# T&AM 570, Intermediate Dynamics

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Course Catalogue Description: *Newtonian mechanics; motion in rotating coordinate systems. Introduction to analytical mechanics; virtual work, Langrangian mechanics. Hamilton's principle. Small vibration and stability theory. Newtonian-Eulerian mechanics of rigid bodies. Gyroscopes.*

In fact this semester little is done on gyroscopes and little explicit on using rotating coordinate systems (as opposed to rotating objects, which are coverred extensively).

**Contents**



# I. Topics and number of lectures (27 lectures, 80 minutes each):

Computers are integrated into many of the lectures and most of the homework assignments. Most often students integrate ODE solutions in MATLAB. By the end of the semester they can do animations of wire models in 3D (projected into 2D).

**A. Newton-Euler particle mechanics and 2D rigid body mechanics.** (11 lectures)
Mechanics of particles, many particles, pendulum, double pendulum, springs, planetary motion, constraints, and non-holonomic constraints.

**B. Lagrange Equations etc.** (5 lectures)
Generalized coordinates, Hamilton's principle, Lagrange equations, vibrations using Lagrange eqs.

**C. 3D rigid body mechanics** (9 lectures)
Geometry of rotation, matrix representations, vector representations (axis-angle), angular velocity, Euler angles, tops, free motion of a rigid body, rolling disk.

**D. Collision mechanics** (1 lecture)

**E. Variable mass systems** (1 lecture)

# II. Lecture Schedule by date (two 80 minute lectures/week):

24 Aug: **Intro to mechanics. The three pillars.**
I) material properties, system models,
II) Geometry of motion and deformation,
III) The laws of mechanics.
0) Force is *the* measure of interaction ⇒ Free Body Diagrams (FBDs)
1) Linear Momentum Balance (LMB)
1) Angualr Momentum Balance (AMB)
1) Energy Balance
(Depending on extra modeling assumptions, the balance laws above can sometimes be derived from each other.)

29 Aug: **Calculus of paths, Intro to particle mechanics.**
Basic particle kinematics $\underline{\mathbf{v}} = \frac{d}{dt}\underline{\mathbf{r}} = \dot{\underline{\mathbf{r}}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{r}}}{\Delta t}$, $\quad \underline{\mathbf{a}} = \frac{d}{dt}\underline{\mathbf{v}} = \dot{\underline{\mathbf{v}}} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{v}}}{\Delta t}$, $\frac{d}{dt} w\underline{\mathbf{v}} = \dot{w}\underline{\mathbf{v}} + w\dot{\underline{\mathbf{v}}}$, etc.
Particle mechanics. LMB ⇒ $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$.
Example: $\underline{\mathbf{F}} = \underline{\mathbf{0}}$, solve for path ⇒ parametric eqn for straight line.
Example: $\underline{\mathbf{F}}$ = Constant (e.g. near-earth gravity), solve for path ⇒ parametric eqn for parabola.
Example: Ballistics with air drag, $-mg\,\hat{\mathbf{j}} - cv\underline{\mathbf{v}} = m\underline{\mathbf{a}}$ ⇒ no known closed form soln. Reduction to 4 first order ODEs. Computer soln.
Mechanics of one particle: Derivation of power balance and AMB from $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$. Central force motion and conservation of Ang. Mom (vector derivation and derivation from Newton's Principia). Conservation of energy for conservative forces.

Aug 31: **More particle mechanics. Intro to Systems of particles.**
Example: 3D harmonic oscillator with zero-rest-length springs.
Example: Celestial mechanics. Set up eqn $\frac{-GMm}{r^3}\underline{\mathbf{r}} = m\ddot{\underline{\mathbf{r}}}$.
Systems of particles: Rudely *assuming* pairwise equal and opposite internal forces between particles ⇒ LMB, AMB and energy balance for system.

Sept 5: **Systems of particles (cont'd)**
Comment on cannon ball physics intuition question (out then down *vs* parabola).
Summary of systems of particles: Center of mass thms for linear momentum , angular momentum and KE. Different forms of the system energy equation,
Example: two masses connected by a linear spring, set up equations and note the def of internal potential energy follows as $\int_{\ell_0}^{\ell} T(\ell')d\ell'$.
The particle model of the universe: assumptions and critique.
The basic simplicity of setting up eqns of motion for the particle model of the universe with forces and interaction forces dependent only on positions and velocities (no constraints).
Set up of eqns for $n$ particles with gravitational interaction (e.g., solar system).

