2D} = m_2\ddot{x}_2$$

↑ substitute in.

CODE to solve

⇒ If Draw FBD of system:



$$\sum F = \dot{L} = m_1 a_1 + m_2 a_2$$

$$T_3 - T_1 = m_1 \ddot{x}_1 + m_2 \ddot{x}_2$$

↳ Not a new equation! (comes from adding LMB from the separate masses)

Set up eqns.

$$\left.\begin{aligned} T_1 &= \dots \\ T_{2s} &= \dots \\ T_{2D} &= \dots \\ T3 &= \dots \end{aligned}\right\} \text{known in terms of } x_1, x_2, \ddot{x}_1, v_2, t, \text{parameters}$$

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = (T_{2s} + T_{2D} - T_1)/m_1$$

$$\dot{x}_2 = v_2$$

$$\dot{V}_2 = (T_3 - T_{2s} - T_{2D})/m_2$$

$$\text{let } z = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

$$\dot{z} = f(z, t, \text{parameters})$$

↑ $F = ma$, $\dot{x} = v$

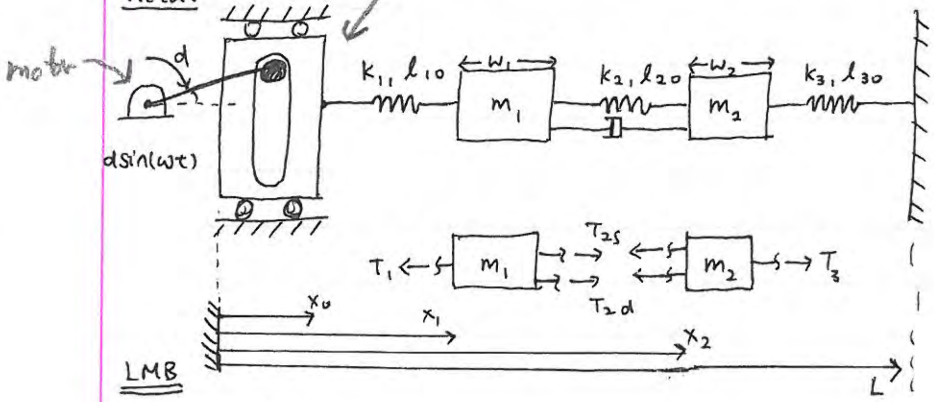
lecture 7

2/12

- 2DoF continued with Matlab
- collisions. "x = x"

"Scotch Yoke" again

Recall



LMB

Mass 1: $\Sigma F = ma$

$$-T_1 + T_{2s} + T_{2d} = m_1 \ddot{x}_1$$

Mass 2:

$$-T_{2s} - T_{2d} + T_3 = m_2 \ddot{x}_2$$

$$T_1 = k_1(\overbrace{x_1 - x_0}^{l_1} - l_{10})$$

* $x_1 - x_0$ is NOT l_{10}, it is l_1

$$T_{2s} = k_2(l_2 - l_{20}) = k_2(\overbrace{x_2 - x_1 - w_1}^{l_2} - l_{20})$$

$$T_{2d} = c_2(\dot{l}_2) = c_2(\dot{x}_2 - \dot{x}_1)$$

$$T_3 = k_3(l_3 - l_{30}) = k_3(\underbrace{L - x_2 - w_2}_{l_3} - l_{30})$$

Put them into Matlab:

$\dot{x}_1 = v_1$

$\dot{x}_2 = v_2$

$\dot{v}_1 = (\ )/m_1$

$\dot{v}_2 = (\ )/m_2$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} \Rightarrow \dot{z} = f(\underset{\substack{\uparrow \\ x's,\, v's}}{z}, t, p) \nwarrow k's, m's, c's$$

$$z_{new} = z_{old} + \dot{z}(\Delta t)$$

Euler's Method (in a nutshell)

```
Loop
  z_{i+1} = z_i + zdot · h  ← Δt
end
```

(that's the whole thing!)

Matlab

can create structures to store different properties. Here we use p, storing mass, k, c, omega, d, w, L.

```
function  zdot=rhs(t,z,p)
     x = z(1:2);   v= z(3:4);
     xdot = v;                        % 1st 2 ODEs
     v1dot = (  )/p.m1;             where () is T2s + T2d - T1  } next 2
     v2dot = (  )/p.m2;                   () is T3 - T2d - T2s  } ODEs.
     vdot = [v1dot  v2dot];                * sub in expressions;
     zdot = [xdot  vdot];                    T1 = p.k1*(x(1) - x0 - p.L10)
end                                              where x0 = p.d * cos (p.omega * t)
                                             T2s = p.k2*(x(2) -(x(1) + p.w1) - p.L20)
                                             T2d = p.c2 * (v(2) - v(1));
                                             T3 = p.k3 *(p.L -(x(2) +p.w2) - p.L30)
```

To predict the future,

→ znew = z + zdot * h

repeatedly in a 'for i=2:n' loop,

updating z and t values for each iteration.

zarray (i,:) = znew

* Details posted online.

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Lecture 8 Feb.14,

2/14.

- 2DoF continued
- Collisions

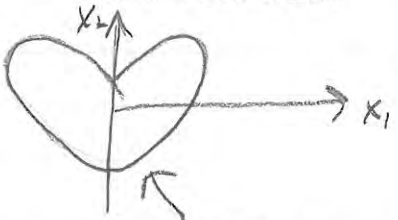
Recall previous setups.

Now, given $m_2 = 0.44m_1$, $k_1 = k_3$, $k_2 = 0$. ← 2 independent oscillations

$d = 0$, $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 2 sine waves with diff frequencies.

cross plot x_2 vs x_1.

$0 \le t \le 6.3 \approx 2\pi$.



⇐ x_1 makes about 1 oscillation

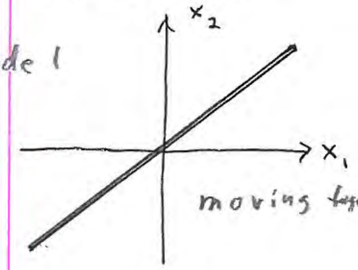
x_2 makes about $\sqrt{1/.44} \approx 1.5$ oscillations

Valentine by ODE solution

Special Solutions

Normal modes

mode 1



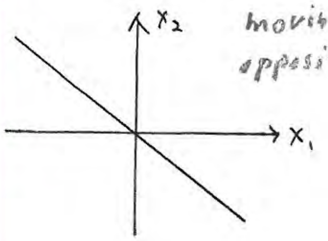
moving together

masses move synchronously.

$$x(t) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \cos(\omega_n t - \phi)$$

↑ list of numbers.

mode 2 or

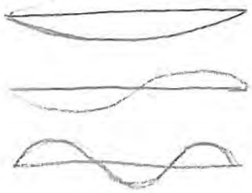


moving oppositely

Demo : musical instrument vibrations

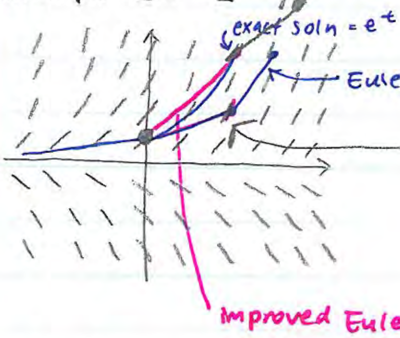
different harmonics,

all normal nodes.



use slope at midpoint to take a full step. ⟹ much closer to exact soln. than Euler's method.

example $\dot{z} = z$



take ½ step & measure slope

Improved Euler's / "midpoint" / "2 pt. RK"

In Matlab,

```
zdottemp= rhs2 (t, z, p) ;
znewtemp = z + zdottemp * h/2;
zdot     = rhs2( t+h/2, znewtemp, p);
znew     = z+ zdot * h ;
```

* Improved Euler's Method is much better as compared to using smaller step size with Euler's Method. Much more accurate, as seen in plots where energy does not increase even if step size is large.

COLLISIONS

Notation

Before	After
v_1 , v_2	v_1', v_2'
v_a, v_b	v_a', v_b'
v_{1b}, v_{2b}	v_{1a} , v_{2a}

confusion

a, b could be names of blocks

OR

could stand for "after" and "before"

Elastic collision : e (coefficient of restitution) = 1 , E_K conserved.

Inelastic/ Plastic : e = 0.

(Full eqs. in posted Matlab code.)

Linear Momentum conserved. Before = After.

$$e = \frac{\text{relative velocity after}}{\text{relative velocity before}}$$

*when e=-1, as if they never collided, merely passed through each other.

Set up matrix $\underbrace{\begin{bmatrix} m1 & m2 \\ -1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} m1 * v1b + m2 * v2b \\ e*(v1b - v2b) \end{bmatrix}}_{\text{rhs, numbers known.}}$

use A\rhs in Matlab to solve.

filename: TAM2030-2-19-2013 ㉑

TAM 2030, Dynamics, Ruina, Cornell

Lecture 10 2/19/2013

Quiz.

With $\vec{F}_{drag} = -c\vec{v}$

$\vec{F}_g = -mg\hat{\jmath}$

For slow launch, trajectory is close to a parabola.

What about a fast launch?

(fast, but earth is still flat)

a) parabola
b) circle
c) ellipse
d) triangle
e) other

Today

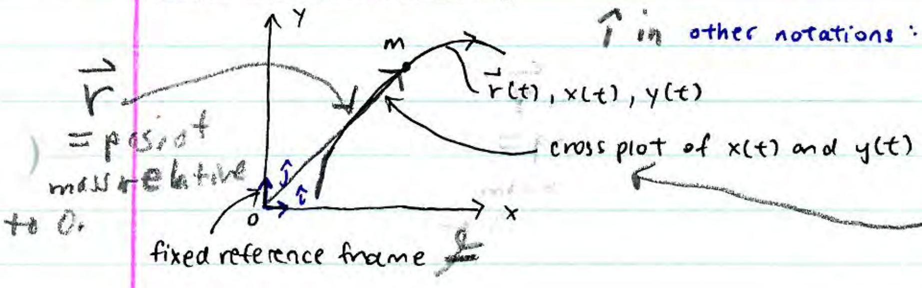
- Spatial Dynamics of a particle.

(Review Chapter 2 on vectors)

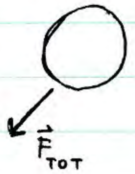
↰ You should. Important.

Trajectory of a particle.

$\hat{\imath}$ in other notations: $\vec{\imath}$, $\underline{i}$, $\mathbf{i}$, $\hat{\imath}$ (use any of these)



FBD



LMB

$\sum \vec{F} = \dot{\vec{L}}$ ↰ capital L

external forces

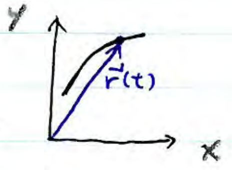
$\vec{F}_{Tot} = m\vec{a}$

$\llcorner\ \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$= \dfrac{d\vec{v}}{dt}$

$\vec{v} = \dfrac{d\vec{r}}{dt}$, $\vec{r}$ is position vector

↰ $= r_x\hat{\imath} + r_y\hat{\jmath}$



Where does F come from?

ex Near earth gravity: $\vec{F}_g = -mg\hat{\jmath}$

ex "Far earth", central force: $\vec{F}_g = mg\dfrac{R^2}{r^2}\dfrac{-\vec{r}}{|\vec{r}|}$, R = radius of earth

$\left(F = \dfrac{GMm}{r^2}\right)$ r = radius of location of mass $= |\vec{r}|$

$= -mg\dfrac{R^2}{r^2}\hat{e}_r$

$\hookrightarrow \hat{e}_r = \dfrac{\vec{r}}{|\vec{r}|}$

$= \dfrac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}}$

Direction is $-\hat{e}_r = -\vec{r}/|\vec{r}|$

scales with m, g, $1/r^2$

has mag. mg at $r = R$

scales with $c|\vec{v}|$
in direction $-\vec{v}/|\vec{v}|$

ex viscous drag : $\vec{F} = c\vec{v}$
$= c|\vec{v}|(\frac{-\vec{v}}{|\vec{v}|})$

ex External Forcing :
$\vec{F} = F_0 \sin(\omega t)\hat{\imath}$

ex quadratic drag: $\vec{F} = c|\vec{v}|^2(\frac{-\vec{v}}{|\vec{v}|})$
$= -c|\vec{v}|\vec{v}$

magnitude = cv^2
direction = $\vec{v}/|\vec{v}|$

A general Problem

Given : m, $\vec{r}_0 = \vec{r}(t=0)$
$\vec{v}_0 = \vec{v}(t=0)$
$\vec{F}_{tot}(t, \vec{r}, \vec{v})$ ← all examples above were of this form

Find : $\vec{r}(t)$, $\vec{v}(t)$, trajectory.

How to solve?

We have :- $\vec{F} = m\vec{a}$
$\Rightarrow \vec{a} = \frac{\vec{F}}{m}$
$\Rightarrow \dot{\vec{v}} = \vec{F}/m$
} Pillar III of mechanics; laws of mechanics

- $\vec{F}(t, \vec{r}, \vec{v})$ – we know the force law } Pillar I, mat. props.
- $\dot{\vec{r}} = \vec{v}$ } Pillar II : geometry

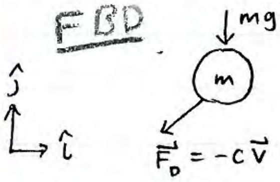
⇒ 2 sets of 1st order ODEs.

A. $\dot{\vec{r}} = \vec{v} \Rightarrow \{\dot{\vec{r}} = \vec{v}\} \cdot \hat{\imath} \Rightarrow \{(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = v_x\hat{\imath} + v_y\hat{\jmath}\} \cdot \hat{\imath}$
$\Rightarrow \dot{x} = v_x$
$\Rightarrow \dot{y} = v_y$
$\Rightarrow [\dot{r}] = [v] \Leftrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
] – one set

$\dot{\vec{v}} = \vec{F}/m$] – the other set

example trajectory with viscous drag.

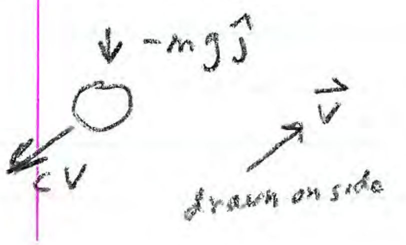
FBD



LMB $\Sigma\vec{F} = \dot{\vec{L}}$
$\vec{F}_{tot} = m\vec{a}$
$-mg\hat{\jmath} - c\vec{v} = m\vec{a}$
$\dot{\vec{v}} = -g\hat{\jmath} - \frac{c}{m}\vec{v}$
$\dot{v}_x\hat{\imath} + \dot{v}_y\hat{\jmath} = -g\hat{\jmath} - \frac{c}{m}(v_x\hat{\imath} + v_y\hat{\jmath})$
$\{\}\cdot\hat{\imath} \Rightarrow \dot{v}_x = -\frac{c}{m}v_x$
$\{\}\cdot\hat{\jmath} \Rightarrow \dot{v}_y = -g - \frac{c}{m}v_y$
] 2 ODES

Also have $\{\dot{\vec{r}} = \vec{v}\}$
$\{\}\cdot\hat{\imath} \Rightarrow \dot{r}_x = v_x$, $\{\}\cdot\hat{\jmath} \Rightarrow \dot{r}_y = v_y$] 2 more ODES

Alternate **FBD**



We have 4 coupled ODEs

↑ not so coupled in our example, (which is an unusually) especially easy problem.

$$\boxed{\begin{array}{l}\dot{\vec{r}} = \vec{v} \\ \dot{\vec{v}} = \vec{F}/m\end{array}} \Rightarrow \begin{bmatrix}\dot{r}_x \\ \dot{r}_y\end{bmatrix} = \begin{bmatrix}v_x \\ v_y\end{bmatrix} \quad \begin{bmatrix}\dot{v}_x \\ \dot{v}_y\end{bmatrix} = \begin{bmatrix}F_x/m \\ F_y/m\end{bmatrix} \Leftrightarrow \dot{z} = f(t, z, p)$$

← the present state

← rules that say how state evolves

MATLAB ODEs to solve

* usually cannot be solved on pencil and paper. This example problem happens to be solvable with analytic methods.

$\dot{v}_x = -\frac{c}{m} v_x$ is decoupled, can find. Simple 1st order constant coefficient differential eqn.

so is $\dot{v}_y = -g - \frac{c}{m} v_y$

$v_x = v_0 e^{-\frac{c}{m}t}$ (soln. You should be able to write this by inspection. See book appendix on ODE solns.)

Section

Today Recitation

2/20/
2013

- English units
- 3D Particle Mechanics

- Read Chapter 10 !!

·Definitions & Formulas

· Ex: Angular Momentum

UNITS

	Length	Force	Mass	Time	
SI	m	N	kg	s	✓
English (1)	ft	pdl	lbm	s	✓
English (2)	ft	lbf	slug	s	(not so common)

1 N = 7.24 pdl = 0.22 lbf

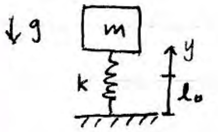
1 kg = 2.2 lbm = 0.069 slug.

in grav. field $F = mg$

1 lbm → 1 lbf = 1bm · g

1 kg → 9.8 N = 1 kg · g

Little Quiz



Find y in equilibrium.

FBD

$m = 6\,lbm$

$k = 2\,lbf/in$

$g = 32.2\ ft/s^2$

LMB in equilibrium, $a = 0$. $-ky - mg = 0$.

$-ky = mg$

Calculation $y = -\frac{mg}{k}$

① $1\,lbm = \frac{1\,lbf}{g}$

$m = 6\left(\frac{1\,lbf}{g}\right)$

$y = -\frac{mg}{k}$

$= -\frac{6\left(\frac{lbf}{g}\right)\cdot g}{2\ lbf/in}$

$= -3\,in$

$\boxed{OR}$

② $m = 6\,lbm\left(\frac{0.069\,slug}{2.2\,lbm}\right)$

$= 0.188\ slug$

$k = 2\ \frac{lbf}{in}\left(\frac{12\,in}{ft}\right) = 24\ lbf/ft$

$y = -\frac{(0.188\,slug)(32.2\,ft/s^2)}{24\ \frac{lbf}{ft}}$

$= -0.25\ ft.$ $\frac{slug\ ft^2/s^2}{lbf}$, $1\,lbf = 1\,slug\ ft/s^2$

$= -3\ in.$

3-D Particle Mechanics

Position $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. or polar coordinates $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

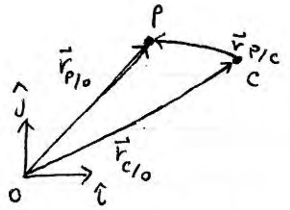
Velocity $\vec{v} = \dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$ $\qquad$ $\dot{\vec{r}} \neq \dot{r}\hat{e}_r + \dot{\theta}\hat{e}_\theta + \dot{\phi}\hat{e}_\phi$

$\uparrow$

! (will cover in coming weeks)

Angular Momentum, $\vec{H}_{/c} = \underbrace{\vec{r}_{P/c}}_{} \times (m\vec{v})$

point P with respect to C.



Moment (Torque), $\vec{M}_{/c} = \vec{r}_{P/c} \times \vec{F}$

AMB (Angular Momentum Balance), $\Sigma \vec{M}^{ext}_{/c} = \dot{\vec{H}}_{/c}$

$\boxed{\text{Ex}}$ A particle of mass m = 2kg moves in space. At a given time, we know:

$$\vec{r}_{P/o} = \vec{r} = (2m)\hat{\imath} + (4m)\hat{\jmath} - (2m)\hat{k}$$

$$\vec{a} = (3\tfrac{m}{s^2})\hat{\imath} - (1\tfrac{m}{s^2})\hat{\jmath} + (5\tfrac{m}{s^2})\hat{k}$$

a) Find $\dot{\vec{H}}_{/o}$

$\dot{\vec{H}}_{/o} = \vec{r}_{P/o} \times (m\vec{a})$ $\qquad$ ← general formula.

$= 36\,Nm\,\hat{\imath} - 32\,Nm\,\hat{\jmath} - 28\,Nm\,\hat{k}$

b) Find $\dot{\vec{H}}_{/c}$, where C is at (2, 4, 0).

$$\dot{\vec{H}}_{/c} = \vec{r}_{P/c} \times (m\vec{a})$$

$$= 2kg\left[\begin{pmatrix}0\\0\\-2\end{pmatrix}^{m} \times \begin{pmatrix}3\\-1\\5\end{pmatrix}\frac{m}{s^2}\right]$$

$$= -4\,Nm\,\hat{\imath} - 12\,Nm\,\hat{\jmath}$$

c) Find $\dot{\vec{H}}_{/D}$, where D is (8, 2, 8).

$$\vec{r}_{P/D} = \vec{r}_{P/o} - \vec{r}_{D/o} = (-6, 2, -10)m$$

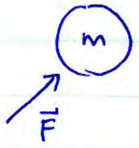
parallel to $\vec{a}$, ∴ cross product = $\vec{0}$.

Lecture Thursday 2/21

- particle in space
- particles in space

Quiz: For $\vec{F}=m\vec{a}$ in 2D, to solve on computer, we have n 1st order ODEs. What is n?

n = a) 1
b) 2
c) 3
d) 4 ✓
e) none of above



$$\vec{F}=m\vec{a} \Rightarrow \vec{a}=\vec{F}/m$$

$$\dot{\vec{v}}=\vec{F}/m$$

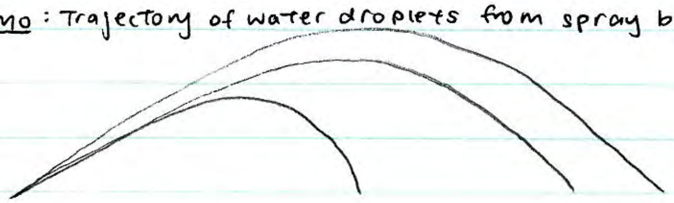
$$\begin{bmatrix}\dot{v}_x\\ \dot{v}_y\end{bmatrix}=\begin{bmatrix}F_x\\ F_y\end{bmatrix}/m \Rightarrow 2 \text{ equations.}$$

$$\dot{\vec{r}}=\vec{v}$$

$$\begin{bmatrix}\dot{r}_x\\ \dot{r}_y\end{bmatrix}=\begin{bmatrix}v_x\\ v_y\end{bmatrix} \Rightarrow 2 \text{ equations}$$

} 2+2= 4 equations

Demo: Trajectory of water droplets from spray bottle. traced on blackboard



Prelim Covers material through homework due next tuesday.
Could have Matlab. Know commands and how to use them.
Study old prelims. Similar kind of questions, probably
Given 3 hours, but should be done in ~1.5hr.

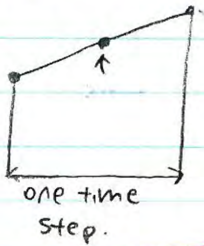
Matlab Mid point method.

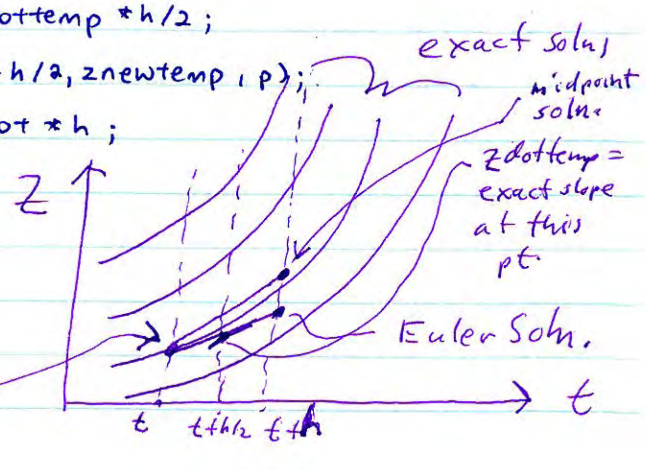
function written from laws of mechanics and geometry.

Main Idea:

```
zdottemp = rhs(t,z,p);
znewtemp = z + zdottemp *h/2;
zdot     = rhs(t+h/2, znewtemp, p);
znew     = z + zdot * h;
```

midpoint method uses slope at 1/2 of an Euler step to take a full step.





*Detailed code posted online.

transpose to row vector. because each time is a row.

zarray(i,:) = znew'; for each t value.

OUTPUT: [tarray zarray] , ~~*~~ each row corresponds to one value of t.

<u>rhs</u> function

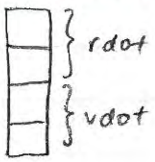
rdot = v

for quadratic drag, use v*abs(v).

vdot = -p.c* v/p.m - p.g * [0 1]';

zdot = [rdot ; vdot];

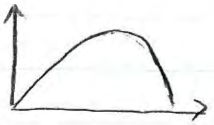
to get y component of acceleration, like [0 -m*g]/m



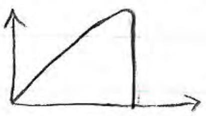
function trajectory.

p.c=0; % no drag.

p.c=1 ;



p.c=40;



vdot = -(1/norm(r))^2 * (r/norm(r)); % gravity. (orbits)

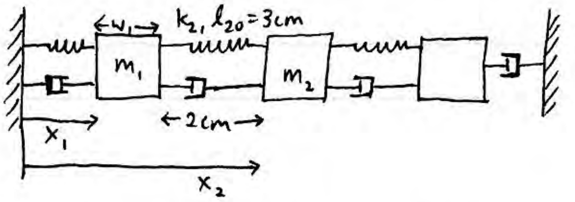
vdot = - p.k * (norm(r) - p.L0) * (r/norm(r)); % spring. (mass on spring)

$|\vec{r}| - l_0$

unit vector in $\vec{r}$ direction

Lecture 2/26/2013

Quiz



a) $-k_2(x_2 - x_1)$

b) $-k_2(x_2 - x_1 - (l_{20} + w_1))$.

c) $k_2(x_2 - x_1 - (l_{20} + w_1))$ ✓

d) $k_2(x_2 - x_1)$

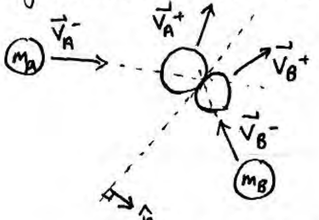
e) depends on x_1 & x_2.

What is the tension in spring 2?

- always take springs to be in tension & lengthening, so $T = k(\Delta l)$ is always correct w/ no fudging of signs, algebra will work signs out.

Collisions in 2D (or 3D)

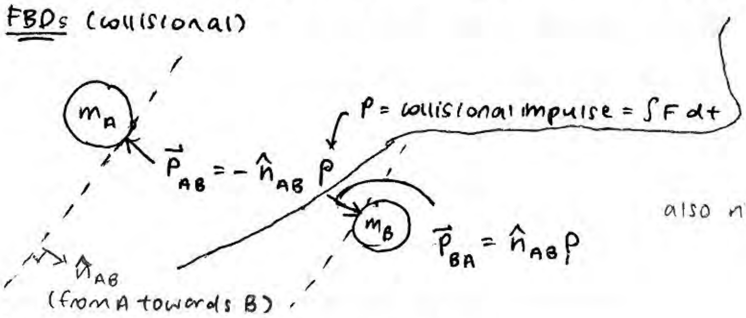
- rolling balls on table, collide.



$\hat{n}_{AB}$ = normal to collisional tangent plane.

Given : m_A, m_B, $\vec{V}_B^-$, $\vec{V}_A^-$, $\hat{n}_{AB}$, Find : $\vec{V}_A^+$, $\vec{V}_B^+$, assume no friction

FBDs (collisional)



also right : $\vec{P}_{AB} = \int \vec{F}_{AB}\,dt$

LMB Impulse - momentum equation

① B : $m_B \vec{V}_B^+ = m_B \vec{V}_B^- + \hat{n}_{AB} P$

② A : $m_A \vec{V}_A^+ = m_A \vec{V}_A^- - \hat{n}_{AB} P$

momentum after = momentum before + impulse

$m\vec{a} = \vec{F}$

$\int m\vec{a}\,dt = \int \vec{F}\,dt$

$\Delta\vec{V}\,m = \vec{P}$

$m(\vec{V}^+ - \vec{V}^-) = \vec{P}$

$m\vec{v}^+ = m\vec{v}^- + \vec{P}$

Aside: deriv. of impulse mom. eqn,

* Assume compressional collision.

2 scalar equations from each,

2+2 = 4 equations in total.

-5 unknowns : P, $\vec{V}_B^+$, $\vec{V}_A^+$

conservation of momentum equation is not independent.

Lin. Mom. Cons. of System follows from ① & ②

$$m_A \vec{V}_A{}^+ + m_B \vec{V}_B{}^+ = m_A \vec{V}_A{}^- + m\vec{V}_B{}^-$$

$$\vec{L}^- = \vec{L}^+ \quad \text{for system}$$

5th eqn : restitution equation. − how component of velocity normal to tangential plane changes.

③ $\hat{n}_{AB} \cdot (\vec{V}_B{}^+ - \vec{V}_A{}^+) = -\underset{\uparrow}{e}\,(\vec{V}_B{}^- - \vec{V}_A{}^-) \cdot \hat{n}_{AB}$ → 1 scalar eqn

coefficient of restitution.

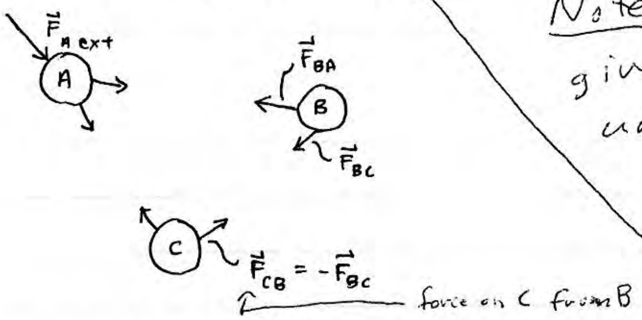
-How much relative normal velocities reverse.

$e=1$ totally elastic. = total reversal

$e=0$ totally plastic. = dead sticky collision.

- use $\underbrace{①, ②, ③}_{\text{5 eqs.}}$ to solve for $\underbrace{\vec{V}_B{}^+, \vec{V}_A{}^+, P.}_{\text{5 unknows.}}$

- Multiple particles in space



Note: Could have other givens and other unknowns, e.g. velocities after collision given (forensics), find velocities before.

to solve on computer : $\vec{F} = m\vec{a}$ for each particle.

In 2 Dim : $\dot{z} = f(z)$

↑ 4n equations, where n = no. of particles.

$$z = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ v_{x1} \\ v_{y1} \\ \vdots \end{bmatrix}$$

} - for 3 particles in 2D ⇒ 12 #s

$$\underset{\substack{\uparrow \\ \vec{r}\ \&\ \vec{v}}}{2} \times \underset{\substack{\uparrow \\ x\ \&\ y \\ \text{comps}}}{2} \times \underset{\substack{\uparrow \\ \text{3 particles}}}{3} = 12$$

Examples of interaction forces :

$$\vec{F}_{BC} = \frac{m_B m_C}{|\vec{r}_C - \vec{r}_B|^2} G \frac{\vec{r}_C - \vec{r}_B}{|\vec{r}_C - \vec{r}_B|}$$

inverse square ~~geometry~~ gravity

$$\vec{F}_{BC} = k(|\vec{r}_C - \vec{r}_B| - l_0) \frac{\vec{r}_C - \vec{r}_B}{|\vec{r}_C - \vec{r}_B|}$$

linear spring, rest length = l_0

$\dot{l} = (\vec{V}_C - \vec{V}_B) \cdot \hat{n}_{BC}$

$$\vec{F}_{BC} = c\underbrace{\left[(\vec{V}_C - \vec{V}_B) \cdot \vec{n}_{BC}\right]}_{\dot{l}} \underbrace{\vec{n}_{CB}}_{\frac{\vec{r}_{CB}}{|\vec{r}_{CB}|} = \frac{\vec{r}_{BC}}{l}}$$

projection onto $\vec{n}_{BC}$ direction

↑ CB

linear dashpot

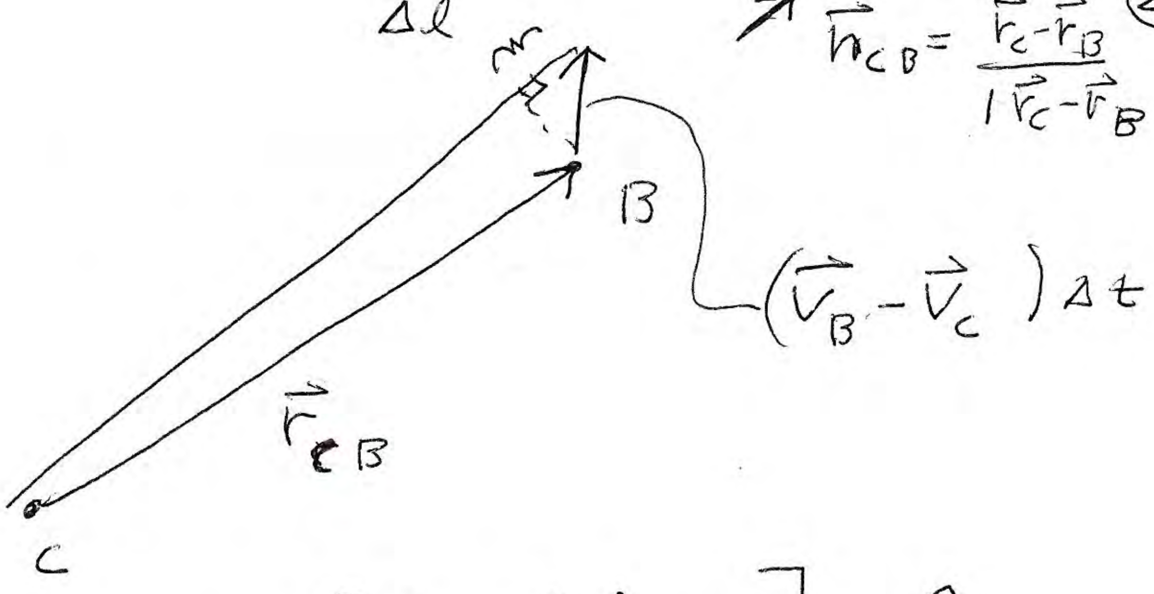
$$l^2 = \vec{r}_{CB} \cdot \vec{r}_{CB}$$

$$\dot{l}^2 = \dot{\vec{r}}_{CB} \cdot \vec{r}_{CB} + \vec{r}_{CB} \cdot \dot{\vec{r}}_{CB}$$

$$2l\dot{l} = 2\vec{r}_{CB} \cdot \dot{\vec{r}}_{CB} \Rightarrow \dot{l} = \frac{\vec{r}_{CB} \cdot \dot{\vec{r}}}{l}$$

[redacted]

$$\nearrow \; \vec{n}_{CB} = \frac{\vec{r}_C - \vec{r}_B}{|\vec{r}_C - \vec{r}_B}$$



$$\Delta l \approx \left[(\vec{V}_B - \vec{V}_C)\,\Delta t \right] \cdot \hat{n}_{CB}$$

$$\mathring{l} \approx \frac{\Delta l}{\Delta t} \approx (\vec{V}_B - \vec{V}_C) \cdot \vec{n}_{CB}$$

$$\Rightarrow \quad \vec{F}_{CB} = c\,\mathring{l}\,\hat{n}_{CB} = c\,(\vec{V}_B - \vec{V}_C) \cdot \frac{(\vec{r}_C - \vec{r}_B)}{|\vec{r}_C - \vec{r}_B|} \, \frac{\vec{r}_C - \vec{r}_B}{(\vec{r}_C - \vec{r}_B]}$$

↑ dashpot

$$\hat{n}_{CB} = \frac{\vec{r}_{CB}}{|\vec{r}_{CB}|}$$

$$\boxed{\vec{F}_{CB} = c\left[(\vec{V}_B - \vec{V}_C) \cdot \hat{n}_{CB} \right] \hat{n}_{CB}}$$

force on C from B by linear dashpot between them

Recitation

2/27/2013

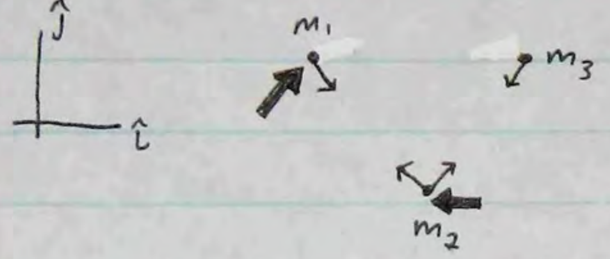
1 - Systems of Particles (CH.12)

2 - Pulleys (Intro to constraints, CH.13)

- pulley Recipe
- Example : 2 masses, 2 pulleys.

- Complexity of questions
  - no. of dimensions (1/2/3)
  - no. of masses (1, 2, 3, ... N) ... ∞ ← modelling rigid bodies

1 -



no. of particles

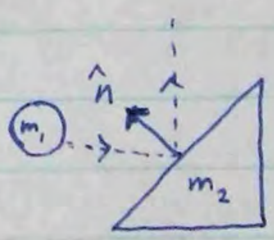
- System LMB : $\Sigma \vec{F}^{ext} = \sum_{i=1}^{N} m_i \vec{a}_i$
- System AMB : $\Sigma \vec{M}^{ext}_{/c} = \sum_{i=1}^{N} \dot{\vec{H}}_{i/c}$

} * assume internal forces are equal and opposite, ∴ only consider external forces.

- System Power Balance : $\underbrace{\Sigma P_j}_{\text{internal AND external } \vec{F}_j \cdot \vec{v}_j} = \frac{d}{dt} \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$

- 2D & 3D collisions

(1) $p = \int_{\text{collision time}} \vec{F}\, dt$ (Impulse)



(2) Restitution

$\hat{n} \cdot (\vec{v}_1^{\,+} - \vec{v}_2^{\,+}) = -e\,(\vec{v}_1^{\,-} - \vec{v}_2^{\,-}) \cdot \hat{n}$

← orthogonal to tangential plane of collision

2 - PULLEYS

(1) FBDs of all masses and pulleys (ideal ropes : tension is constant everywhere)

(2) LMB for each mass and pulley.

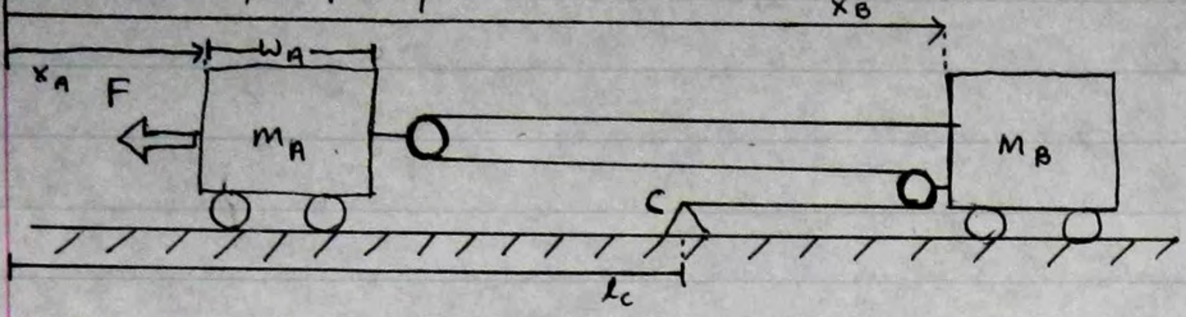
(3) Write down constraint equations (rope length is CONST)

* pick a reference point away from the system

(4) Take $\frac{d^2}{dt^2}$ of (3), killing constant terms. ⇒ Then SOLVE (2) and (4).

EXAMPLE 2/27/2013

- 2 masses, 2 pulleys.

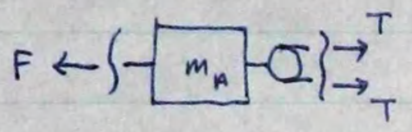


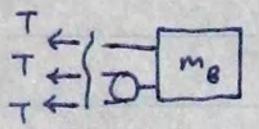
Assume: Frictionless carts and pulleys. Ideal rope of length l_0.

Constant applied force. Massless pulleys, no radius.

(a) compare $\ddot{x}_A$ and $\ddot{x}_B$

(1) FBDs





(2) LMB : $\Sigma F = ma$

A

$-F + 2T = m_A \ddot{x}_A$

B

$-3T = m_B \ddot{x}_B$

To find tension in rope.

(3) $2(x_B - (x_A + w_A)) + (x_B - l_c) = l_0$ ← length of string

(4) $3\ddot{x}_B - 2\ddot{x}_A = 0$

(5) $\ddot{x}_B = \frac{2}{3}\ddot{x}_A$

Lecture 2/28/2013

- 2D particles in space : Matlab
- Pulleys

⇒ Matlab will be posted online.

Euler method :

```
zdot = feval (rhsname, t, z, p);
znew = z + zdot * h ;
```

extra argument for name of right-hand-side file

gets passed into that file

can also be used for midpoint method.

Earth Moon :

```
FAB = (p.G * p.mA * p.mB / r^2) * (rAB / r);
FBA = - FAB; % action & reaction
vAdot = FAB / p.mA; % F=ma
vBdot = FBA / p.mB;
```

G m_A m_B

norm$(r_B - r_A)$

rAB means from A to B.

2 element vector of comps. of $\vec{r}_{AB} = \vec{r}_{B/A}$

syntax to solve :

```
[tarray zarray] = solvemidpoint ('rhs1', tspan, z0, p);
```

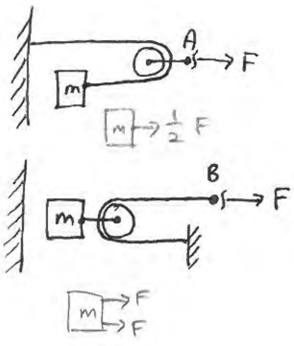
ODE soln. function

ODEs to solve are in this file

* shg ⇒ function to show graph
(brings graphics up front)

PULLEYS

Quiz :



F = F
m = m

What is $\frac{a_B}{a_A}$? assume massless pulley, string.

a) 2
b) 4
c) 16 ✓ $(2 \times 2) \times (2 \times 2) = 16$
d) 1/2
e) none of the above.

For B : acceleration of m is $\overset{\overset{2}{\downarrow}}{\frac{2F}{m}}$, acceleration of B is $\overset{\overset{2}{\downarrow}}{2} \times$ acceleration of m $= \frac{4F}{m}$

For A: acceleration of m is $\underset{\underset{2}{\uparrow}}{\frac{F}{2m}}$, acceleration of A is $\underset{\underset{2}{\uparrow}}{\frac{1}{2}}$ acceleration of m $= \frac{F}{4m}$.

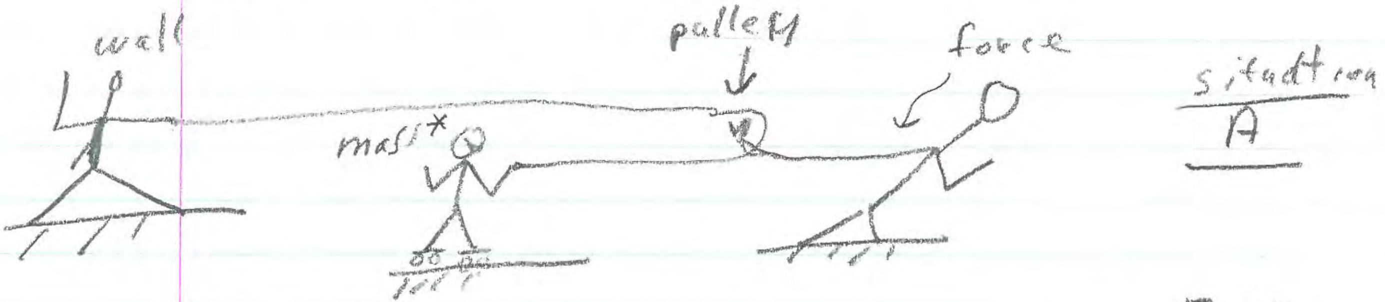
Pulley Solutions

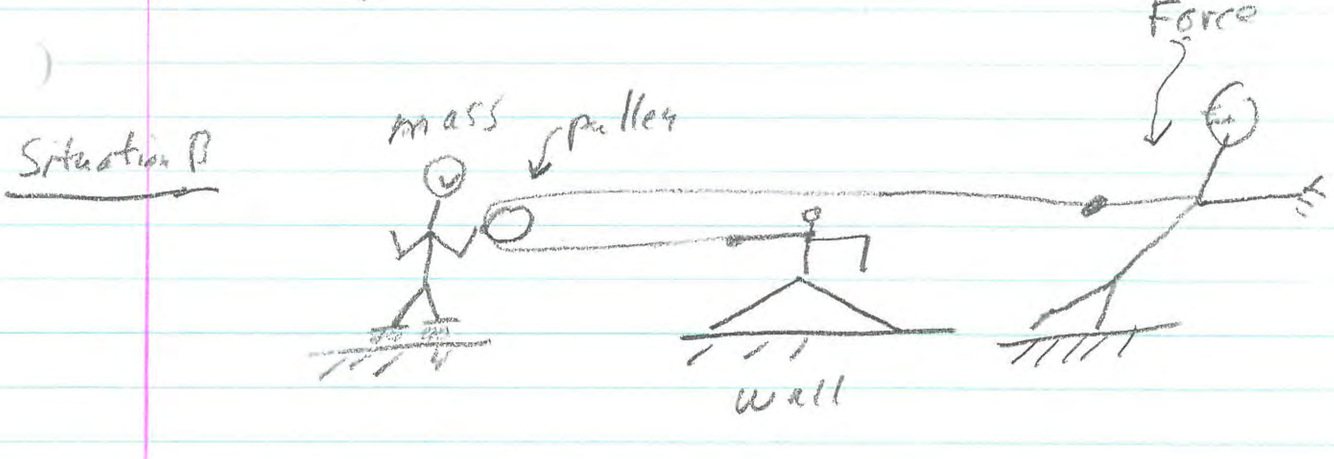
1) Draw FBDs

2) LMB

3) Use geometry / kinematics . Extra geometric constraints.

Demo of pulleys using 3 people for mass, wall and force



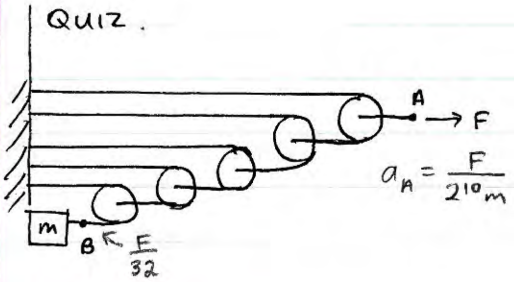


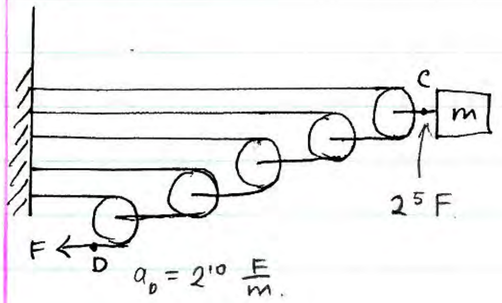
pulley = let string slip between fingers

*professor = inert mass

Lecture 3/5/2013

QUIZ.





Neglecting inertia and friction, inextensible strings.

$\frac{|a_A|}{|a_D|} = ?$

$2^5 = 32$
$2^{10} = 1024 \approx 1000$
$(2^{10})^2 \approx 10^6$

a) ≈ 1000

b) ≈ 1/1000

c) ≈ 1,000,000

d) ≈ 1/1,000,000 ✓ $\frac{1}{2^{20}} \frac{F}{m} = \frac{|a_A|}{|a_B|}$

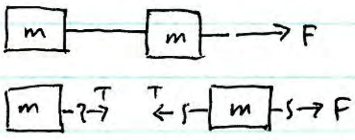
e) None of the above.

CONSTRAINED MOTION

- forces not det. by position, velocity, time
  eg. string, rods, hinges, contact, welds

- different from cases where material properties tell us forces
  eg. springs, dashpot, gravity, air friction

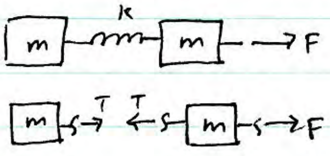
ex. ① This:



⇒ harder problem finding T.

FBDs look the same.

② versus this:



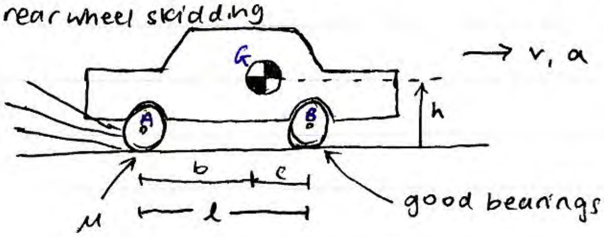
⇒ $T = k(\Delta \ell)$

① Find T by a) Finding acceleration

b) Avoid finding T. "Finnesse" the problem.

ex car (2D), accelerating with rear wheel drive. Given μ. ⇒ given ϕ with $\tan\phi = \mu$.

rear wheel skidding

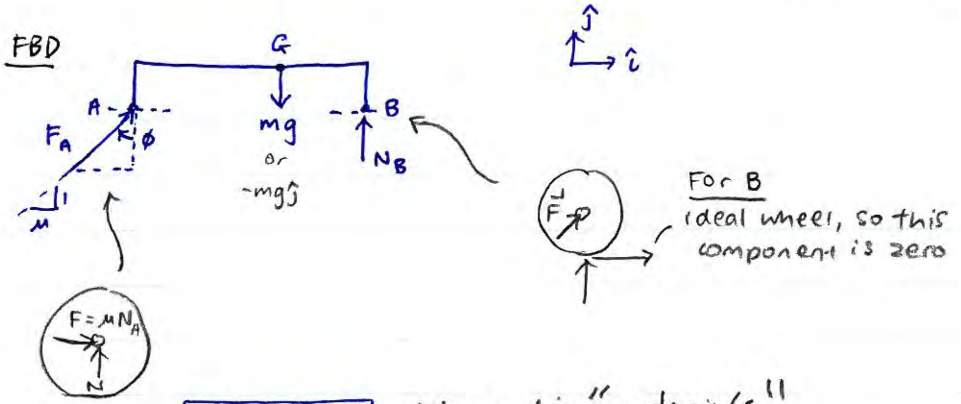


good bearings, negligible inertia

kinematic constraint : "stiff suspension"

car does not rotate, wheels stay on ground & no wheelies

FBD



For B
ideal wheel, so this component is zero

$$\boxed{\begin{matrix}\vec a_G = a_G \hat i \\ \omega = 0\end{matrix}}$$

Kinematic "constraints" (assumptions)

Mecnanics

LMB

$$\sum_{ext} \vec F = \dot{\vec L}$$

$$-mg\hat j + N_B\hat j + N_A\hat j + \mu N_A \hat i = m a_G \hat i \qquad — ①$$

AMB

$$\sum \vec M_{/c} = \dot{\vec H}_{/c}$$, where c is any point.

$$\sum \vec r_{i/c}\times\vec F = \vec r_{G/c}\times(m\underset{\uparrow\, a_G\hat i}{\vec a_G}) + \underbrace{I^G \cancel{\omega}\hat k}_{\text{2D rigid object}}$$

ω = 0 for now (no motion of car parts rel. to C.O.M.)

(other terms added for other cases).

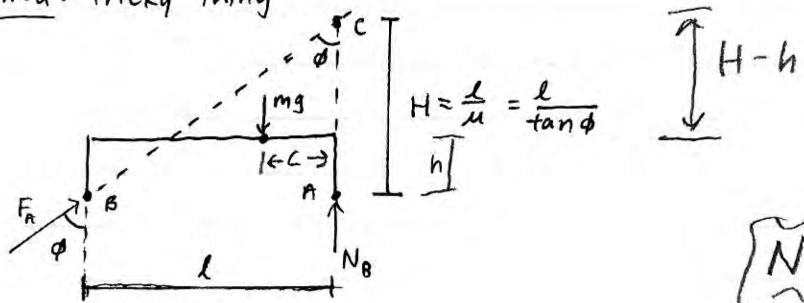
$$\underbrace{\vec r_{A/c}\times(N_A\hat j + \mu N_A\hat i) + \vec r_{B/c}\times(N_B\hat j) + \vec r_{G/c}\times(-mg\hat j)}_{\text{sum of moments} \; = \sum \vec M_{/c}} = \underbrace{\vec r_{G/c}\times(ma_G\hat i)}_{\dot{\vec H}_{/c}} \qquad — ②$$

(2D, non-rotating rigid object)

For any point C of your choice, we have 3 equations — 2 for LMB
for 3 unknowns (N_A, N_B, a_G). 1 for AMB ↳ terms all in $\hat{k}$ direction.

Method : write 3 eq. in 3 unknowns and solve.

Method : Tricky thing



AMB :

$$\Sigma \vec{M}_{/C} = \vec{H}_{/C}$$

$$\underset{\uparrow}{\vec{r}_{G/C}} \times (-mg\hat{j}) = \underset{\uparrow}{\vec{r}_{G/C}} \times (m a_G \hat{i})$$

$$(-c\hat{i} - (H-h)\hat{j})$$

1 equation, 1 unknown!

Note: if $\mu h > b$
$\Rightarrow N_B < 0$
$\Rightarrow$ suction
$\Rightarrow$ assumption of no wheelie was wrong

Details: $(-c\hat{i} - (H-h)\hat{j}) \times (-mg\hat{j}) = [-c\hat{i} - (H-h)\hat{j}] \times (m a_G \hat{i})$

$\hat{j} \times \hat{j} = \vec{0} \Rightarrow$ $$cmg \underbrace{\hat{i} \times \hat{j}}_{\hat{k}} = -(H-h) m a_G \underbrace{\hat{j} \times \hat{i}}_{-\hat{k}}$$

$$\Rightarrow \left\{ cmg\hat{k} = (H-h) m a_G \hat{k} \right\}$$

$$\{\ \} \cdot \hat{k} \Rightarrow c\cancel{m}g = (H-h)\cancel{m} a_G$$

$$a_G = \frac{cg}{(H-h)} = \frac{cg}{(\frac{l}{\mu} - h)} = \boxed{\mu g \frac{c}{(l - \mu h)} = a_G}$$

Note: if $\mu h = b$, almost wheelie,
$a_C = \mu g \frac{c}{(l-b)}$
$= \mu g \frac{c}{c}$
$a_G = \mu g$
(all weight on rear wheel)

Note: $a_G \neq \mu g \frac{c}{l}$ = answer you would get if you used statics to calculate N_A & N_B

Recitation

3/6/2013

I. 1-D Motion with 2D or 3D forces

pg 662 – useful formulas

acceleration and velocity are in same direction.

Definitions:

Angular Momentum — $\vec{H}_{/C} = \vec{r}_{G/C} \times (m\vec{v}_G)$

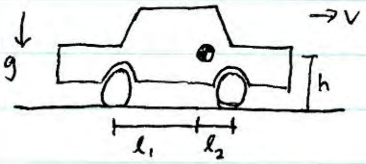
Rate of change of AM — $\dot{\vec{H}}_{/C} = \vec{r}_{G/C} \times (m\vec{a}_G)$

Kinetic Energy — $E_K = \frac{1}{2} m_{TOT} V_G^2$

Rate of change of KE — $\dot{E}_K = m_{TOT} V_G a_G$

Example

Front-wheel drive vs Rear-wheel drive



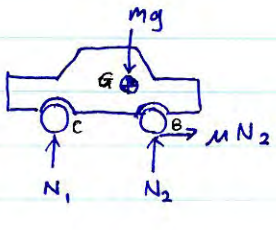
center of mass usually towards front of car as engine is somewhere in front.

Assumptions: Ideal wheels, stiff suspension (∴ 1-D motion), wheels do not slip. same front and rear tires.

What is the max. acceleration for a FWD car? A RWD car?

FWD

FBD



LMB

$$\sum \vec{F}^{ext} = m\vec{a}_G$$

$$(N_1 + N_2 - mg)\hat{j} + \mu N_2 \hat{i} = \ddot{x}_G m \hat{i}$$

$$LMB \cdot \hat{i} \Rightarrow \ddot{x}_G = \frac{\mu N_2}{m} \quad \text{— (1)}$$

$$\cdot \hat{j} \Rightarrow N_1 + N_2 = mg \quad \text{— (2)}$$

AMB

$$\sum \vec{M}_{/C}^{ext} = \dot{\vec{H}}_{/C}$$

Pick rear wheel as point C, where it touches the ground.

- N_1, μN_2 drops out.

LHS

$$\sum \vec{M}^{ext}_{/C} = \vec{r}_{B/C} \times (N_2 \hat{j}) + \vec{r}_{G/C} \times (-mg\hat{j})$$

$$\vec{r}_{B/C} = (l_1+l_2)\hat{i}, \quad \vec{r}_{G/C} = l_1\hat{i} + h\hat{j}$$

$$\sum \vec{M}^{ext}_{/C} = N_2(l_1+l_2)\hat{k} + (-mgl_1)\hat{k}$$

RHS

$$\dot{\vec{H}}_{/C} = \vec{r}_{G/C} \times (m\vec{a}_G)$$

$$= -m\ddot{x}_G h\hat{k}$$

AMB/C

$$N_2(l_1+l_2) - mgl_1 = -m\ddot{x}_G h$$

$$N_2 = \frac{m(gl_1 - h\ddot{x}_G)}{l_1+l_2} \quad \text{— (3)}$$

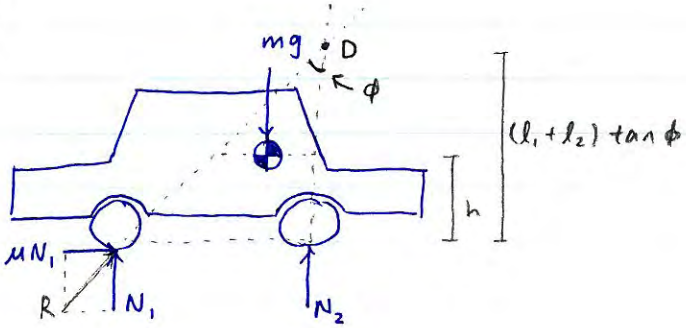
(3) into (1)

$$\boxed{\ddot{x}_G = \frac{gl_1}{\frac{l_1+l_2}{\mu} + h}} \quad \text{MAX. FWD } a_G$$

↳ to ↑ $\ddot{x}_G$, center of mass toward front of car, ↑ l_1.
center of mass lower, ↓ h.

RWD

FBD



AMB/D

$$\sum \vec{M}^{ext}_{/D} = -mgl_2\hat{k}$$

$$\dot{\vec{H}}_{/D} = \vec{r}_{G/D} \times m\vec{a}_G$$

$$\ddot{x}_G = \frac{gl_2}{\frac{l_1+l_2}{\mu} - h}$$

↑ h or ↑ l_2 to ↑ rear wheel acceleration.

Lecture

3/7/2013

- Laws of Mechanics
- Circular Motion

Laws of Mechanics

LMB : $\sum_{ext} \vec{F} = m\vec{a}$

$$= \sum m_i \vec{a}_i$$

$$= m_{TOT} \vec{a}_G$$

AMB: $\sum \vec{M}_{/c} = \sum \dot{\vec{H}}_{/c}$

$$\left.\begin{array}{l} = \sum \vec{r}_{i/c} \times (m_i \vec{a}_i) \\ \int \vec{r} \times \vec{a} \, dm \end{array}\right] \begin{array}{l} \leftarrow \text{point masses} \\ \leftarrow \text{distributed masses.} \end{array}$$

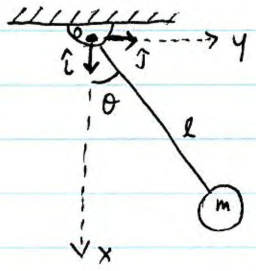
$$= \vec{r}_{G/c} \times m_{tot} \vec{a}_G + \sum \vec{r}_{i/G} \times (\vec{a}_{i/G} m_i) \leftarrow \text{point masses}$$

$$\text{or } \underbrace{\int \vec{r}_{/G} \times \vec{a}_{/G} \, dm}_{\vec{H}_{/G} = I^G \dot{\omega} \hat{k}} \leftarrow \text{distributed masses.}$$

for 2 D rigid object

Circular Motion

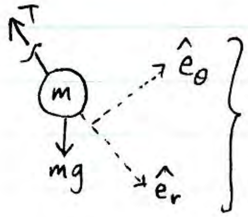
Example : Simple Pendulum



$\ddot{\theta} = \frac{-g}{\ell} \sin \theta$

$$\left.\begin{array}{l} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{\ell} \sin \theta \end{array}\right] \text{ can be plugged into MATLAB. Difficult to solve by hand.}$$

FBD



coordinate axes change as θ changes

AMB$_{/O}$

$$\Sigma \vec M_{/O} = \dot{\vec H}_{/O}$$

the mass

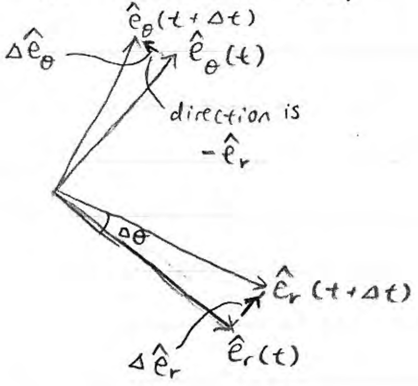
Finding this is tricky part.

$$\vec r_{G/O} \times mg\hat\imath = \vec r_{G/O} \times m\underline{\vec a_G}$$

$$\vec r_{G/O} = l\hat e_r$$

$$\vec v_{G/O} = \frac{d}{dt}(l\hat e_r) = l\dot{\hat e}_r \quad , \quad \dot{\hat e}_r = \dot\theta\,\underline{\hat e_\theta} \quad \Rightarrow \boxed{\dot{\vec r}_{G/O} = l\dot\theta\hat e_\theta}$$

direction of tangential motion at that point



$$\frac{\Delta\hat e_r}{\Delta t}$$

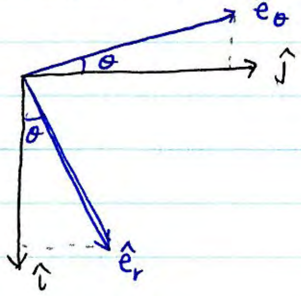
$$\vec a_G = \dot{\vec v}_{G/O}$$
$$= \frac{d}{dt}(l\dot\theta\hat e_\theta)$$
$$= l\ddot\theta\hat e_\theta + l\dot\theta\dot{\hat e}_\theta \quad , \quad \dot{\hat e}_\theta = -\dot\theta\hat e_r$$
$$= l\ddot\theta\hat e_\theta - l\dot\theta^2\hat e_r$$

$$l\hat e_r \times (mg\,i) = l\hat e_r \times m\,[l\ddot\theta\hat e_\theta - l\dot\theta^2\hat e_r]$$

$$\{-l mg\sin\theta\,\hat k = ml^2\ddot\theta\hat k + 0.\} \cdot k \times \frac{1}{ml^2}$$

$$\ddot\theta = -\frac{g}{l}\sin\theta$$

Polar Coordinates



$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_r = \dot{\theta}(-\sin\theta)\,\hat{i} + \dot{\theta}\cos\theta\,\hat{j}$$
$$= \dot{\theta}\,\hat{e}_\theta$$

$$\hat{e}_\theta = -\dot{\theta}\cos\theta\,\hat{i} - \dot{\theta}\sin\theta\,\hat{j}$$
$$= -\dot{\theta}\,\hat{e}_\theta$$

Back to Mechanics

LMB

$$\sum_{ext}\vec{F} = m_{tot}\,\vec{a}_G$$

$$mg\hat{i} - T\hat{e}_r = m(a_r\hat{e}_r + a_\theta\hat{e}_\theta) = m(-\ell\dot{\theta}^2\hat{e}_r + \ell\ddot{\theta}\,\hat{e}_\theta)$$

$\{\}\cdot\hat{e}_r$ $\quad mg\cos\theta - T = -m\ell\dot{\theta}^2$

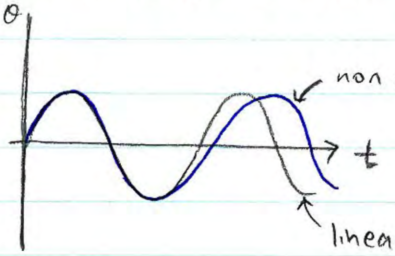
$\{\}\cdot\hat{e}_\theta$ $\quad -mg\sin\theta = m\ell\ddot{\theta}$

$$\ddot{\theta} = \frac{-g}{\ell}\sin\theta$$ ⟶ non linear. Period longer than it would be in linear motion.

$$T = mg\cos\theta + m\ell\dot{\theta}^2$$

$$\vec{T} = -T\hat{e}_r$$

For $\theta \ll 1$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.



non linear. starts the same, then deviates as period ↑

linear model (sine wave).

Lecture 3/12/2013

- Mechanics Review
- Circ Motion (continued)
- Before next lecture, READ 15.1 and 15.2 – (Quiz at start of next lecture! on 15.1)

Quiz:

$$Q = \frac{\overset{\text{energy}}{(\text{Energy capacity})} \cdot \overset{\text{distance / time}}{(\text{Vel})}}{\underset{\text{energy / time}}{\text{Power}}} \gg Q \text{ for all other robots}$$

[cornell robot in Zurich last week]

looking at units ⇒ distance.

Is it good at a) walking far

b) minimizing total energy / dist.

c) " " " / time

d) " " " / (vel × weight)

e) other

$\frac{E}{P} = \text{time}$

$\text{vel} \cdot \text{time} = \underline{\underline{\text{distance}}}$

* several useful tables at the back of the book. eg. 1047 pg. row 7 column 4.

[Summary of different ~~situations~~ expressions for $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/c}$, $\dot{\vec{H}}_{/c}$, E_K for different systems]

MECHANICS

Draw FBD

LMB

$$\Sigma \vec{F}_{ext} = \left[\begin{array}{ll} \Sigma m_i \vec{a}_i & \text{particle} \\ \int \vec{a}\, dm & \text{conti......} \\ \vec{a}_G m_{TOT} & \text{any system} \end{array}\right.$$

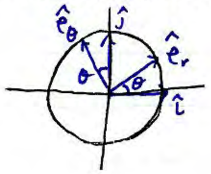
AMB

$$\Sigma \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\hookrightarrow \dot{\vec{H}}_{/c} = \left[\begin{array}{ll} \Sigma \vec{r}_{i/c} \times (m_i \vec{a}_i) & \text{particle} \\ \int \vec{r}_{/c} \times \vec{a}\, dm & \text{conti.} \\ \vec{r}_{G/c} \times m \vec{a}_G + \underbrace{\boxed{I^G}}_{\text{moment of inertia about center of mass}} \dot{\omega} \hat{k} & \text{2D rigid object} \end{array}\right.$$

[see pg 1047 for other systems]

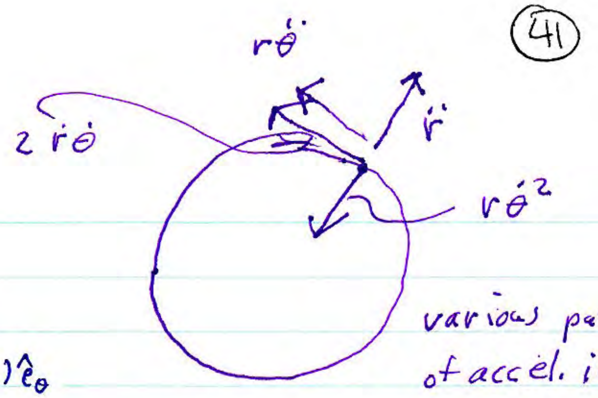
Polar coordinates



$$\hat{e}_r = \frac{\vec{r}}{|\vec{r}|}, \quad \hat{e}_\theta = \hat{k} \times \hat{e}_r$$

$$\dot{\hat{e}}_r = \dot{\theta} \hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta} \hat{e}_r$$



various parts of accel. in polar coord.

$\vec{r} = x\hat{i} + y\hat{j} = r\hat{e}_r$

$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = \dot{r}\hat{e}_r + \dot{\theta} r\hat{e}_\theta$

$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (\ddot{r} - \underbrace{r\dot{\theta}^2}_{\text{centripetal acceleration}})\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

For circular motion, r is constant. $\dot{r} = \ddot{r} = 0$.

Think about pendulum problem! demonstrated in class.

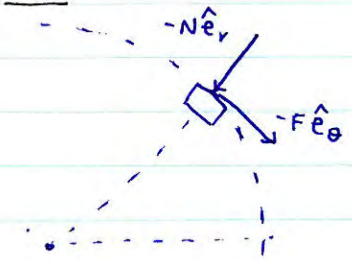
EXAMPLE

Block sliding in a cylinder. No gravity - with friction.

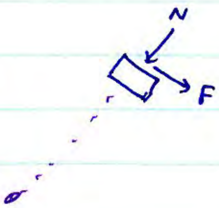
Assume $\dot{\theta} > 0$.

FBD

[see Ch. 2 for vector notation.]



or

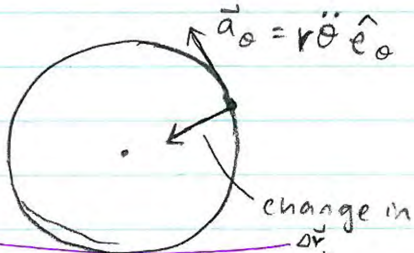


LMB

$\Sigma\vec{F} = m\vec{a}$

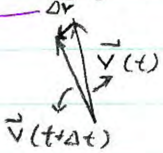
$-N\hat{e}_r - \underset{\hookrightarrow \mu N}{F}\hat{e}_\theta = m[\overbrace{-r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta}^{\dot{r}=\ddot{r}=0}]$

Aside



$\vec{a}_\theta = r\ddot{\theta}\hat{e}_\theta$ ← rate of change of speed tangent to circular motion.

change in velocity vector.



$|\Delta\vec{v}| \approx (\underbrace{\dot{\theta}\Delta t}_{\Delta\theta})\cdot v$ ↳ $\dot{\theta} r$

$= \dot{\theta}^2 r \Delta t$

↳ $\frac{(r\dot{\theta})^2}{r} = \frac{v^2}{r}$ in $-\hat{e}_r$ direction.

the part due to change in direction

$$\left\{-N\hat{e}_r - \mu N\hat{e}_\theta = m[-r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta]\right\}$$

$\{\}\cdot\hat{e}_r \Rightarrow -N = -mr\dot{\theta}^2$

$\{\}\cdot\hat{e}_\theta \Rightarrow -\mu N = mr\ddot{\theta}$ $\Big\}$ $-\mu m r\dot{\theta}^2 = mr\ddot{\theta}$

define $\omega = \dot{\theta}$

↑ Beware, ω has a slightly different meaning usually.

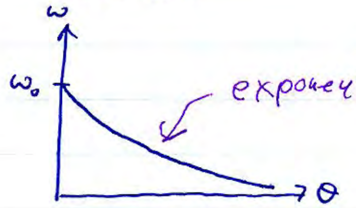
$$\boxed{\dot{\omega} = -\mu\omega^2} \quad *$$

can think of $\omega(t) = \omega(\theta(t))$.

$$\dot{\omega} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

$* \Rightarrow$ $$\frac{d\omega}{d\theta} = -\mu\omega$$

$$\omega = \omega_0 e^{-\mu\theta}$$



exponential decay with θ

$$\Rightarrow \frac{d\theta}{dt} = \omega_0 e^{-\mu\theta}$$

θ = log like thing.

$e^{\mu\theta}\, d\theta = \omega_0\, dt$

$\Rightarrow e^{\mu\theta} = \mu\omega_0 t + C$

$\Rightarrow \mu\theta = \ln(\omega_0 t + C)$

$\theta = \frac{1}{\mu}\ln(\omega_0 t + c)$

$\theta(0) = 0 \Rightarrow c = 1$

$$\underbrace{\theta = \frac{1}{\mu}\ln(1 + \omega_0 t)}_{\text{log like thing}}$$

this problem has

⇝ quadratic drag.

⇝ Goes slower and slower, but does not stop.

Goes infinite distance

[Completely analogous to bullet w/ quadratic drag.]

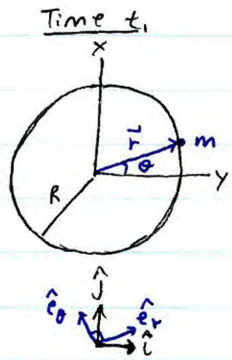
Section

3/13

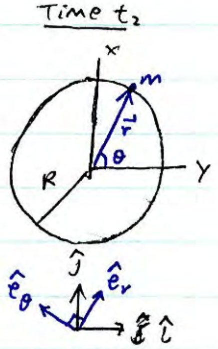
- circular motion

We can ~~ts~~ solve rotation problems in cartesian coordinates, but polar coordinates are often FAR EASIER!

Key Idea: Polar unit vectors change over time.



$\vec{r} = R\hat{e}_r$



- Like $\{\hat{\imath}, \hat{j}, \hat{k}\}$, the polar basis $\{\hat{e}_r, \hat{e}_\theta, \hat{k}\}$ forms a right-handed system, i.e. $\hat{e}_r \times \hat{e}_\theta = \hat{k}$ and $\hat{e}_r \cdot \hat{e}_\theta = 0$

$\hat{\imath} \times \hat{j} = \hat{k}$ $\quad$ $\hat{\imath} \cdot \hat{j} = 0$.

- $\hat{e}_r = \cos\theta \; \hat{\imath} + \sin\theta \, \hat{j}$

$\hat{e}_\theta = -\sin\theta \, \hat{\imath} + \cos\theta \, \hat{j}$

- How do we write $\vec{v}$ and $\vec{a}$?

$\vec{v} = \dot{\vec{r}}$ and $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$ ALWAYS.

So $\vec{v} = \frac{d}{dt}(R\hat{e}_r)$

$= R\frac{d}{dt}\hat{e}_r$

$= R\frac{d}{dt}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{j})$

$= R(-\sin(\theta)\dot{\theta}\hat{\imath} + \cos(\theta)\dot{\theta}\hat{j})$

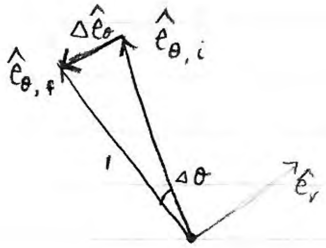
$= R\dot{\theta}\hat{e}_\theta$

$\vec{a} = \frac{d}{dt} R\dot{\theta}\hat{e}_\theta$

$= R(\ddot{\theta}\hat{e}_\theta + \dot{\theta}\frac{d}{dt}\hat{e}_\theta)$

$= R(\ddot{\theta}\hat{e}_\theta - \dot{\theta}^2\hat{e}_r)$

$= -R\dot{\theta}^2\hat{e}_r + R\ddot{\theta}\hat{e}_\theta$

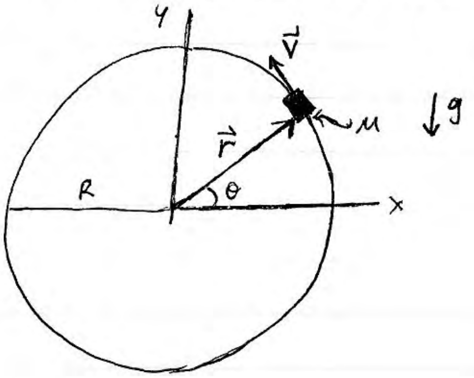


$\Delta\hat{e}_\theta = -\Delta\theta\,\hat{e}_r$ (in $-\hat{e}_r$ direction).

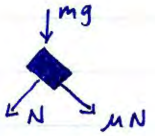
$\Rightarrow \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

EXAMPLE (same as lecture from 3/12)

Bead on hoop with friction.



FBD



LMB

$$\sum \vec{F}^{ext} = m\vec{a}$$

$$\{-\mu N\hat{e}_\theta - mg\hat{j} - N\hat{e}_r = (-R\dot{\theta}^2\hat{e}_r + R\ddot{\theta}\hat{e}_\theta)m\}$$

Find N

$$\{\}\cdot\hat{e}_r \quad -mg(\hat{j}\cdot\hat{e}_r) - N = -mR\dot{\theta}^2$$

$$-mg\sin\theta - N = -mR\dot{\theta}^2$$

$$N = \underbrace{mR\dot{\theta}^2}_{\frac{mv^2}{R}} - mg\sin\theta$$

Find $\dot{v} = R\ddot{\theta}$ (from LMB $\cdot\,\hat{e}_\theta$)

$$-\mu N - mg(\hat{j}\cdot\hat{e}_\theta) = mR\ddot{\theta}$$

$$-\mu(mR\dot{\theta}^2 - mg\sin\theta) - mg\cos\theta = mR\ddot{\theta}$$

$$\dot{v} = R\ddot{\theta} = g(\sin\theta - \cos\theta) - \mu\frac{v^2}{R} \quad , \dot{\theta} = \frac{v}{R}$$

Neglecting gravity, Find $v(\theta)$ in terms of θ, μ, v.

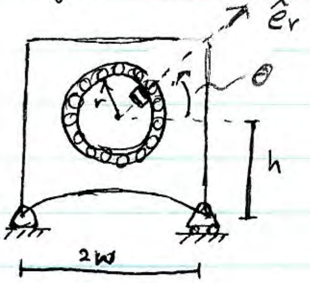
$$\dot{v} = \frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt} = \frac{dv}{d\theta}\dot{\theta} = -\mu\frac{v^2}{R}$$

Lecture 3/14/2013

- Circular Motion (cont'd)
- Rotation of rigid object (w/ animation and ODE45).

Example – given motion, find forces. (inverse dynamics) ~ no diff eq.

Spinning of "unbalanced" machine

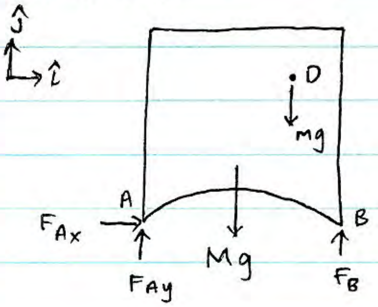


spins with constant ω, $\ddot{\theta} = 0$.

$\theta = \omega t$

$w \neq \omega$! double-u is not omega

FBD system:



LMB:

$$\Sigma \vec{F} = \dot{\vec{L}}$$

$$F_{Ax}\hat{i} + (F_{Ay} + F_B - (m+M)g)\hat{j} = \Sigma m_i \vec{a}_i$$

* all points in drum, when sum up $m_i\vec{a}_i$, cancel out.

$$\Sigma m_i \vec{a}_i = -m\dot{\theta}^2 r \underbrace{\hat{e}_r}_{\cos\theta\hat{i} + \sin\theta\hat{j}}.$$

AMB:

$$\Sigma \vec{M}_{/A} = \dot{\vec{H}}_{/A}$$

$$\Sigma(\vec{r} \times \ldots) \text{, Statics stuff} = \vec{r}_{D/A} \times (m\vec{a}_D).$$

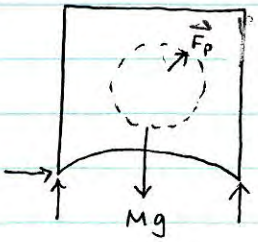
$$\left.\begin{aligned} &\tfrac{w}{2}\hat{i} \times (-Mg\hat{j}) + w\hat{i} \times (F_B\hat{j}) \\ &+ (w/2\,\hat{i} + h\hat{j} + r\hat{e}_r) \times (-mg\hat{j}) \end{aligned}\right] = (w\hat{i} + h\hat{j} \overset{}{\underset{+r\hat{e}_r}{\uparrow}}) \times -(\dot{\theta}^2 r\hat{e}_r)m.$$

⇒ 3 eqns to solve for F_{Ax}, F_{Ay}, F_B in terms of:

$t, \theta, r, M, m, g, \omega, w, h$

ALT. APPROACH:

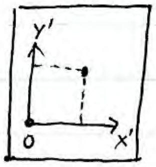
2 FBD:

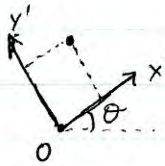


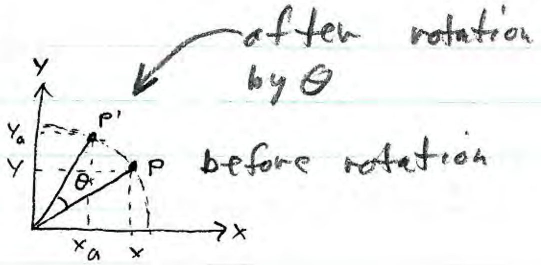
↖ Statics. Nothing moving.

simple dynamics.

Rigid Object rotation about point O.







As object rotates, coordinates of a material pt. in body coordinates x'y',

a) depend on θ

b) constant in time ✓

c) not enough information

d) something else.

↗ R, rotation matrix.

$$\begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

-rotate through matrix multiplication.

First col. of R is coords of $\hat{i}'$ in XY frame.
2nd col. is coords of $\hat{j}'$ in XY frame.

MATLAB

(code to be pasted)

```
r_ref=[x0,y0];
r = R* r_ref ; % rotated drawing
[t z]=
ode45(@rhs, t, z0, [],p);
```

z → each row is values of z at time t.

[] ← options. Data structure created by odeset. ↑ accuracy. etc.

* animation in class, details of code online.

nan : "pick up pen" , new, disconnected drawing.

Lecture 3/26/2013

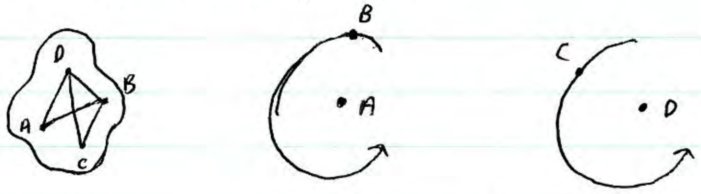
- Angular velocity of rigid object.
- " momentum " " "
- Rate of change of . . .

ANGULAR VELOCITY of 2D RIGID OBJECT — KEY FACTS

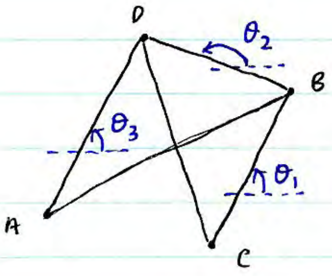
1) All pairs of points on a rigid object go in circles around each other.

2) Rate of change of any orientation line on object is same for all others $\Rightarrow \omega$

Translating and rotating object.



motion of B rel. to A , C rel. to D.



$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \omega$ ← angular velocity of object.

$\vec{\omega}$ = ang. velocity vector

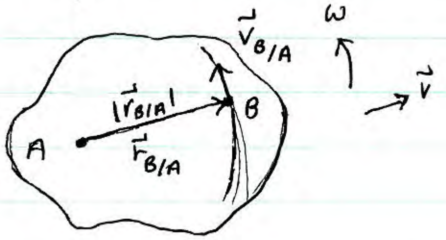
$= \omega \vec{k}$

$= \dot{\theta}_1 \hat{k} = \dot{\theta}_2 \hat{k} = \dot{\theta}_3 \hat{k} = \dots$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \frac{d}{dt} \vec{r}_{B/A}$$

$$\boxed{\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}}$$

Why?



$|\vec{v}_{B/A}| = \dot{\theta} \, |\vec{r}_{B/A}| =$ "$\dot{\theta} r$"

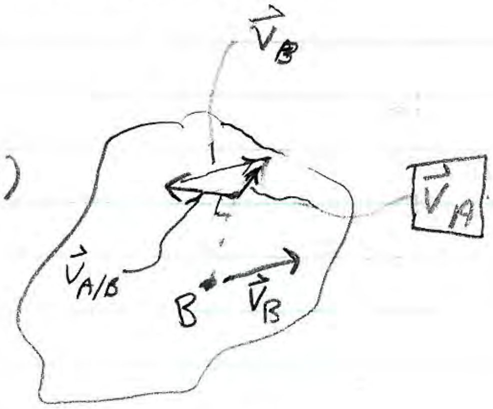
direction of $\vec{v}_{B/A}$ is $\perp$ to $\vec{r}_{B/A}$.

it is the right vector; $\vec{v}_{B/A}$ $\Leftarrow$ [so $\vec{\omega} \times \vec{r}_{B/A}$ is in right direction & has right magnitude.

Velocity of any point on a rigid body : A and B on object.

QUIZ 1) Given $\vec{V}_A$, ω , $\vec{r}_B$, $\vec{r}_A$

Find $\vec{V}_B$.



a) $\vec{V}_A + \overbrace{\vec{V}_B - \vec{V}_A}^{\vec{V}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}}$ (correct but useless)

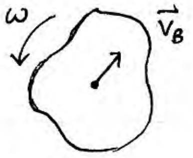
b) $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) + \vec{V}_A$

c) $\vec{V}_A + \vec{\omega} \times (\vec{r}_B - \vec{r}_A)$ ✓

d) $\vec{V}_A + \omega \vec{r}_{B/A}$

e) cannot be determined

2) Given $\vec{V}_B$ & $\vec{\omega}$, find a point A on the object with $\vec{V}_A = \vec{0}$. $\vec{r}_A$ = ?



$\vec{r}_B = \vec{r}_{B/O}$ = given

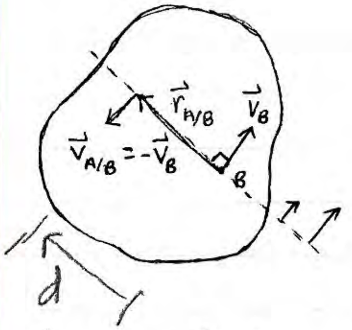
a) $\vec{r}_B + \frac{\vec{\omega} \times \vec{V}_B}{\omega^2}$ ✓

b) ×

c) ×

d) ×

e) cannot be determined.



$d = \frac{|\vec{V}_B|}{\omega}$

$$\vec{V}_A = \vec{V}_B + \vec{V}_{A/B}$$

$$\vec{0} = \vec{V}_B + \vec{V}_{A/B}$$

$$\left\{ \begin{aligned} \vec{V}_B &= -\vec{V}_{A/B} \\ &= -\vec{\omega} \times \vec{r}_{A/B} \end{aligned} \right\}$$

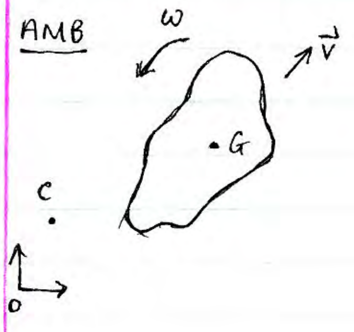
$$\{\} \times \vec{\omega} \Rightarrow \vec{\omega} \times \vec{V}_B = -\vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})$$

$$= -[-\omega^2 \vec{r}_{A/B}]$$

$$\vec{r}_{A/B} = \frac{\vec{\omega} \times \vec{V}_B}{\omega^2}$$

$$\vec{r}_A = \vec{r}_B + \frac{\vec{\omega} \times \vec{V}_B}{\omega^2}$$

AMB



$$\vec{H}_{/C} = \vec{r}_{G/C} \times (m\vec{v}_G) + \omega\hat{k} I^G$$

$$= \sum \vec{r}_{i/C} \times (m_i\vec{v}_i) \quad \rightsquigarrow \text{theory in text}$$

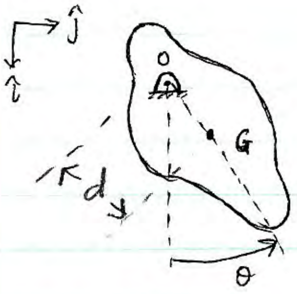
$$\dot{\vec{H}}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G \alpha k$$

$$\alpha \equiv \dot{\omega}$$

$$I^G = \int |\vec{r}_{/G}|^2 \, dm$$

(polar moment of inertia)

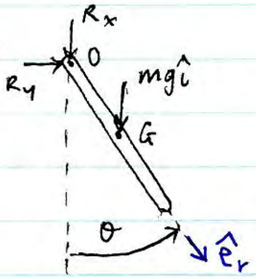
Example PENDULUM.



Main newish governing eqn:

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

FBD



use polar coord formula for accel. expressed in terms of $\vec{\omega}$ & $\vec{\alpha}$.

AMB$_{/O}$: $\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$

$$\underset{d\hat{e}_r}{\underset{\uparrow}{\vec{r}_{G/O}}} \times (mg\hat{\imath}) = \underset{d\hat{e}_r}{\underset{\uparrow}{\vec{r}_{G/O}}} \times m\underbrace{\vec{a}_G}_{-\omega^2\vec{r}_{G/O} + \underset{\ddot{\theta}}{\underset{\uparrow}{\dot{\omega}}}\hat{k}\times\vec{r}_{G/O} \;=\; \ddot{\theta}\, d\hat{e}_\theta - \omega^2 d\hat{e}_r} + I^G\alpha\hat{k}$$

$$\left\{ \begin{aligned} -dmg\sin\theta\,\hat{k} &= dm\,[0 + \dot{\omega}d]\,\hat{k} + I^G\alpha\hat{k} \\ &= (d^2m + I^G)\,\alpha\hat{k} \end{aligned} \right\}$$

$$\left.\begin{aligned} \hat{k}\cdot\{\} \Rightarrow & \boxed{\dot{\omega} = \frac{-dmg\sin\theta}{I^G + d^2m}} \\ & \dot{\theta} = \omega \end{aligned}\right\}$$

"The pendulum equation

Recitation

3/27/2013

PLANAR RIGID BODIES

- The $\dot{\vec{Q}}$ Formula
- $\vec{v}$, $\vec{a}$, $\vec{L}$, $\dot{\vec{L}}$, I^G, $\vec{H}$, $\dot{\vec{H}}$, E_K
- Ex. Physical Pendulum

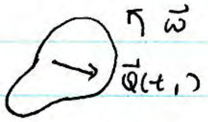
2 dimensions, ∞ # of particles (rigid body)

I. The $\dot{\vec{Q}}$ Formula

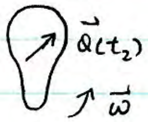
consider a flat rigid object with angular velocity $\vec{\omega}$.

$\vec{Q}$ is a vector $\overbrace{\text{drawn on}}^{= \text{"fixed on"}}$ the object.

Time t_1



Time t_2



$$\boxed{\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}}$$

True for any vector that's fixed in the "body frame".

II. $\vec{v}$, $\vec{a}$, etc. for Rigid Bodies

- $\vec{v}_{G/O} \equiv \dot{\vec{r}}_{G/O} = \vec{\omega} \times \vec{r}_{G/O}$
- $\vec{a}_{G/O} \equiv \ddot{\vec{r}}_{G/O} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{G/O}) + \dot{\vec{\omega}} \times \vec{r}_{G/O}$

$\vec{a}_{G/O} = \dot{\vec{\omega}} \times \vec{r}_{G/O} - \omega^2 \vec{r}_{G/O}$ [another way to write: $\vec{a} = \ddot{\theta} r \hat{e}_\theta - \dot{\theta}^2 r \hat{e}_r$]

$$\vec{L} = m\vec{v}_{G/O}, \quad \dot{\vec{L}} = m\vec{a}_{G/O}$$

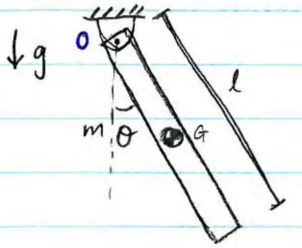
$$\vec{H}_{/O} = \vec{r}_{G/O} \times m\vec{v}_{G/O} + I^G \omega \hat{k}$$

$$\dot{\vec{H}}_{/O} = \underbrace{\vec{r}_{G/O} \times m\vec{a}_{G/O}}_{\substack{\text{acceleration (motion) of center of mass.} \\ \text{eg. Earth revolving around Sun}}} + \underbrace{I^G \dot{\omega} \hat{k}}_{\substack{\text{how fast it's spinning. is accelerating} \\ \text{eg. Earth rotating on its axis}}}$$

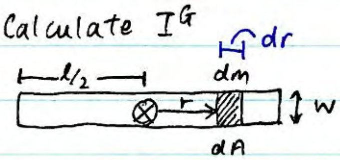
$$I^G = \int_{\text{all mass}} r^2 \, dm$$

$$E_K = \frac{1}{2} m v_{G/O}^2 + \frac{1}{2} I^G \omega^2$$

Ex. Physical Pendulum (like HW 15.4.34)



a) Calculate I^G



Assume constant density

$$\rho = \frac{m}{A} = \frac{dm}{dA}$$

$$I^G = \int_{\substack{\text{all}\\ \text{mass}}} r^2 \, dm$$

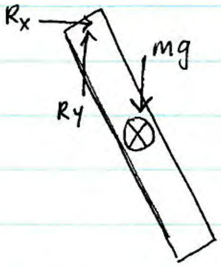
$$dA = w\,dr$$

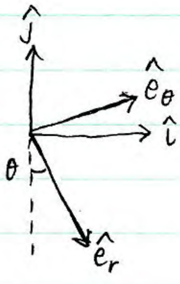
$$\rho = \frac{m}{\ell \cdot w} = \frac{dm}{dA} \Rightarrow dm = \frac{m}{\ell . w} dA = \frac{m}{\ell} dr$$

$$I^G = \int_{-\ell/2}^{\ell/2} \frac{m}{\ell} r^2 \, dr$$

$$= \frac{m}{3\ell} \left[r^3 \right]_{-\ell/2}^{\ell/2} = \frac{m\ell^2}{12}$$

b) Draw FBD, define unit vectors





$$\hat{e}_r = \sin\theta \, \hat{\imath} - \cos\theta \, \hat{\jmath}$$

$$\hat{e}_\theta = \cos\theta \, \hat{\imath} + \sin\theta \, \hat{\jmath}$$

c) Use AMB to find equations of motion.

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times -mg\hat{\jmath} = \vec{r}_{G/O} \times m\vec{a}_{G/O} + I^G \dot{\omega} \hat{k}$$

$$-mg\frac{\ell}{2} \sin\theta \, \hat{k} = \frac{m\ell}{2} \hat{e}_r \times \left[\dot{\vec{\omega}} \times \left(\frac{\ell}{2} \hat{e}_r\right) - \omega^2 \frac{\ell}{2} \hat{e}_r \right] + \frac{m\ell^2}{12} \dot{\omega} \hat{k}$$

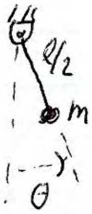
$$-mg\frac{l}{2}\sin\theta\,\hat{k} = \frac{ml}{2}\hat{e}_r\times\left[\underbrace{\dot\omega\hat{k}\times\left(\frac{l}{2}\hat{e}_r\right)}_{\dot\omega\frac{l}{2}\hat{e}_\theta}\right] + \frac{ml^2}{12}\dot\omega\,\hat{k}$$

$$= \frac{ml}{2}\hat{e}_r\times\frac{\dot\omega l}{2}\hat{e}_\theta + \frac{ml^2}{12}\dot\omega\,\hat{k}$$

$$-mg\frac{l}{2}\sin\theta = \frac{ml^2\overset{\swarrow\ddot\theta}{\dot\omega}}{3}$$

$$\boxed{\ddot\theta + \frac{3g}{2l}\sin\theta = 0.}$$

Note: for pt. mass at $l/2$ we would get



$$\ddot\theta + \frac{2g}{l}\sin\theta = 0$$

$2 > 3/2 \Rightarrow$ swings w/ higher frequency

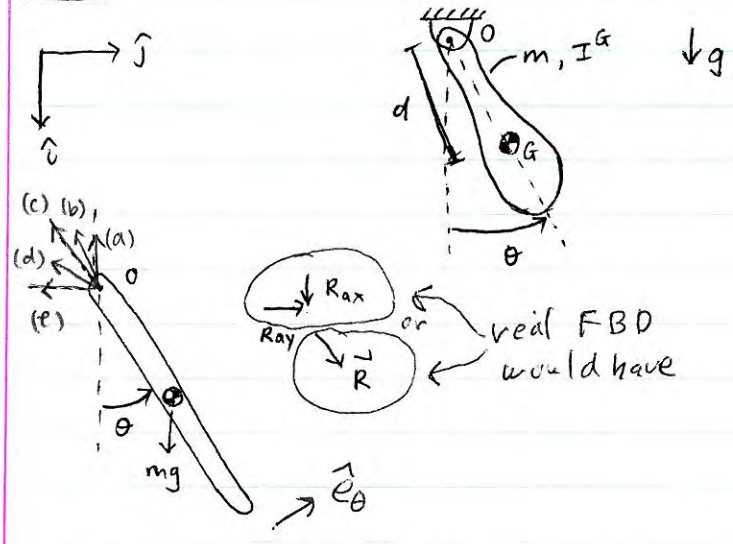
Lecture

3/28/2013

- Pendulum (con't).
- 2D object in general motion.

Recall

Quiz :



Which direction is $\vec{R}$ in ?

Same eqns:

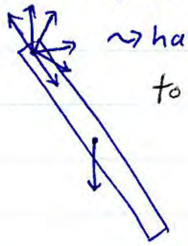
$AMB_{/O}$ $\quad \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O} \swarrow \vec{r}_{G/O} \times m\vec{a}_G + I^G \alpha \hat{k}$

$AMB_{/G}$ $\quad \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G} \swarrow I^G \alpha \hat{k}$

LMB $\quad \sum \vec{F} = m\vec{a}_G$

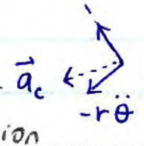
$AMB_{/O} \Rightarrow \ddot{\theta} < 0$, i.e. $\vec{\omega}$ ↺

$AMB_{/G} \Rightarrow$



⤳ has to be on this half plane to provide Torque. to make $\ddot{\theta} < 0$ $\qquad \vec{R} \cdot \vec{e}_\theta > 0$

LMB :



net acceleration

, $a_y < 0$. '←' direction (net force will also be in same direction)

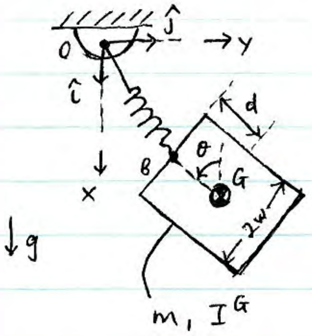
So answer is (b).

⇒

$\Leftarrow \begin{cases} \vec{R} \cdot \hat{j} < 0 \\ \vec{R} \cdot \hat{e}_\theta > 0 \end{cases}$

Example of 2D rigid object in space.

<u>Block suspended by spring.</u>



$$\frac{m(2w)^2}{12} + \frac{m(2d)^2}{12}$$

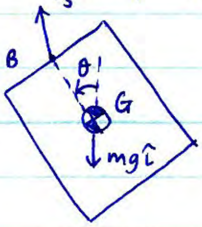
Given: g, l_0, k, m, d, I^G, w Given present state: $\underbrace{\underset{\llcorner \vec{r}_{G/O}}{\vec{r}_G}, \vec{V}_G, \theta, \underset{\llcorner \omega}{\dot{\theta}}}_{\text{"z" in programcode.}}$

Find motion.

i.e. $\underbrace{\dot{\vec{r}}_G, \dot{\vec{V}}_G, \dot{\theta}, \dot{\omega}}_{\text{zdot. solve on computer.}}$

$$\left.\begin{aligned} \dot{\vec{r}}_G &= \vec{V}_G \\ \dot{\theta} &= \omega \end{aligned}\right] \text{- 3 of our 6 ODES.}$$

FBD



$\vec{F}_s = -T \; \hat{\lambda}_{OB} \Rightarrow T = k(\Delta l) = k(l - l_0) = k(|\vec{r}_{B/O}| - l_0)$

$$\hat{\lambda}_{OB} = \frac{\vec{r}_{B/O}}{|\vec{r}_{B/O}|}$$

$$\vec{r}_B = \vec{r}_G + \vec{r}_{B/G}$$

$$\llcorner \vec{r}_{B/G} = -d\sin\theta\,\hat{j} - d\cos\theta\,\hat{i}$$

LMB

$$\vec{F}_s + mg\hat{i} = m\dot{\vec{V}}_G$$

$$\left.\dot{\vec{V}}_G = [\vec{F}_s + mg\hat{i}]\,\frac{1}{m}\right] \text{ 2 more ODEs}$$

AMB

$$\Sigma \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$\vec{r}_{B/G} \times \vec{F}_s = \ddot{\theta}\, I^G \hat{k}$ {can calc. $\vec{r}_{B/G}$ from calc. above.}

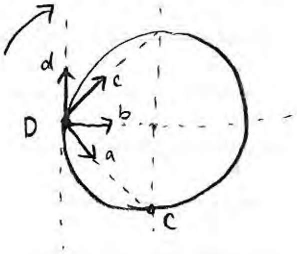
$\Rightarrow \ddot{\theta}$ can be calculated.] 6th and last ODE

//

Lecture 4/2/2013

- Rolling
- Heat treatment of aluminum

QUIZ



Rolling wheel

a) a

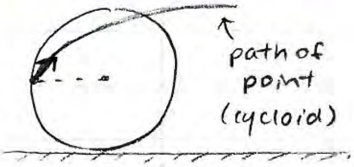
b) b

c) c ✓ By experiment in class of rolling disk.

d) d

e) cannot be determined.

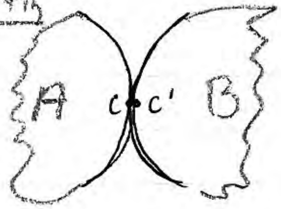
(which has $\vec{0}$ vel. at ~~this time~~)



$\vec{V}_D = \perp \vec{r}_{D/C'}$ (D is moving circles around C)

pt C. No velocity at this point. No lifting up or sliding (translational)

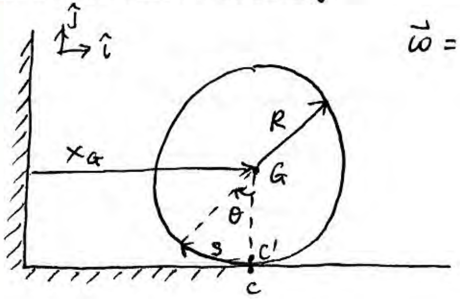
Any contacting objects



If contact (of rigid objects) with no sliding,

$\vec{V}_c = \vec{V}_{c'}$ (no penetration, no slip)

Kinematics of rolling



$\vec{\omega} = -\dot{\theta}\hat{k}$ standard CCW definition

position when θ=0

distance travelled = s = arc length = $X_G + X_0$

Contact

$$\vec{V}_c = \vec{V}_{c'}$$

$$\vec{0} = \underbrace{\vec{V}_G}_{V_G \hat{\imath}} + \underbrace{\vec{V}_{c'/G}}_{\vec{\omega} \times \underbrace{\vec{r}_{c'/G}}_{-R\hat{\jmath}}}$$

$$\vec{0} = V_G \hat{\imath} + (-\dot{\theta}\hat{k}) \times (-R\hat{\jmath})$$

$$0 = V_G - \dot{\theta}R$$

$\boxed{V_G = \dot{\theta}R}$ ~ true for all time $\Rightarrow$ $\boxed{V_G = -\overset{\omega = -\dot{\theta}}{\omega} R}$

True for all time:

$$a_G = \ddot{\theta} R$$

$$x_G = \theta R + \cancel{C}$$ constant $+ X_0$

$$\vec{V}_D = \vec{V}_G + \vec{V}_{D/G}$$

$\boxed{ex}$ $\vec{V}_D = \vec{V}_G + \vec{\omega} \times \vec{r}_{D/G}$

$= V_G \hat{i} + (-\dot{\theta}\hat{k}) \times (-R\hat{i})$

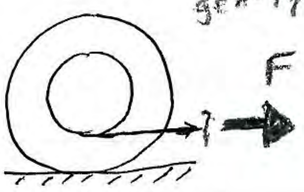
$= (\dot{\theta}R)\hat{i} + (\dot{\theta}R)\hat{j}$

$= (\dot{\theta}R)(\hat{i} + \hat{j})$ ~ same as picture at start of lecture

(answer c) $\vec{V}_D = \vec{\omega} \times \vec{r}_{D/C}$

$\boxed{\text{Quiz}}$ pulled gently

Spool



a) left

b) still

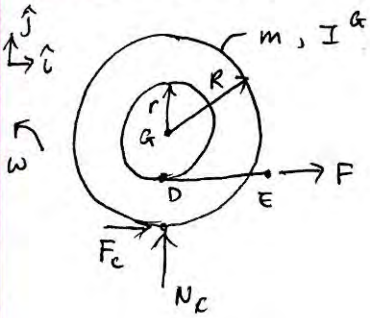
c) right ✓

d) cannot be determined

e) up.

(experiment w/ real spool)

where does it go?



$m, I^G = \frac{1}{2} m r^2$

IMPT!

ω is counterclockwise, sign convention as positive

$$\Sigma \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$\vec{r}_{D/G} \times F\hat{i} = \vec{r}_{G/C} \times (\vec{a}_G m) + I^G \overset{\alpha}{\dot{\omega}} \hat{k}$$

↓ same if put $\vec{r}_{E/G}$

↳ $\vec{a}_G = -\dot{\omega} R \hat{i}$

$$-(R-r)F\hat{k} = \dot{\omega} R^2 m \hat{k} + I^G \dot{\omega} \hat{k}$$

$$\dot{\omega} = \frac{-(R-r)F}{mR^2 + I^G}$$

↑ $\circlearrowleft_{+} < 0$

$\Rightarrow \vec{\omega} = \circlearrowright$

$R - r > 0$

$F > 0$

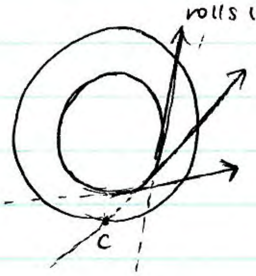
$\dot{\omega} < 0$

$\Rightarrow$ negative ccw ω

$\Rightarrow$ CW ω

If line of action of F passes to right of C ⇒ CCW

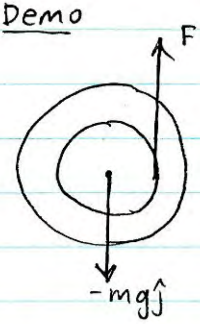
If line of action of F passes to right of C ⇒ CW



does not roll (line of action of force intersects point C).

$\Sigma \vec{M}_{/C} = 0$

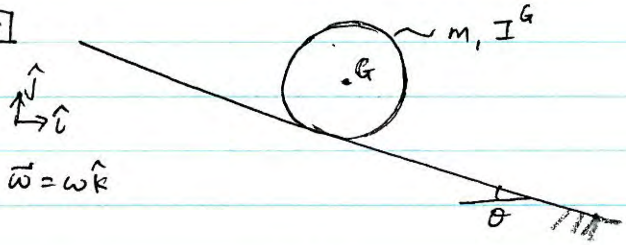
<u>Demo</u>



When dropped, reel falls straight down.

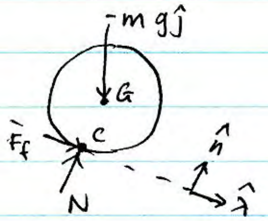
only forces are vertical

$\boxed{ex}$



ball rolls down slope

FBD



$F_f \neq \mu N$ because there is no slip

$\Sigma \vec{M}_{/C} = \vec{H}_{/C}$

$-mgR\sin\theta\,\hat{k} = \vec{r}_{G/C} \times (ma_G\hat{\lambda}) + I^G\alpha\hat{k}$

Recitation

4/3/2013

I. Mini Quiz : wheel FBDs

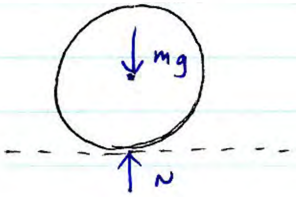
II. Recap : laws of mechanics for 2-D rigid objects

III. Rolling without slip

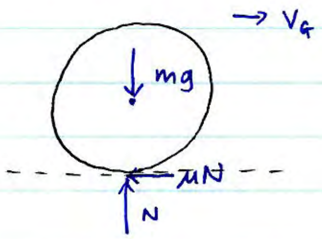
IV. Ex. Bowling Ball

I. Assume point contact, gravity, no driving forces / torques

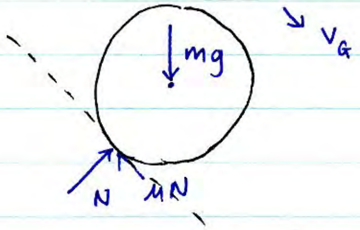
a) Flat ground, no friction



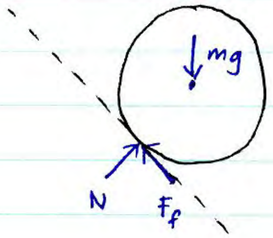
b) Flat ground, rolling WITH slip.



c) Hill, rolling WITH slip.

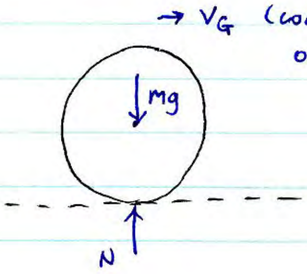


d) Hill, rolling without slip.



$F_f \leq \mu N$

e) Flat ground, rolling without slip.



(constant velocity, or sitting still)

II. RECAP

LMB. : $\Sigma \vec{F}^{ext} = \dot{\vec{L}}$, $\dot{\vec{L}} = m\vec{a}_G$

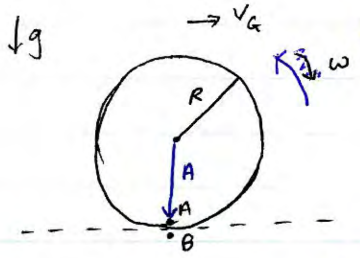
AMB. : $\Sigma \vec{M}^{ext}_{/C} = \dot{\vec{H}}_{/C}$, $\dot{\vec{H}}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G \dot{\omega} \hat{k}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ L $\ddot{\theta}$

Power Balance : $P = \dot{E}_K$, $\dot{E}_K = \frac{1}{2} m v_G^2 + \frac{1}{2} I^G \omega^2$

L when no dissipative forces such as { friction.
[sliding

III. Rolling without slip.



[need this sign convention if you want $\vec{\omega} = \omega \hat{k}$ and $\vec{V}_{A/B} = \vec{\omega} \times \vec{r}_{A/B}$]

No slip condition

$$\boxed{\vec{V}_A = \vec{V}_B} = \vec{0}.$$

Equivalently, for $\vec{v}_B = \vec{0}$,

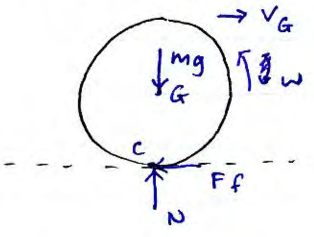
$$\boxed{\begin{aligned} V_G &= -\omega R \\ a_G &= -\dot{\omega} R \end{aligned}}$$

Proof:
$$\begin{aligned} \vec{V}_A &= \vec{V}_G + \vec{V}_{A/G} \\ &= \vec{V}_G + \vec{\omega} \times (\vec{r}_{A/G}) \\ &= V_G \hat{\imath} + \omega \hat{k} \times R(-\hat{\jmath}) \\ &= (V_G + \omega R)\,\hat{\imath} \end{aligned}$$

$$V_G + \omega R = 0.$$
$$V_G = -\omega R$$

Look at (e): Level ground, unforced rolling without slip. Find F_f.

Candidate FBD.



LMB
$$-mg\hat{\jmath} + N\hat{\jmath} - F_f\hat{\imath} = ma_G\hat{\imath} = m(-\dot{\omega}R)\hat{\imath}$$
$$F_f = m\dot{\omega}R$$

AMB$_{/C}$
$$\begin{aligned} \Sigma \vec{M}^{ext}_{/C} &= \dot{\vec{H}}_{/C} \\ \vec{0} &= \vec{r}_{G/C} \times m\vec{a}_G + I^G \dot{\omega} \hat{k} \\ &= -R\hat{\jmath} \times m(-\dot{\omega}R)\hat{\imath} + I^G \dot{\omega} \hat{k} \\ &= (-m\dot{\omega}R^2 + I^G \dot{\omega})\,\hat{k} \end{aligned}$$

$$0 = \underbrace{(I^G - mR^2)}_{\neq 0 \text{ as } I^G \neq mR^2.} \dot{\omega}$$

$$\Rightarrow \dot{\omega} = 0.$$

$$F_f = m\dot{\omega}R$$
$$= 0.$$

Pure undriven rolling on level ground means no rolling resistance.

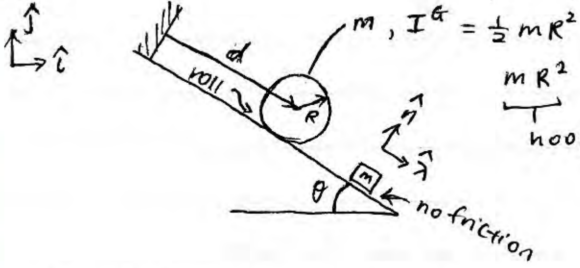
This assumes: * point contact (not distributed)
** No couples/torques/moments between ground and disk

Why does a real disk slow down?
Ans: rolling friction, a concept we ignore away in this course.
But, if you are interested, look up "rolling friction" in Wikipedia. [It comes from violation of * above.]

Lecture 4/4/2013

- Rolling

Recall



m, $I^G = \underbrace{\tfrac{1}{2}mR^2}_{}$ uniform disk

$\underbrace{mR^2}$ hoop.

Assume. Rolling without sliding.

Race : fastest to slowest.

a) block, disk, hoop ✓

b) hoop, disk, block

c) block, hoop, disk

d) disk, hoop, block

e) cannot be determined, none of above

This problem has 1 deg. of freedom, no energy dissipation.

Intuitively, Block $KE = \frac{1}{2}mv^2$, spends all energy on translational motion.

Hoop/, Disk $E_K = \frac{1}{2}mv^2 + \frac{1}{2}\, I \underset{\uparrow\, \omega^2 = \frac{v^2}{R^2}}{\omega^2}$ $\Big\}$ some energy for hoop & disk used for rotation, more so for hoop.

$= \frac{1}{2}mv^2 + \frac{1}{2}\frac{I}{R^2}v^2$

$E_{K final} = mgh$

$E_{K final} = \frac{1}{2}mv^2 + \frac{1}{2}\frac{I}{R^2}v^2$

Block : $I = 0$

Hoop : $I = mR^2$

Disk : $I = \frac{1}{2}mR^2$

smaller v for given E_K — smaller still

∴ Block fastest, then disk, then hoop.

For a 1 DoF system, use $\boxed{\text{energy}}$ to get eq. of motion.

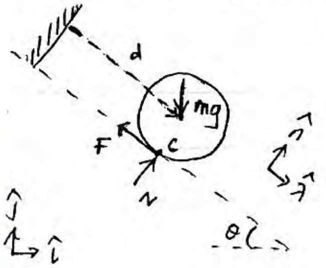
$$P = \dot{E}_K$$

$$-mg\dot{h} = \frac{d}{dt}\left[\frac{1}{2}mv^2 + \frac{I}{2R^2}v^2\right.$$

$$+\cancel{m}g\cancel{\dot{d}}\sin\theta = \cancel{m}\cancel{v}\dot{v} + \frac{I}{mR^2}\cancel{v}\dot{v} \qquad (\dot{d} = v)$$

$$\dot{v} = \frac{g\sin\theta}{\left(1 + \frac{I}{mR^2}\right)}$$ ⤳ independent of mass. because I (prop. to) to mass

Work out using Momentum



Friction

F_f without sliding ~ have to be calculated

μN with sliding

$|F_f| < \mu N$ ——— Friction

- It does no work as velocity of the point F_f acts on is zero. (in contact with ground) without sliding.

AMB/c

$$\sum M_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{r}_{G/c} \times (-mg\hat{j}) = \vec{r}_{G/c} \times m\vec{a}_G + I\alpha\hat{k}$$

$$-mgR\sin\theta\ \hat{k} = R\hat{n} \times m(\ddot{d}\hat{\lambda}) + I\alpha\hat{k}$$

$$= -R\ddot{d}\,m\,\hat{k} + I\alpha\hat{k}$$ ~ 2 unknowns, α and $\ddot{d}$

Kinematic constraint:

$$\alpha = -\frac{\ddot{d}}{R}$$ (found in last class)

$$-mgR\sin\theta = -R\ddot{d}m - \frac{I\ddot{d}}{R} = \ddot{d}\left(-Rm - \frac{I}{R}\right)$$

$$\ddot{d} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

Find F

$$\sum \vec{M}_{/G} = I\alpha\hat{k}$$

$$-RF\hat{k} = I\left(\frac{g\sin\theta}{1 + I/mR^2}\right)\left(-\frac{1}{R}\right)\hat{k}$$

$$\vdots$$

$$F = \frac{I}{R^2}\ddot{d}$$

Could also find using LMB! $\sum \vec{F} = m\vec{a}$

Find N

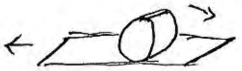
LMB : $\sum \vec{F} = m\vec{a}_G$

$$\{N\hat{n} - F\hat{\lambda} - mg\hat{j} = m\ddot{d}\hat{\lambda}\}$$

$\{\}\cdot\hat{n}$ $\quad N = mg\cos\theta.$

Demo in class

- Pull out paper from cylinder. ω when it leaves the paper then it slides. Eventually it rolls. Final rolling vel. = 0. Why? [Hardish HW. problem.]

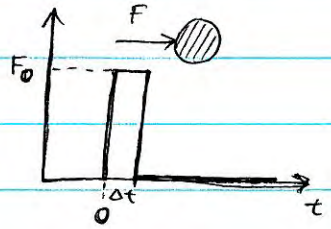


Lecture 4/9/2013

- collision Mechanics

  - sometimes you have big forces acting for a short time.

Ex Particle starts from rest.



What happens between $0 \le t \le \Delta t$?

$$F = ma$$

$$F = m\frac{dv}{dt}$$

$$\int_0^{\Delta t} F\,dt = m\int_0^{\Delta t} \frac{dv}{dt}\,dt$$

$$\overset{|||}{P} = m\,\Delta v$$

⇒ there is a change in velocity

$F_0 \cdot \Delta t$ = Area under F vs t curve.

Ex) F is constant : $v = v_0 + at$

$$= 0 + \frac{F_0}{m}t$$

$$= \frac{F_0}{m}t$$

$$x(\Delta t) = x_0 + \int_0^{\Delta t} v\,dt$$

$$= 0 + \int_0^{\Delta t} \frac{F_0}{m}t\,dt$$

$$\Delta x = \frac{F_0}{m}\frac{(\Delta t)^2}{2}$$

$$= \frac{P\Delta t}{2m} = 0\,!$$

↑ lim $\Delta t \to 0$, P fixed.

⇒ there is no change in position.

If time duration is short, change in position is negligible.

↑ same w/ θ and impulsive Moments.

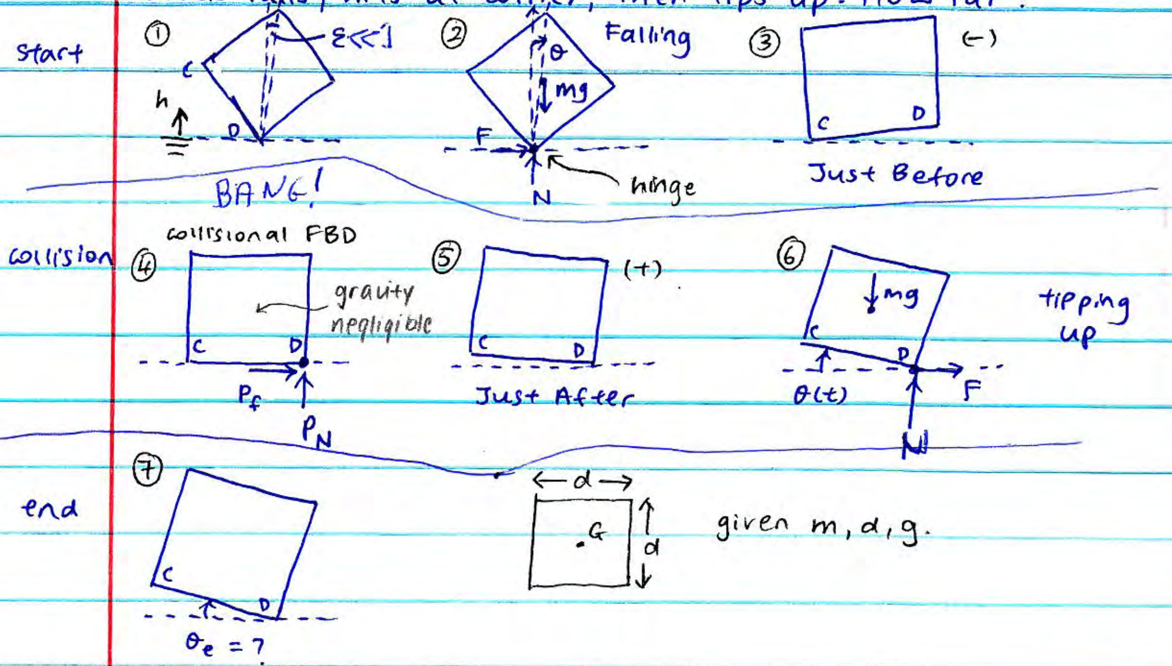
Collisional FBD

- neglect non collisional forces.

- show collisional impulses.

EXAMPLE

Block falls, hits at corner, then tips up. How far?



given m, d, g.

phase ② $\vec{\omega} = -\dot{\theta}\hat{k}$ — $AMB_{/C}$ or Energy conservation.

phase ③-⑤ collision calculation — $AMB_{/D}$

phase ⑥ $AMB_{/D}$ or energy conservation.

(AMB would require solving ODEs on computer)

QUIZ : $I^G = ?$ $\left(\int r_G^2\, dm\right)$

a) $md^2/2$

b) $md^2/3$

c) $md^2/6$ ✓

d) $md^2/12$

e) other.

$$\int x^2+y^2\, dm = \frac{\rho d^4}{12} + \frac{\rho d^4}{12} = md^2/6$$

Three succesive calculations

② cons. of E $\Rightarrow \omega^-$

④ given $\omega^- \underset{AMB/D}{\Rightarrow} \omega^+$

⑥ given $\omega^+ \underset{\text{cons. of E}}{\Rightarrow} \theta_e$

at ② : $E_{TOT①} = E_{TOT③}$

$$v^- = \omega^- d \frac{\sqrt{2}}{2}$$

$$\underbrace{mg\, d\frac{\sqrt{2}}{2}}_{E_{P1}} = \underbrace{mg\frac{d}{2}}_{E_{P3}} + \underbrace{\frac{1}{2}m(v_G^-)^2 + \frac{1}{2}I^G(\omega^-)^2}_{\underbrace{E_K(\omega^-)}_{E_{K3}}} \quad \ldots$$

* datum for Ep is ground.

$\Rightarrow \omega^- =$ (expression involving g, d, m)

Before collision; rotation about C

at ④ : AMB/D. $\vec{H}_{/D}^{\,-} = \vec{H}_{/D}^{\,+}$

After collision; rotation about D

Before coll.

$$\underbrace{\vec{r}_{G/D} \times m\vec{v}_G^{\,-} + I_G\omega^-\hat{k}}_{\text{Before coll.}} = \underbrace{\vec{r}_{G/D} \times m\vec{v}_G^{\,+} + I_G\omega^+\hat{k}}_{\text{After coll.}}$$

After coll.

$\frac{d}{2}(-\hat{\imath}+\hat{\jmath})$ D

$\omega^-\hat{k}\times\vec{r}_{G/C}$ C

$\frac{d}{2}(\hat{\imath}+\hat{\jmath})$

D

$\omega^+\hat{k}\times\vec{r}_{G/D}$ D

$\frac{d}{2}(-\hat{\imath}+\hat{\jmath})$

$\vdots$

$\omega^+ = \omega^-/4 = - - - - -$

at ⑥ : energy cons. $E_{p5} + E_{k5} = E_{p7} + E_{k7}$

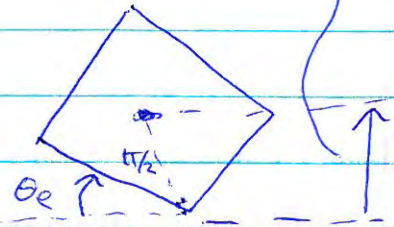
$\omega^+ \Rightarrow \theta_{end}$

state 7

$$\underbrace{\frac{mgd}{2} + \frac{1}{2}mv^{+2} + \frac{1}{2}I^G\omega^{+2}}_{\text{state 5}} = \overbrace{mg\underbrace{d\frac{\sqrt{2}}{2}\sin\left(\theta_e+\frac{\pi}{4}\right)} + 0}^{\text{state 7}}$$

$\boxed{\omega^+ d\frac{\sqrt{2}}{2} = v^+}$

state 5



$h_e = d\frac{\sqrt{2}}{2}\sin\left(\theta+\frac{\pi}{4}\right)$

$$\boxed{\theta_e = \sin^{-1}\left[\underbrace{\text{big. expression w/ } m, d, g}\right] - \frac{\pi}{4}}$$

$\boxed{h_e - d/2 = \left(\frac{1}{4}\right)^2\left(\frac{\sqrt{2}}{2}d - \frac{d}{2}\right)}$

Simple form for final result.

Why?
(see next page)

actually, if you do all the algebra, m, g & d all must drop out by dimensional analysis.

Experts only :

Want to do this w/ minimal algebra?

Measure P. E. w/ datum at configuration ④, the lowest point of G in the motion.

Before collision only ω^- contributes to $\vec{H}_{/D}$ because $\vec{V}^-$ is directed at D. After coll can use I^D ⇒

$$\omega^- \underset{\frac{1}{6}md^2}{I^G} = \omega^+ \underset{\frac{1}{6}md^2 + m\left(\frac{d\sqrt{2}}{2}\right)^2 = \frac{2}{3}md^2}{I^D}$$

$$\Rightarrow \omega^+ = \frac{1}{4}\omega^- \Rightarrow E_k^+ = \frac{1}{4}E_k^-$$

$$\Rightarrow E_{p7} \approx \frac{1}{16} E_{p1}$$ (energy decreases by factor of 16)

$$\Rightarrow mg\left(h_e - \frac{d}{2}\right) = \left(mg\left(\frac{\sqrt{2}}{2}d - \frac{d}{2}\right)\right)\cancel{4}\ \frac{1}{16}$$

$$\Rightarrow \underbrace{h_e - \frac{d}{2}}_{\Delta h} = \frac{\sqrt{2}}{\cancel{8}32}d - \frac{d}{\cancel{8}32}$$

$$\frac{h_e}{d} - \frac{1}{2} = \frac{1}{32}(\sqrt{2} - 1)$$

$$\frac{\Delta h}{d/2} \approx \frac{(\sqrt{2}-1)}{16}$$

$$\Theta \approx \frac{.414...}{16} \approx .025$$
$$\approx 1.8^\circ$$

$$\frac{\Delta h}{d/2} \approx \theta_e$$

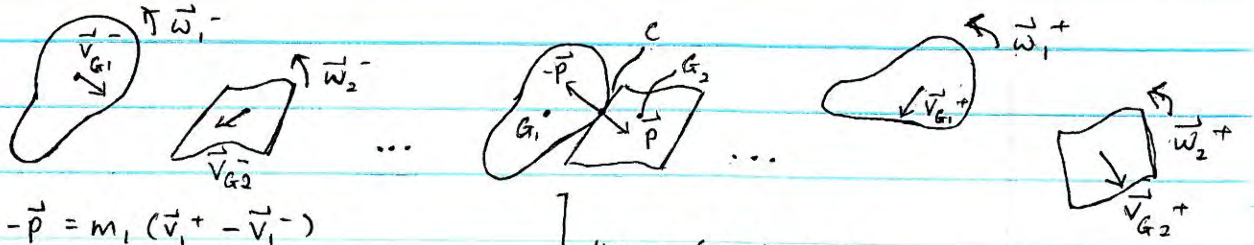
Recitation

4/10/2013

I – 2D Rigid object collisions

II – Ex. Hitting a Pool Ball (Frictionless collision)

I. Rigid Object collisions



$$\left.\begin{aligned} -\vec{P} &= m_1(\vec{v}_1^+ - \vec{v}_1^-) \\ \vec{P} &= m_2(\vec{V}_2^+ - \vec{V}_2^-) \end{aligned}\right\} \text{4 eqn, 6 unknowns}$$

$$\left.\begin{aligned} \vec{r}_{c/G_1} \times (-\vec{P}) &= I_1^G(\omega_1^+ - \omega_1^-)\hat{k} \\ \vec{r}_{c/G_2} \times (\vec{P}) &= I_2^G(\omega_2^+ - \omega_2^-)\hat{k} \end{aligned}\right\} \overset{+}{2}\text{ eqn, }\overset{+}{2}\text{ unknowns.}$$

Not solvable in general, 8 unknowns, more than 6 eqn.

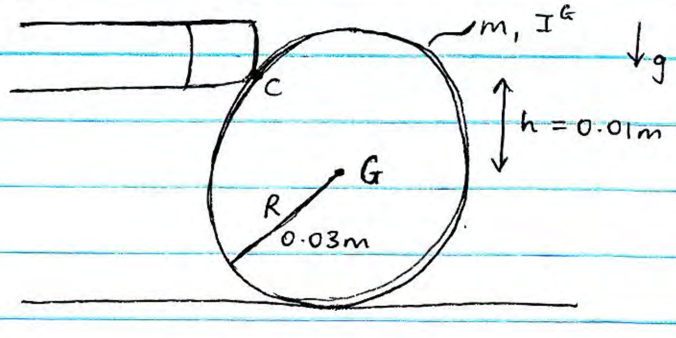
Special Cases

① Plastic (sticking) collision * see lecture notes.

② Frictionless collisions. $\Rightarrow \vec{P} = P\hat{n}$ (orthogonal to collision surface).
(rotation rates not going to change).

$$\Rightarrow (\vec{V}_2^+ - \vec{v}_1^+) = -e(\vec{V}_2^- - \vec{V}_1^-)$$

II. EXAMPLE



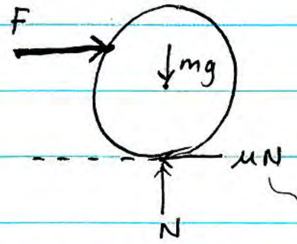
$m_b = 0.2\text{ kg}$

$V_b^- = \omega_b^- = 0.$

$V_b^+ = 1\text{ m/s}$

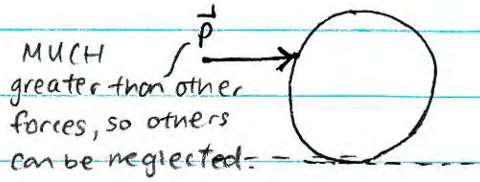
↖ refers to "ball"

a) Full FBD.



rolling with slipping

b) collisional FBD



MUCH greater than other forces, so others can be neglected.

c) Find impulse, $\vec{P}$.

$$\vec{P} = m_b(\vec{v}_b^{\,+} - \vec{v}_b^{\,-})$$

$$= 0.2\text{kg}(1\ \text{m/s})\,\hat{\imath}$$

$$P = 0.2\ \text{kgm/s}$$

Assume: $\vec{P} = P\hat{\imath}$

{Alternative: include normal impulse from ground and only ~~include~~ calc. $\hat{\imath}$ comp of $\vec{P}$ }

d) Find ω_b^+ (use $\vec{r}_{C/G} = -l\hat{\imath} + h\hat{\jmath}$)

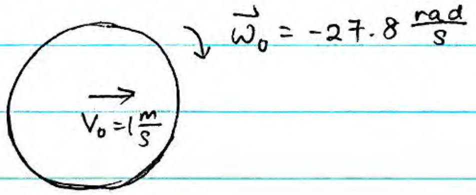
$$\vec{r}_{C/G} \times \vec{P} = I_b^G(\omega_b^+ - \omega_b^-)\hat{k}$$

$$(-l\hat{\imath} + h\hat{\jmath}) \times (\overset{\text{kg}\cdot\text{m/s}}{0.2}\,\hat{\imath}) = \frac{2}{5}m_bR^2(\omega_b^+)\hat{k}$$

$$-0.002\,\frac{\text{kg}\cdot\text{m}}{\text{s}} = \frac{2}{5}(\overset{\text{kg}}{0.2})(\overset{\text{m}}{0.03})^2(\omega_b^+)$$

$$\omega_b^+ = -27.8\ \frac{\text{rad}}{\text{s}}$$

e) After the strike, does the ball roll with or without slip?



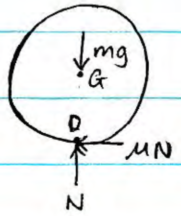
It is rolling with slip.

Without slip, $v = -\omega R$.

In this case, $V_0 \neq \underbrace{-\omega_0 R}_{-27.8 \times 0.03 \,=\, 0.95}$.

∴ rolling WITH slip.

f) How long does it take till it's rolling without slip?



LMB $\quad -\mu N\hat{\imath} + (N - mg)\hat{\jmath} = m\ddot{x}\,\hat{\imath}$

$$\ddot{x} = \frac{-\mu N}{m} = -\mu g$$

AMB/O $\quad \Sigma\vec{M}_{/O}^{\,ext} = \dot{\vec{H}}_{/O}$

$$0 = \vec{r}_{G/O} \times m\vec{a} + I_b^G\dot{\omega}\hat{k}$$

$$\Rightarrow \dot{\omega} = \frac{mR\ddot{x}}{I_b^G}$$

$$v(t) = V_0 - \mu g t$$

$$w(t) = w_0 - \frac{5}{2}\frac{\mu g}{R} t.$$

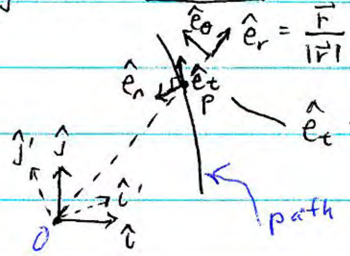
Applies until t^* w/ $v(t^*) = -w(t^*) R$. when rolling without slip.

then for $t > t^*$ prob. reverts to pure rolling

Lecture: 4/11/2013

- Polar coordinates
- Path coordinates

The big idea: $\boxed{\vec{a} = \vec{a}}$ ~ different ways of representing the same vector using different coordinate systems.



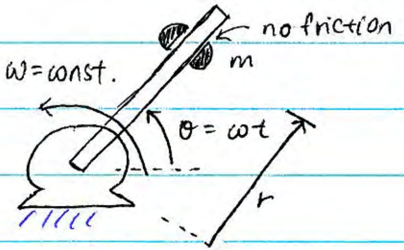
$\hat{e}_r = \frac{\vec{r}}{|\vec{r}|}$

$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|}$ = tangent to path

$$\left.\begin{aligned} \vec{a} &= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} \\ &= \ddot{x}'\,\hat{i}' + \ddot{y}'\,\hat{j}' \\ &= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta \\ &= \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n \end{aligned}\right\} \vec{a} = \vec{a}$$

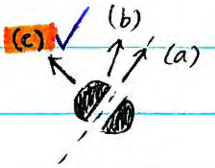
Ex "New Gun"

$g = 0$
$\mu = 0$



Quiz

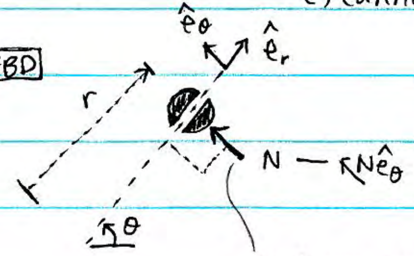
$\vec{a} = ?$



a)
b)
c) ✓
d) None of above
e) Cannot be determined

Dir. of accel. at instant shown is ?

FBD



so large that gravity can be neglected (because motor moves quickly)

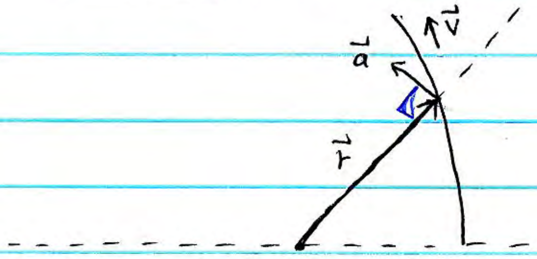
$\boxed{\text{LMB}}$

$$\Sigma \vec{F} = m\vec{a}$$

$$\left\{ N\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right]\right\}$$

$\{\}\cdot\hat{e}_r$ $\quad 0 = \ddot{r} - r\dot{\theta}^2$

$\boxed{\ddot{r} = \omega^2 r}$, where ω is constant.



constant coefficient 2^{nd} order linear ODE :

$$\ddot{r} = \omega^2 r \quad , \quad \text{I.C.} \quad r(0) = r_0$$
$$\dot{r}(0) = 0$$

$$r = r_0 \cosh(\omega t) = r_0 \cosh(\theta) \approx r_0 \frac{e^\theta}{2}$$
$$= r_0 \frac{e^{\omega t} + e^{-\omega t}}{2}$$

$\uparrow$ $\theta >> 1.$

Given $r_0 = .5\,m/s$, $\omega = 5$, $\theta_{max} = \pi$

$\Rightarrow \vec{V}_{final} = 85\,mph.$

$$\vec{r}(t) = r\hat{e}_r$$

$\hat{e}_r$: $\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

θ: ωt

r: $r_0 \cosh(\omega t)$

$$\vec{v}(t) = \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

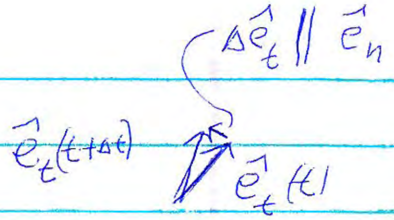
for, e.g., plotting

Radial acceleration has 2 components. For our case, no force along radius.

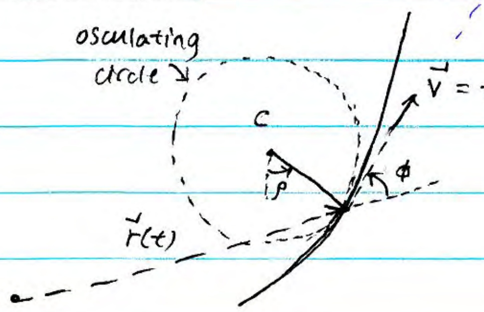
∴ centripetal term must exactly cancel radial $\ddot{r}$ term (changing radius)

* remember in circular motion, there is a radial acceleration,

eg.



Path coords.



$$\vec{v} = |\vec{v}|\,\frac{\vec{v}}{|\vec{v}|}$$

$$= v\hat{e}_t$$

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \dot{v}\hat{e}_t + v\dot{\hat{e}}_t$$

└ $\dot{\phi}\,\hat{e}_n$

↑ rate of change of dir. of $\vec{v}$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

└ ρ = radius of curvature

ex) Given $\vec{v}$ & $\vec{a}$ what is radius of curvature?

$\hat{e}_t = \vec{v}/|\vec{v}|$

$\hat{e}_n = \hat{k} \times \hat{e}_t$

$\{\vec{a} = \vec{a}\}$

$\{\quad = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n\}$

$\{\}\cdot\hat{e}_n \Rightarrow \quad \vec{a}\cdot\hat{e}_n = \frac{v^2}{\rho}$

$\rho = \frac{\vec{a}\cdot\hat{e}_n}{v^2}$

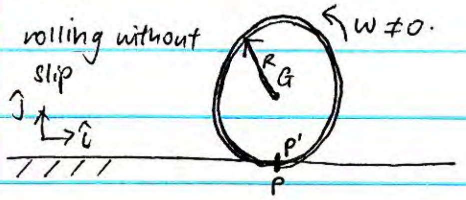
$$\boxed{\rho = \frac{\vec{a}\circ(\hat{k}\times\vec{v})}{|\vec{v}|^3}}$$

#3

Lecture 4/16/2013

- Rolling Quiz
- Brief Review
- Relative Motion

Quiz



a) $\vec{V}_p \neq \vec{V}_{p'}$ & $\vec{a}_p = \vec{a}_{p'}$

b) $\vec{V}_p = \vec{V}_{p'}$ & $\vec{a}_p \neq \vec{a}_{p'}$ ✓

c) $\vec{V}_p \neq \vec{V}_{p'}$ & $\vec{a}_p \neq \vec{a}_{p'}$

d) $\vec{V}_p = \vec{V}_{p'}$ & $\vec{a}_p = \vec{a}_{p'}$

e) none of above, cannot be determined.

p' has centripetal acceleration.

derivative of $\vec{V}_p = \vec{V}_{p'}$

is not $\vec{a}_p = \vec{a}_{p'}$. But, Tangential acceleration matches.

Why not? because p' is sequence of points below G, which is not a material point.

Review of course, so far:

Skills: Matlab, ODEs, vectors

Basics: 3 Pillars – Mat. Properties, Geometry, Laws

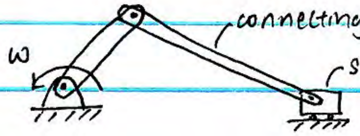
Organization: Particles("Free") – 1D, 2D(3D)

↳ force = f(position, velocity)

Constraints – Find or finesse constraint forces.

- Pulleys.
- Rigid Objects
  - finesse, by using ω, I^G
  - straight lines (car, box, etc)
  - circles (pendulum, particle on hoop)
  - general motion (2D)
  - more general const. motions
    - ex) rolling
    - ex) Newgun
    - ex) more to come

Demo – slider crank.



connecting rod

slider – usually piston! turns crank by ~~piston~~ explosion

motion is not sine wave. (geometric problem).

Looking at complicated machines

① Get "on board" a part of machine, look at connecting part.

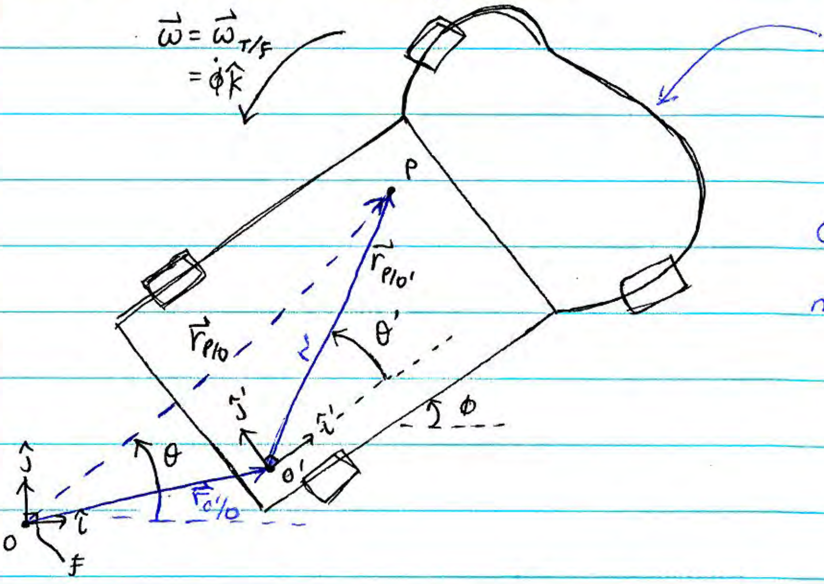
eg. motion of connecting rod relative to crank.

Relative Motion

Given motion of a rigid object (a moving reference frame) and given motion relative to that object, FIND:

- things about motion in fixed frame.

ex pick up truck (looking down)



moving Frame T (for Truck)

O' $\hat{i}'$, $\hat{j}'$ make up moving ref. frame T

Generally, given: $x', y', r', \dot{r}', \dot{x}', \dot{y}', \theta', \dot{\theta}', \ddot{x}', \ddot{y}'$ (motion rel. to truck)

also: $x_{O'/O}, y_{O'/O}, \dot{x}_{O'/O}, \dot{y}_{O'/O}, \ddot{x}_{O'/O}, \ddot{y}_{O'/O}, \omega, \dot{\omega}$ (motion of truck)

GOAL: Find $\vec{r}_{P/O}$, $\vec{v}_{P/F}$, $\vec{a}_{P/F}$

$$\begin{cases} \vec{r}_{P/O} = \{x_{O'}\hat{i} + y_{O'}\hat{j} + x'\hat{i}' + y'\hat{j}'\} \\ \vec{v}_{P/O} = \frac{d}{dt}\{\ \} \\ \vec{a}_{P/O} = \frac{d}{dt}\{\vec{v}_{P/O}\} \end{cases}$$

Note: $\hat{i}' = \hat{i}'(t)$

$\hat{j}' = \hat{j}'(t)$

Need to calc. $\dot{\hat{i}}'$, $\dot{\hat{j}}'$

$\ddot{\hat{i}}'$, $\ddot{\hat{j}}'$

OR $\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$ ~ $\overbrace{x'\hat{i}' + y'\hat{j}'}^{\text{using } T} = \overbrace{x_{P/O'}\hat{i} + y_{P/O'}\hat{j}}^{\text{using } F}$

$\vec{v}_{P/O} = \vec{v}_{O'/O} + \vec{v}_{P/O'}$ ~ $\vec{v}_{P/O'} = \dot{x}_{P/O'}\hat{i} + \dot{y}_{P/O'}\hat{j}$

$= \vec{v}_{O'/O} + \vec{v}_{P/T} + (\vec{\omega}_{T/F} \times \vec{r}_{P/O'})$ ~ P moving in circles around O'

$\vec{V}_{P|T} = \dot{x}'\hat{i} + \dot{y}'\hat{j}$ ⇒ velocity relative to frame.

└ truck frame

= velocity you would see of P if you were on the truck and could not look outside the truck.

Note: $\vec{V}_{P/T} \neq \vec{V}_{P/O'}$

↑ rel. to rotating truck ref. frame

$\llcorner\ \vec{V}_{P/O'} = (X_P - X_{O'})\hat{i} + (Y_P - Y_{O'})\hat{j}$

$= \vec{V}_{P/f} - \vec{V}_{O'/f}$

Key concept: vel. relative to moving ref. frame.

Recitation 4/17/2013

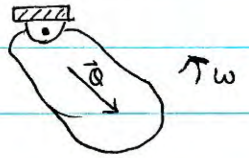
I. Mini Quiz

II. Vocab & Symbols

III. New, general $\dot{\vec{Q}}$ formula

IV. 3-term $\vec{v}$ & 5 term $\vec{a}$

I. Mini Quiz : $\dot{\vec{Q}}$



Name 2 situations when we CAN'T use $\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}$.

(1) Translating body. Formula will be incomplete. (Kind of, subtle.)

(2) If $\vec{Q}$ changes in body frame.

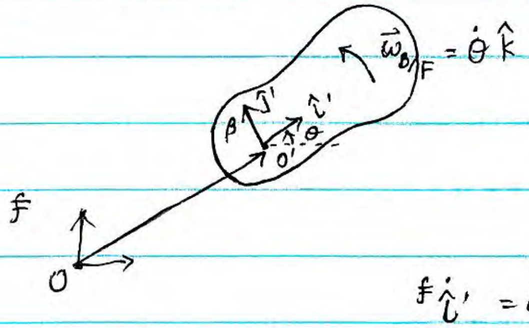
II. · A <u>FIXED FRAME</u> $\mathcal{F}$ on a coordinate system that doesn't move. We'll
use O to mean origin of $\mathcal{F}$.

O is origin.

· The <u>BODY FRAME</u> $\mathcal{B}$ is a coordinate system that moves with the object.
$\mathcal{B}$ can translate and rotate with respect to $\mathcal{F}$, but it is fixed on the body.

· "Absolute" velocity and acceleration are measured wrt. $\mathcal{F}$.

The general picture



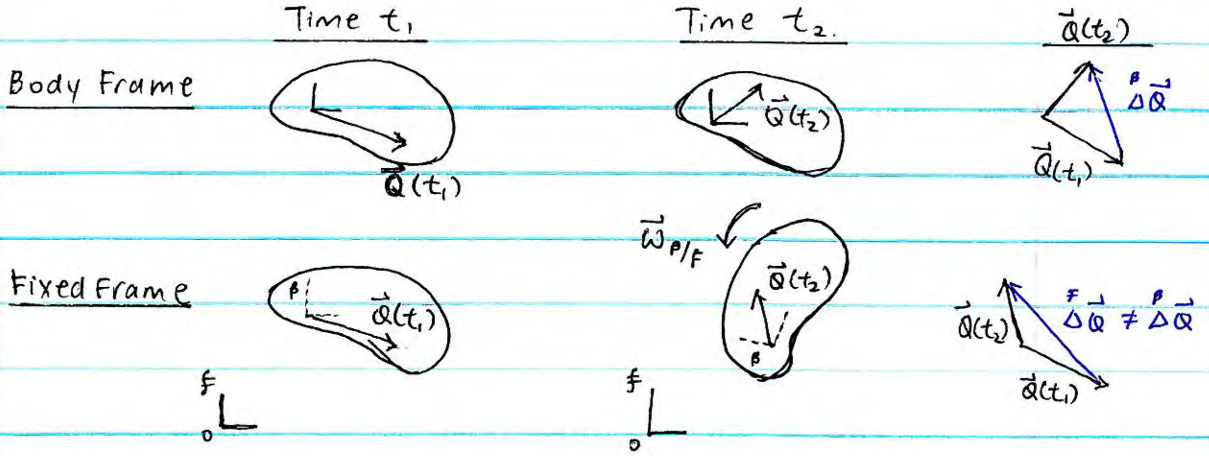
"$\vec{\omega}$"

$${}^{\mathcal{F}}\dot{\hat{\imath}}' = \vec{\omega}_{B/\mathcal{F}} \times \hat{\imath}'$$

$${}^{\mathcal{F}}\dot{\hat{\jmath}}' = \vec{\omega}_{B/\mathcal{F}} \times \hat{\jmath}'$$

III. New, General $\dot{\vec{Q}}$ formula.

The Big IDEA : Derivatives in F and B are different.



$$\boxed{{}^{F}\dot{\vec{Q}} = {}^{B}\dot{\vec{Q}} + \vec{\omega}_{B/F} \times \vec{Q}}$$

$$\underbrace{\text{"}\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}\text{"}}_{\text{sloppy notation but easy to remember}}$$

IV. 3-term $\vec{v}$ & 5-term $\vec{a}$

Find ${}^{F}\vec{v}_{P/O} \equiv {}^{F}\frac{d}{dt}\vec{r}_{P/O}$.

(a) Write $\vec{r}_{P/O}$ in terms of O' stuff.

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

(b) Differentiate, using ${}^{F}\dot{\vec{Q}}$ formula.

$${}^{F}\dot{\vec{r}}_{P/O} = {}^{F}\vec{v}_{O'/O} + \underbrace{\vec{\omega}_{B/F} \times \vec{r}_{P/O'} + {}^{B}\dot{\vec{r}}_{P/O'}}_{\text{derivative of } \vec{r}_{P/O'} \text{, using } {}^{F}\dot{\vec{Q}} \text{ formula.}}$$

↳ 3-term $\vec{v}$ formula.

$${}^{F}\vec{V}_{P/O} = {}^{F}\vec{V}_{O'/O} + {}^{B}\vec{V}_{P/O'} + \vec{\omega}_{B/F} \times \vec{r}_{P/O'}$$

$$\underbrace{\text{"}\vec{V} = \vec{V}_{O'} + \vec{V}_{rel} + \vec{\omega} \times \vec{V}\text{"}}_{\text{sloppy memorable version}}$$

Lecture

4/18/2013

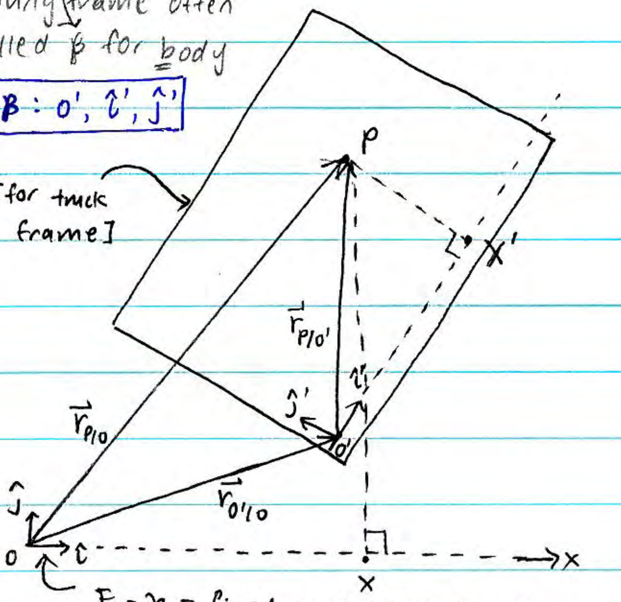
- Relative Motion.

capital script B

Moving frame often called $\mathcal{B}$ for body

$\boxed{\tau, \mathcal{B} : o', \hat{\imath}', \hat{\jmath}'}$

τ [for truck frame]



$\mathcal{F} = \mathcal{N} =$ fixed on Newtonian frame

O' has position $\vec{r}_{O'/O}$

vel. $\vec{V}_{O'/O}$

acceleration $\vec{a}_{O'/O}$

] rel. to $\mathcal{F}$ is implicit

τruck has $\vec{\omega}_\tau$ & $\dot{\vec{\omega}}_\tau$

$\llcorner$ means $\vec{\omega}_{\tau/\mathcal{F}}$

$\boxed{\text{QUIZ}}$

For picture shown, given $\vec{\omega}_\tau = 5/s^2\ \hat{k}$ $\qquad \dot{x}' = 1\,m/s =$ constant

$\dot{\vec{\omega}}_\tau = \vec{0}$ $\qquad \dot{y}' = 0 =$ constant

$\vec{V}_{O'/O} = 3\ m/s\ \hat{\imath}$ $\qquad x'(t) = 4\,m$ at t of interest

$\vec{a}_{O'/O} = 2\ m/s^2\,\hat{\imath}$ $\qquad y'(t) = 3\,m.$ " " " "

Find $\vec{v}_p$, $\vec{a}_p$ using $\vec{V}_{P/\tau}$, $\vec{a}_{P/\tau}$.

a) $\vec{V}_P = \vec{0}$, $\vec{a}_P = 0$

b) $\vec{V}_P = \vec{0}$, $\vec{a}_P \neq 0$

} $\vec{V}_P \neq \vec{0}$

c) $\vec{V}_{P/\tau} = \vec{0}$, $\vec{V}_{P/\mathcal{F}} \neq 0$. wrong as $\dot{x}' = 1m/s$.

d) $\vec{a}_{P/\tau} = \vec{0}$, $\vec{a}_P \neq 0$. ✓

e) more than one of above, or none of above.

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$\vec{V}_{P/O} = \vec{V}_{O'/O} + \underbrace{\vec{V}_{P/B} + \underset{\llcorner \vec{\omega}_{B/F}}{\vec{\omega}}}_{} \times \vec{r}_{P/O'}$$

$$\vec{V}_{P/O'} = \vec{V}_{P/F} = \vec{V}_{P/F} - \vec{V}_{O'/F} = (\dot{x}_P - \dot{x}_{O'})\hat{i} + (\dot{y}_P - \dot{y}_{O'})\hat{j}$$

$$\vec{a}_{P/O} = \vec{a}_{O'/O} + \vec{a}_{P/B} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'})}_{= -\omega^2 \vec{r}_{P/O'} \text{ in 2D}} + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega} \times \vec{V}_{P/B}$$

① Bug glued to truck, truck accelerating, not rotating.

$$\vec{a}_{P/O} = \vec{a}_{O'/O}$$

② Bug glued to truck, truck only rotating at constant ω, O' is still.

like circular motion acceleration of bug $\vec{a}_P = -\omega^2 \vec{r}_{P/O'}$

③ Truck is still, bug runs around

$$\vec{a}_{P/O} = \vec{a}_{P/B}$$

④ $2\dot{\theta}\dot{r}$ ← similar to this term.

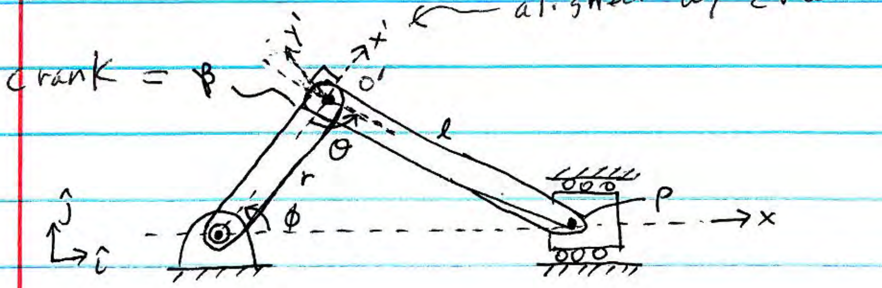
Bug walking at constant speed on x' axis, truck rotating at constant rate, passing through O' point.

⇒ curved path.

$$\vec{a}_{P/O} = 2\vec{\omega} \times \vec{V}_{P/B}$$

---

ex) Slider-Crank



Given: $r, \ell, \theta, \phi, \dot{\phi}, \ddot{\phi}$

Find: $\dot{\theta}, \ddot{\theta}$

First $\boxed{\vec{V}_P = \vec{V}_P}$

$$\Big\{ \underbrace{V_P \hat{\imath}}_{\substack{\uparrow \\ \text{slider}}} = \vec{V}_{O'/O} + \vec{\omega}_B \times \vec{r}_{P/O'} + \vec{V}_{P/B} \Big\} \qquad 2 \text{ eqs.}$$

$\vec{V}_{O'/O}$: $\dot{\phi}\hat{k} \times \vec{r}_{O'/O}$

$\vec{\omega}_B$: $\dot{\phi}\hat{k}$

$\vec{r}_{P/O'}$: calculate from picture

$\vec{V}_{P/B}$: $(\dot{\theta}\hat{k}) \times \vec{r}_{P/O'}$

unknowns: $V_P, \dot{\theta}$

$\{\}\cdot\hat{\jmath} \Rightarrow$ 1 equation to solve for $\dot{\theta}$.

Next: $\boxed{\vec{a}_P = \vec{a}_P}$

$$\begin{aligned}
\Big\{ \underbrace{a_P \hat{\imath}}_{\text{slider}} = \ & \vec{a}_{O'/O} && = -\dot{\phi}^2 \vec{r}_{O'/O} + \ddot{\phi}\hat{k} \times \vec{r}_{P/O} \\
& + \vec{a}_{P/O'} && = -\dot{\theta}^2 \vec{r}_{P/O'} + \ddot{\theta}\hat{k} \times \vec{r}_{P/O'} \\
& + -\omega^2 \vec{r}_{P/O'} && = -\dot{\phi}^2 \vec{r}_{P/O'} \\
& + \dot{\vec{\omega}} \times \vec{r}_{P/O'} && = -\ddot{\phi}\hat{k} \times \vec{r}_{P/O'} \\
& + 2\vec{\omega}_{B/F} \times \vec{V}_{P/O'} \Big\} && = 2(\dot{\phi}\hat{k}) \times \Big[\underbrace{(\dot{\theta}\hat{k}) \times \vec{r}_{P/O'}}_{\vec{V}_{P/O'}} \Big]
\end{aligned}$$

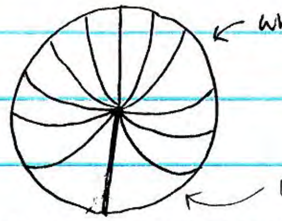
$\{\}\cdot\hat{\jmath} \Rightarrow$ one eqn. for $\ddot{\theta}$ in terms of given quatities

Today Lecture 4/23/2013

- Bicycle
- Particle motion relative to a frame.

- Photograph of bicycle. (Scanning film finish-line photo)

Small space = small time



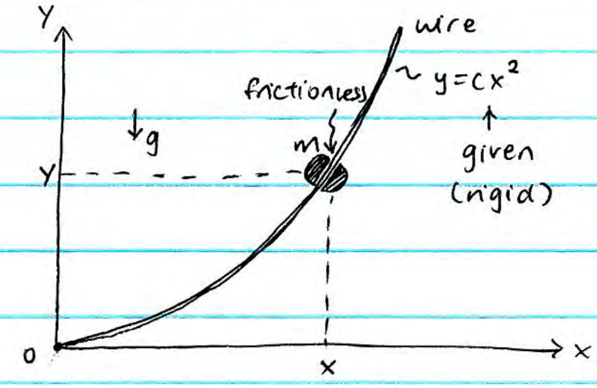
[used originally in horse races; New to bicycle races]

SCANNING SLOT CAMERA:

camera has image line (rather than plane). film slides by slot, as bicycles pass finish line., [Compare w/ photo on book cover (where camera follows bike)]

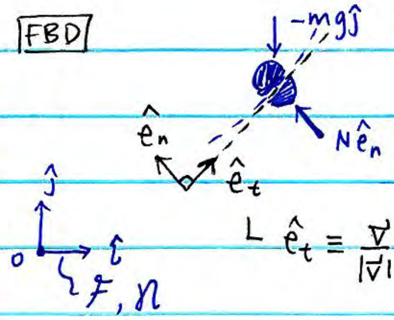
1) Project idea: Use Matlab to draw shape of spokes, use calculations.

Ex Bead on stationary wire:



How does bead move? ⇒ $\ddot{x}$? in terms of $x, \dot{x}, c, m, g$.

FBD



$$L \quad \hat{e}_t = \frac{\vec{v}}{|\vec{v}|} = \frac{\dot{x}\hat{\imath} + 2cx\dot{x}\hat{\jmath}}{\sqrt{\dot{x}^2 + (2cx\dot{x})^2}}$$

LMB $\Sigma \vec{F} = m\vec{a}$

$$-mg\hat{j} + N\hat{e}_n = m\frac{d^2}{dt^2}(\vec{r})$$

$$\llcorner \vec{a} = \frac{d}{dt}\left(\frac{d}{dt}\vec{r}\right) = \frac{d}{dt}\cdot\frac{d}{dt}(\overbrace{x\hat{i} + \underset{\llcorner cx^2}{y}\hat{j}}^{\vec{r}})$$

$$\vec{a} = \frac{d}{dt}(\overbrace{\dot{x}\hat{i} + 2cx\dot{x}\hat{j}}^{\vec{v}})$$

$$\vec{a} = \ddot{x}\hat{i} + 2c(\dot{x}^2 + x\ddot{x})\hat{j} \quad (*)$$

$$\{-mg\hat{j} + N\hat{e}_n = m\ddot{x}\hat{i} + (2c\dot{x}^2 + 2cx\ddot{x})\hat{j}\} \quad (**)$$

Quiz: How many eqns and unknowns? in this eqn.

a) 2,2 ✓

b) 2,3

c) 3,2

d) 2,1

e) other

2 eqn: vector in 2D

unknown: $N, \ddot{x}$

would use $\hat{e}_t$

use $\vec{a}$ from $*$ or $**$

$$\{\}\cdot\vec{v} \Rightarrow \vec{v}\cdot(-mg\hat{j}) = \vec{v}\cdot(m\vec{a})$$

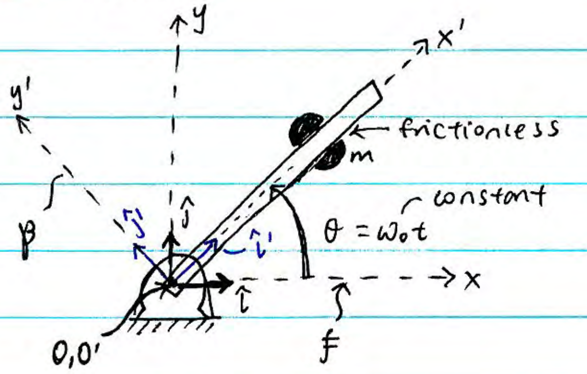
algebra

$$\boxed{\ddot{x} = f(x, \dot{x}, m, g, c)}$$

$$\{\vec{F} = m\underset{\llcorner \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n}{\vec{a}}\}$$

$$\{\}\cdot\hat{e}_t \Rightarrow \underset{\uparrow -mg\hat{j} + N\hat{e}_n}{\vec{F}}\cdot\hat{e}_t = m\dot{v} \quad \dot{v} = \ddot{s}$$

Ex New Gun



FBD

$*\quad \vec{V}_{P/O'} \quad \boxed{\text{VS}} \quad \vec{V}_{P/B}$

$\llcorner\ \vec{V}_{P/\mathcal{F}} - \vec{V}_{O'/\mathcal{F}} \qquad \llcorner = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$

$\boxed{\text{LMB}} \quad \Sigma\vec{F} = m\vec{a}$

$$
\begin{array}{lll}
\{ N\hat{\jmath}' = m[\,\vec{a}_{O'/O} & = m\ [\,\vec{0} & \} \\
\qquad + \vec{a}_{P/B} & \quad + \ddot{x}'\hat{\imath}' & \\
\qquad + -\omega^2\vec{r}_{P/O'} & \quad + -\omega_0^2\, x'\hat{\imath}' & \\
\qquad + \dot{\vec{\omega}} \times \vec{r}_{B/O'} & \quad + \vec{0} & \sim \omega \text{ constant} \\
\qquad + 2\vec{\omega} \times \vec{v}_{P/B} & \quad + 2\omega_0\hat{k} \times \dot{x}'\hat{\imath}' & = 2\omega_0\dot{x}'\hat{\jmath}'
\end{array}
$$

$\{\} \cdot \hat{\imath}' \qquad 0 = (\ddot{x}' - \omega_0^2\, x')\cancel{m}$

$$\boxed{\ddot{x}' = \omega_0^2\, x'}$$

$$x' = x_0 \cosh(\omega_0 t) + \frac{V_0}{\omega_0}\sinh(\omega_0 t)$$

Recitation

4/24/2013

5-term $\vec{a}$

- Formula Recap
- Ex. Double Pendulum $\vec{a}$
  - Method 1: write $\vec{r}$ and differentiate
  - Method 2: 5-term $\vec{a}$ formula

I. RECAP

$$\underset{(0)}{{}^{F}\vec{a}_P} = \underset{(1)}{{}^{F}\vec{a}_{O'/O}} + \underset{(2)}{\vec{\omega}_B \times (\vec{\omega}_B \times \vec{r}_{P/O'})} + \underset{(3)}{\dot{\vec{\omega}} \times \vec{r}_{P/O'}} + \underset{(4)}{{}^{B}\vec{a}_P} + \underset{(5)}{2\,\vec{\omega}_B \times {}^{B}\vec{V}_P}$$

(0): Acceleration of point P as seen in fixed frame.

(1): Acceleration of point O' as seen in fixed frame.

(2): Centripetal acceleration

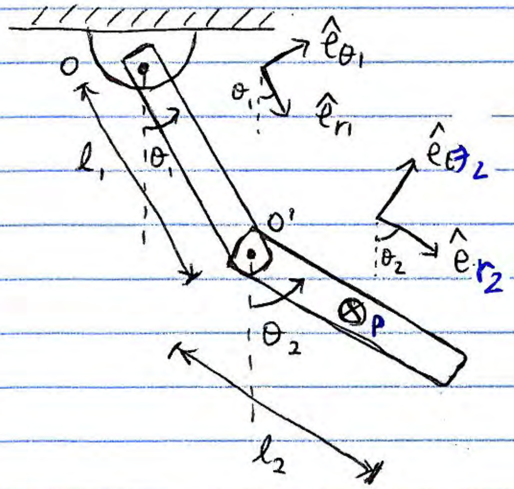
(3): Tangential acceleration

(4): Acceleration of point P as seen in body frame.

(5): Coriolis acceleration (see text p. 858 - 859).

* Remember, we can always find $\vec{a}$ by writing $\vec{r}(t)$ and taking derivatives. But the 5-term $\vec{a}$ formula can save time!

II. $\boxed{\text{Ex}}$ Double pendulum $\vec{a}$



$$\dot{\hat{e}}_{ri} = \dot{\theta}_i\,\hat{e}_{\theta i}$$
$$\dot{\hat{e}}_{\theta i} = -\dot{\theta}_i\,\hat{e}_{ri}$$

* P center of mass of bottom bar.

Method 1:

a) write position, $\vec{r}_{P/O} = l_1 \hat{e}_{r_1} + \frac{1}{2} l_2 \hat{e}_{r_2}$

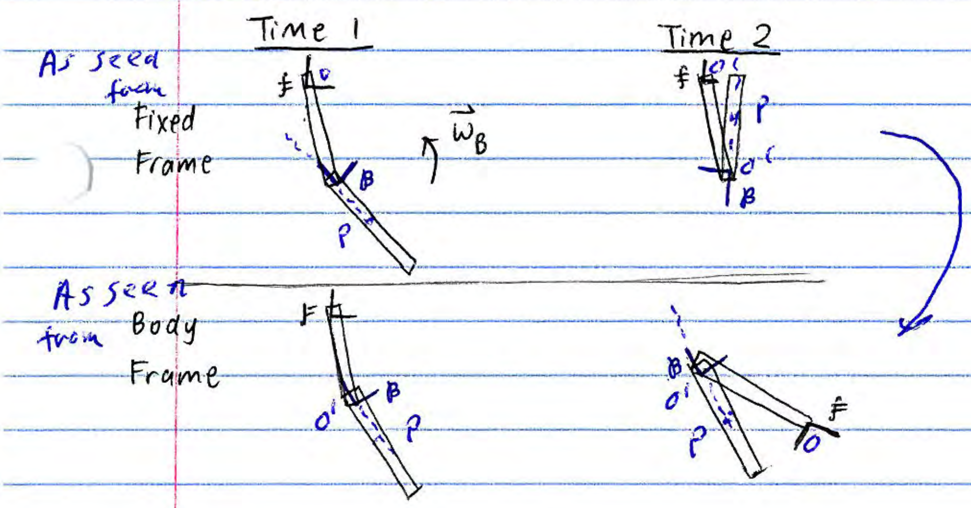
b) calculate $\vec{v}_{P/O} \equiv {}^{F}\frac{d}{dt} \vec{r}_{P/O}$

$$\vec{v}_{P/O} = l_1 \dot{\hat{e}}_{r_1} + \frac{l_2}{2} \dot{\hat{e}}_{r_2} = l_1 \dot{\theta}_1 \hat{e}_{\theta 1} + \frac{l_2}{2} \dot{\theta}_2 \hat{e}_{\theta 2}$$

c) Calculate $\vec{a}_P \equiv {}^{F}\frac{d}{dt} \vec{v}_{P/O}$

$$\vec{a}_P = l_1 \frac{d}{dt}(\dot{\theta}_1 \hat{e}_{\theta 1}) + \frac{l_2}{2} \frac{d}{dt}(\dot{\theta}_2 \hat{e}_{\theta_2})$$
$$= l_1 (\ddot{\theta}_1 \hat{e}_{\theta_1} - \dot{\theta}_1^2 \hat{e}_{r_1}) + \frac{l_2}{2}(\ddot{\theta}_2 \hat{e}_{\theta 2} - \dot{\theta}_2^2 \hat{e}_{r_2}) \qquad *$$

Method 2: 5-term formula



Same situations, different points of view

a) are any terms in the formula ZERO?

$${}^{B}\vec{a}_P \ , \ 2\vec{\omega}_B \times {}^{B}\vec{v}_P$$ (point P is not moving in Body Frame).

b) what do we use for $\vec{\omega}_B$?

$$\vec{\omega}_B = \dot{\theta}_2 \hat{k}$$

c) Find (2): $\vec{\omega}_B \times (\vec{\omega}_B \times \vec{r}_{P/O'})$

$$= \dot{\theta}_2 \hat{k} \times [\dot{\theta}_2 \hat{k} \times (\cancel{l_1 \hat{e}_{r_1} +} \frac{l_2}{2} \hat{e}_{r_2})]$$

$$= \dot{\theta}_2 \hat{k} \times \cancel{l_1 \dot{\theta}_1 \hat{e}_{\theta_1} +} \left(\frac{l_2}{2} \dot{\theta}_2 \hat{e}_{\theta 2} \right)$$

$\vec{\omega}_B \times (\vec{\omega}_B \times \vec{r}_{P/O'}) \Rightarrow (2) = -\frac{l_2}{2}\dot{\theta}_2^2\,\hat{e}_{r_2}$

d) Find (1) , $\vec{a}_{O'/O}$

$$* \; \vec{a}_{O'/O} = l_1(\ddot{\theta}_1\,\hat{e}_{\theta_1} - \dot{\theta}_1^2\,\hat{e}_{r_1})$$

* O' moves in circles of radius l_1 about O.

e) Find (3), $\dot{\vec{\omega}}_B \times \vec{r}_{P/O'}$

$\llcorner \ddot{\theta}_2\,\hat{k}$

$$\ddot{\theta}_2\,\hat{k} \times \frac{l_2}{2}\hat{e}_{r_2} = \frac{l_2}{2}\ddot{\theta}_2\,\hat{e}_{\theta_2}$$

$$\vec{a}_P = l_1(\ddot{\theta}_1\,\hat{e}_{\theta_1} - \dot{\theta}_1^2\,\hat{e}_{r_1}) + \frac{l_2}{2}(\ddot{\theta}_2\,\hat{e}_{\theta_2} - \dot{\theta}_2^2\,\hat{e}_{r_2})$$, again

↑

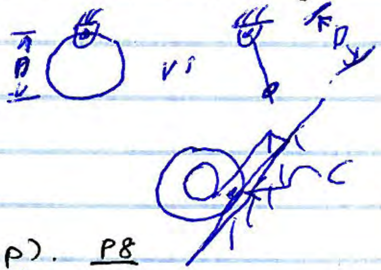
This is the same as what we got from Method 1 ; (*)

Lecture 4/25/2013

- Prelim Demos

HOOP as in Qn 7, compared to point mass pendulum.

- If length of point mass pendulum = diameter of hoop, frequency/period of motion is about the same.
- If length is shorter, moves faster than hoop.
- If length is longer, moves slower than hoop.



SPOOL Demo. rolling/sliding (up) the table (ramp). P8

- as the spool rolls down the ramp, bottom point slides upward.

ARROW P9

- cotton balls : if thrown sideways, does not really straighten it out
- feathers : if thrown sideways, friction force on feathers causes arrow motion to straighten out.

- Bike

Demo

- tie a string to the pedal and pull backwards.

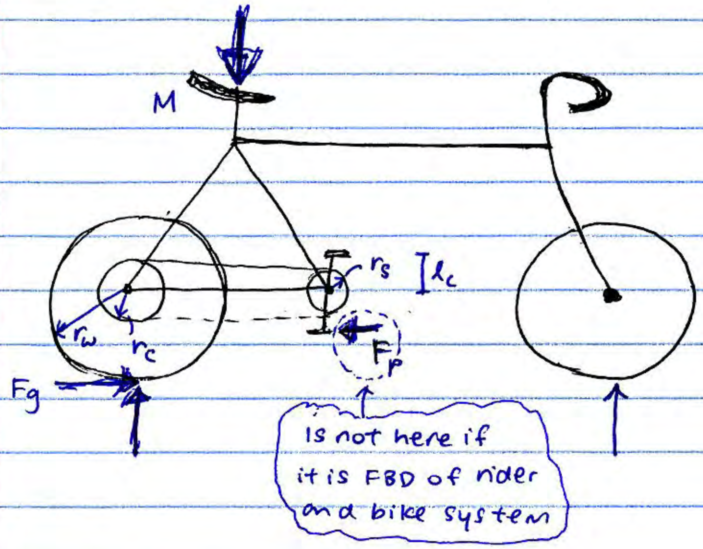
Quiz

- Does the bike move : a) right (got most of the votes)
  b) left ✓
  c) stay still
  d) depends
  e) None of above

Result

- Bike moves to the left. (backwards)
- But there is a critical angle ~~af~~ depending on the gear ratio where the bike does not move.
- Greater than that critical angle, bike moves forward.

FBD of Bike



Is not here if it is FBD of rider and bike system

* neglect mass of wheels and crank.

l_c = crank length

r_s = radius of front sprocket

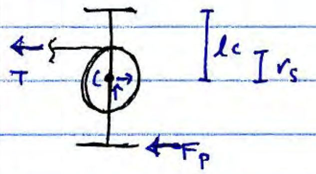
r_c = radius of rear cluster, or rear cassette

r_w = radius of rear wheel

F_p = backwards force on bottom pedal

F_g = forwards force from ground on rear wheel

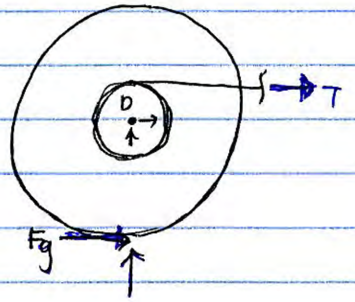
FBD crank



$$\sum \vec{M}_c = \vec{0}$$

$$\Rightarrow \boxed{T = F_p \frac{l_c}{r_s}}$$

FBD rear wheel



$$\sum \vec{M}_D = \vec{0}$$

$$F_g = T\frac{r_c}{r_w}, \quad T$$

$$\boxed{F_g = \frac{l_c}{r_s}\frac{r_c}{r_w}F_p}$$

LMB whole bike

$$\left\{\sum \vec{F} = M\underbrace{\vec{a}_b}_{a_b\hat{\imath}}\right\}\cdot\hat{\imath}$$

$$F_g - F_p = M a_b$$

$$a_b = (F_g - F_p)/M$$

$$= \left(\frac{l_c r_c}{r_s r_w} - 1\right)F_p/M.$$

If gearing is low enough (r_c big, l_c big, r_w small, r_s small) bike ~~goes~~ accelerates right. No real bike, none that you can buy, happens to have such low gears.

If $\frac{l_c r_c}{r_s r_w} > 1$ ⇒ bike accelerates right., <1, accelerates left (backward)

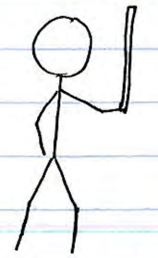
LECTURE 4/30/2013

- Pendulum with moving support.

Inverted pendulum.

- To balance it, you have to move your hand in just the right way, in response to motion of stick.

Demo



Aside: Recall

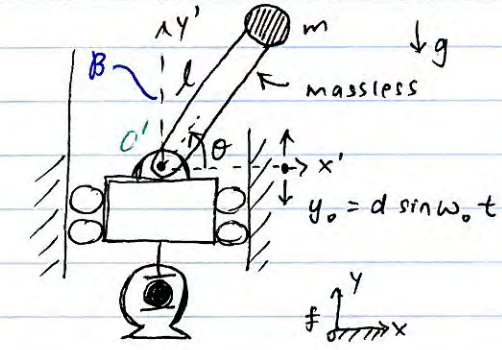
Kinematics: geometry

5 term $\vec{a}$

polar/path coord $\vec{a}$.

kinetics: mechanics of motion.

$\vec{F} = m\vec{a}$ etc

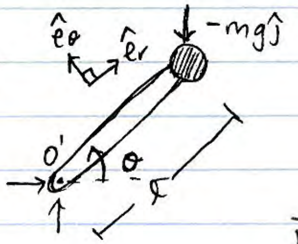


w/ physical demo

vertical motion of O'.

$\Rightarrow \vec{a}_{O'} = -d\omega_o^2 \sin(\omega_o t)$

FBD



AMB $_{/O'}$

$$\Sigma \vec{M}_{/O'} = \dot{\vec{H}}_{/O}$$

$\vec{r}_{P/O'} \times (-mg)\hat{j}$

$$-mg\ell\cos\theta\,\hat{k} = \vec{r}_{P/O'} \times m\vec{a}_{P/F}$$

$$-mg\ell\cos\theta\,\hat{k} = \vec{r}_{P/O'} \times m(\vec{a}_{O'/O} + \vec{a}_{P/B})$$

5 term $\vec{a}$ formula, other 3 terms = 0, as moving frame not rotating. $\vec{\omega} = \vec{0}$, $\dot{\vec{\omega}} = \vec{0}$

$$-mg\ell\cos\theta\,\hat{k} = \underset{\llcorner \ell\hat{e}_r}{\vec{r}_{P/O'}} \times m(-d\omega_o^2 \sin\omega_o t\,\hat{j} + \underbrace{(-\ell\dot{\theta}^2)\hat{e}_r + \ell\ddot{\theta}\hat{e}_\theta}_{\vec{a}_{P/B}}$$

$$\{-mg\ell\cos\theta\,\hat{k} = (-\ell\cos\theta\, d\omega_o^2 \sin\omega_o t\,\hat{k} + \ell^2\ddot{\theta}\,\hat{k})m\}$$

$$\{\}\cdot\hat{k} \qquad \underbrace{\ddot{\theta} = \frac{-g + d\omega_o^2 \sin\omega_o t}{\ell}\cos\theta}$$

note! pendulum ~~to~~ equation with gravity replaced by $g + (-d\omega_o^2 \sin\omega_o t)$.

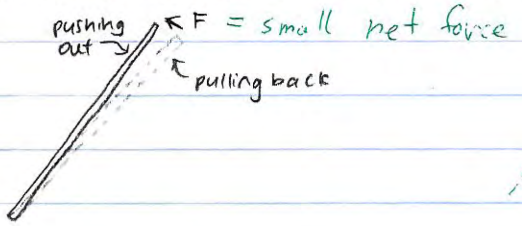
IN MATLAB,

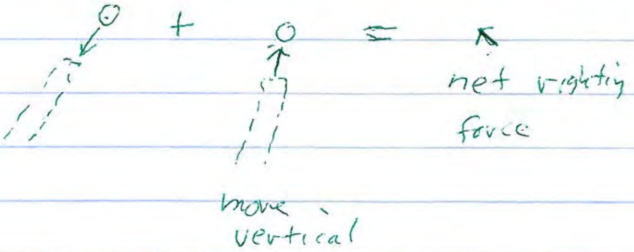
```
thetadot = force(t, p) * sin(theta)/(p.m * p.L);
```

~ neglecting gravity.

* details posted online.

Intuitively, the rod pushes the point mass at the end of it outwards and back, and the direction (angle) of the rod changes, being more vertical as it pushes the mass out, and less so as it pulls the point mass back. There is thus a net force





~ Stabilising effect.

Video: someone standing up with markers on their body.
horizontal motion exaggerated by a factor of 10.
- shows feedback loops in human beings.

Recitation 5/1/2013

I. **Review**

(a) units !

means "bad, wrong equation"

Mini Quiz

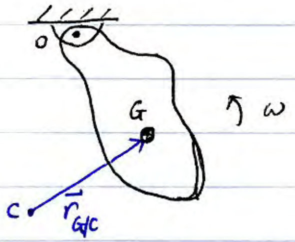
(i) $(m + \frac{1}{2})\vec{a} = \vec{F}_f$ × $\frac{1}{2}$ is dimensionless.

(ii) $\dot{\theta}\cos(\theta+5t) = \frac{v}{h}$ × $(\theta+5t)$ has to be dimensionless. 5 has units of $1/\text{sec}$.

(iii) $\frac{m l_1 l_2}{I^G + \frac{2m}{l_1}} = \ddot{\theta}$ × I^G and $\frac{m}{l_1}$ don't have the same units.

Which expressions above are valid?

(b) Kinetic energy for a rotating rigid body.



What is E_K for this rotating blob?

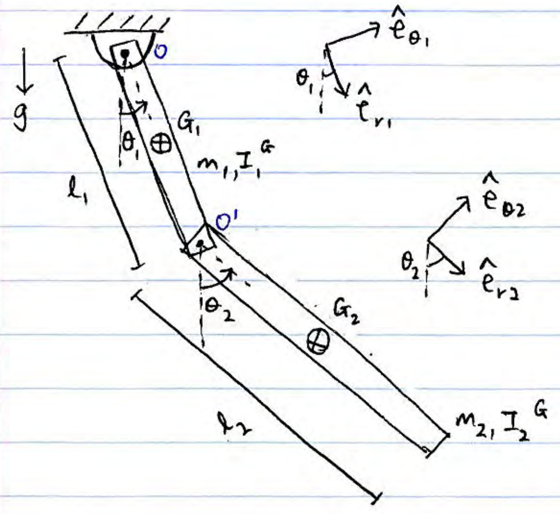
$E_K = \frac{1}{2} m v_G^2 + \frac{1}{2} I^G \omega^2$ (generally)

(c) How to write $\vec{H}$?

$$\vec{H}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\vec{\alpha}$$

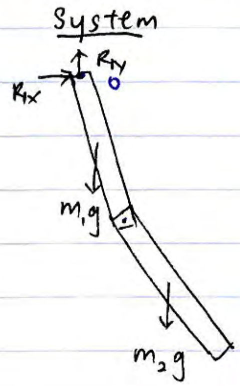
$\dot{\omega}\hat{k} = \ddot{\theta}\hat{k}$

II. **Double Pendulum**

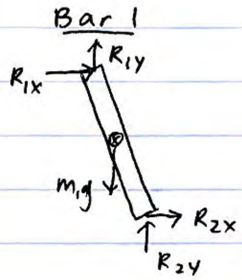


a) Draw 3 FBDs.

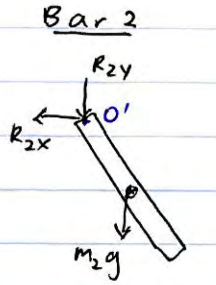
System



Bar 1



Bar 2



b) Pick 2 AMB points.

O for whole system, O' for Bar 2.

c) Write out general AMB equations. Don't plug stuff in yet.

$$\sum \vec{M}^{ext}_{/O} = \dot{\vec{H}}_{/O} = \dot{\vec{H}}_{1/O} + \dot{\vec{H}}_{2/O}$$

$$= \vec{r}_{G1/O} \times m_1 \vec{a}_{G1} + I^{G1} \ddot{\theta}_1 \hat{k} + \vec{r}_{G2/O} \times m_2 \vec{a}_{G2} + I^{G2} \ddot{\theta}_2 \hat{k}$$

$$\sum \vec{M}^{ext}_{/O'} = \vec{r}_{G2/O'} \times m_2 \vec{a}_{G2} + I^{G2} \ddot{\theta}_2 \hat{k}$$

$$\vec{r}_{G1/O} = \frac{l_1}{2} \hat{e}_{r1}$$

$$\vec{r}_{G2/O} = l_1 \hat{e}_{r1} + \frac{l_2}{2} \hat{e}_{r2}$$

$$\vec{r}_{G2/O'} = \frac{l_2}{2} \hat{e}_{r2}$$

$$\vec{a}_{G1} = \frac{l_1}{2} (\ddot{\theta}_1 \hat{e}_{\theta 1} - \dot{\theta}_1^2 \hat{e}_{r1})$$

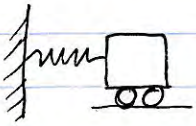
$\vec{a}_{G2}$ = see recitation from 4/24 (5 term formula)

⇒ big very messy expressions
(evaluated using computer algebra in lecture on May 2, Matlab code posted)

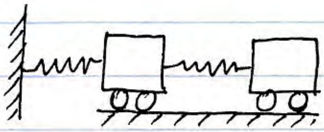
Lecture 5/2/2013

- 2 - Dof systems w/ kinematic constraints

Recall



⇒ 1 2nd order ODE
(2 first order)



or ballistics

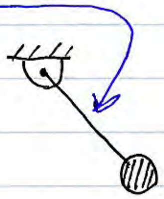
⇒ 2nd order (coupled) ODEs.
solve by Matlab.

Easy because you can calculate all forces from knowing position & velocity.

position & vel.

* string more complex than spring. Cannot tell tension in string by just looking at it.

different from:

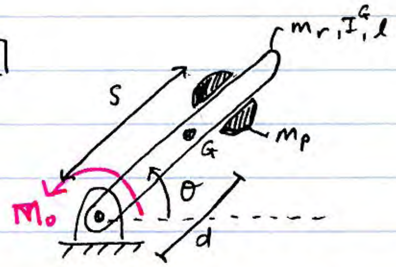


1 Dof

geometric constraints result in forces that cannot be solved without solving dynamics. eqs.

hardest problems: 2 Dof linked rigid objects, 3 Dof, etc.

ex



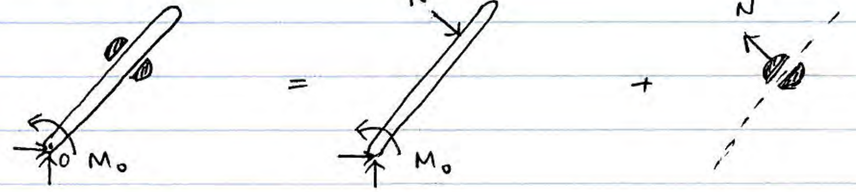
New gun with no motor constraint
⇒ don't know θ as function of time.

2 Dof : s, θ.

Parameters (given) : $m_p, m_r, \ell_r, I, d, M_o$

Dynamics Variables : $\theta, s, \dot{\theta}, \dot{s}, \underbrace{\ddot{\theta}, \ddot{s}}_{\text{find these (eqns of motion)}}$

Draw FBDs



(2 independent)
(3 unknown reaction forces)

If we don't care about reaction forces:

<u>AMB</u>$_{/O}$ of system

$$\Sigma \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$① - \quad M_o \hat{k} = \underbrace{\vec{r}_{P/O} \times (m_p \vec{a}_P) + \vec{r}_{G/O} \times (m_r \vec{a}_G) + I^G \ddot{\theta} \hat{k}}_{\text{eqn with parameters}, \dot{\theta}, \theta, s, \dot{s}, \ddot{\theta}, \ddot{s}} \qquad \left(\text{particle has no } I^G\right).$$

<u>LMB</u> of particle in $\hat{e}_r$.

$$\{\Sigma \vec{F} = m_p \vec{a}_P\} \cdot \hat{e}_r$$

$$② - \quad \{\vec{0} = \underbrace{m_p \vec{a}_p}_{}\} \cdot \hat{e}_r$$

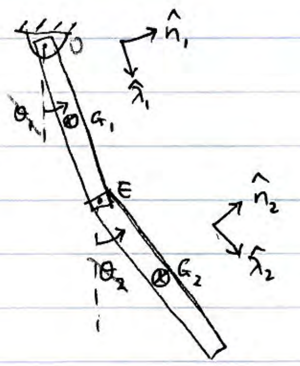
we know in terms of parameters & $\theta, \dot{\theta}, \ddot{\theta}, s, \dot{s}, \ddot{s}$

①, ② are 2 eqns for $\ddot{\theta}$ and $\ddot{s}$ ⇒ eqns of motion.

* set up and solve in Matlab.

$\boxed{\text{ex}}$ Demo in class : Double ~~double~~ pendulum.

- chaotic
- 2 similar setups exponentially diverge from each other.



get $\ddot{\theta}_1$, $\ddot{\theta}_2$.

Derive through Matlab symbolic algebra.

Find $\hat{\lambda}$, $\hat{n}$,

$\vec{a}_{G1/O}$

$\vec{a}_{E/O}$

$\vec{a}_{G2/E}$

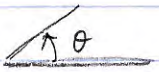
$\vec{a}_{G2/O}$

$\vec{M}_{/O}$

$\vec{M}_{/E}$

$\dot{\vec{H}}_{/O}$

$\dot{\vec{H}}_{/E}$



Matlab code is posted.

eqn1 = $\vec{M}_{/O} - \dot{\vec{H}}_{/O}$

eqn2 = $\vec{M}_{/E} - \dot{\vec{H}}_{/E}$

} set to 0.

eqn1 = eqn1(3)

eqn2 = eqn2(3)

} pull out $\hat{k}$ component.

[r1 r2] = solve(eqn1, eqn2, th1ddot, th2ddot);