Sept 7: **Particles cont'd. Intro to constraints.**
Hints on HW 4 (particle hanging from spring).
Motivation for leaving the world of $10^2 3$ particles to join the world of constraints: 1) too slow on computer, 2)too complicated 3) don't know the details of model or initial conditions accurately, 4) its wrong.
⇒ rigid body model & other kinematic constraints.
Pre-D'Ambert method: supplement $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$ with geometric constraints and unknown constraint forces.
example: 1D motion of two particles connected by a rod. Two approaches: 1) three coupled second order ODEs with 2 unknown accelerations and one unknown constraint force, 2) Elimination of constraint force by equation manipulation.

Sept 12: **Intro to rigid body constraint.**
Naive approach no-one uses: particles connected by massless truss.
Kinematics of $\underline{\mathbf{v}}$ and $\underline{\mathbf{a}}$ of points on a rigid body, differentiation of time-varying unit vectors, etc.
⇒ 2D rigid body eqns of motion (e.g.,2D moment of inertia).

Sept 14: **2D rigid body mechanics**
example: braking car, how fast does it stop. Three approaches:
LMB & $\text{AMB}_{/COM}$ ⇒ 3 messy equations
AMB relative to three nice points ⇒ 3 simpler equations
AMB relative to a really nice pt. ⇒ 1 equation in 1 unknown
example: swinging bar in 2D. Two approaches:

LMB & AMB$_{/COM}$ & constraint equation.
AMB$_{/\text{hinge}}$ $\Rightarrow$ standard pend eqn directly.
example: stear stability of a car with locked rear wheels, initial set up of equations.

Sept 19: **Chaplygin Sleigh.** A simplified model of car braking and grocery carts.
Full detailed set up of 2D equations of motion of this non-holonomic rigid body system.
Reduction to 2 non-linear coupled first order ODEs + auxiliary equations to integrate to find position and orientation.
Stability of straight-line motion. Linearized eqns of motion and resulting exponential solutions.

Sept 21: **Sleigh (cont'd) & double pendulum**
Phase plane description of non-linear ODEs for Sleigh.
Double pendulum. Intro to generalized coordinates (angles).
Two approaches:
Two rigid bodies and two hinge constraints $\Rightarrow$ lots of equations can be solved i) by algebraically eliminating the constraint equations, ii) "on the fly" solution numerically.
See HW 6
Generalized coordinates and judicious choice of AMB eqns that automatically eliminate constraints.

Sept 26: **Double pendulum (cont'd)**
Set up of general pendulum eqn to this general form: $[M][\dot{\omega}] + [f(\omega, \theta)] + [g(\theta)] = [0]$ where [] is a matrix of some kind, $\theta$ is a list of generalized coordinates, $\omega$ is a list of generalized velocities ($\dot{\theta}$ in this problem), $f(\omega, \theta)$ are terms that, in this dissipation free problem are centripital and coriolis terms, and $g(\theta)$ are terms from gravity.
See HW 7
(missing notes for this lecture)

Sept 28: **Comments on numerics. Intro to 3D rigid body kinematics**
Some numerics issues relevant to HW 7 on double pendulum: convergence checks; checks on solving the right equations: energy conservation (with and without $g$), angular momentum conservation (when $g = 0$), compare with special cases where you know the solution (one or the other bar is massless), small oscillations and the linearized equations.
Rotations in 3D, the first of many lectures. Goal: simplify expressions for linear momentum, angular momentum and energy for use in equations of motion, represent the orientation of a rigid body.
Representing rotation as a list of 3 orthonormal vectors.
Rotation as the array of numbers that gives the linear combinations of the rotated base vectors in terms of the fixed base vectors.
otation as the matrix that transforms from body-fixed to space-fixed coordinates of vectors.

Oct 3: **Rotations cont'd**
$[R]$ is 9 numbers with 6 orthonormal conditions. How to express as just 3 numbers?
$R^T R = RR^T = I$
Dyads and dyadics. Relation to matrix notation.
Representing R as the sum of three dyads (using mixed base vectors).
Representing R as the sum of 9 dyads (all using original base vectors)

Oct 5: **Geometric approach to 3D rotations**
Watching points on a unit sphere.
Represent rotation by the motion of an arbitrary point and the motion of the point that started where that point ends up.
Euler's thm: rotation is rotation about an appropriate axis representable, like a convential vector, with magnitude $\theta$ and direction $\mathbf{n}$.
Rotation is not a (conventional) vector: sequential rotations cannot be found from vector addition.
Compound rotations are a rotation (by construction on the sphere)
Derivation of representation of rotation with dot and cross products: rotation of $\underline{\mathbf{r}} = [R]\underline{\mathbf{r}} = \cos\theta\ \underline{\mathbf{r}} + (1-\cos\theta)(\mathbf{n} \cdot \underline{\mathbf{r}})\mathbf{n} + \sin\theta \mathbf{n} \times \underline{\mathbf{r}}$.
See HW 8.

Oct 12: **Rotations cont'd**
Vectors, matrices and dyadics and the solution of HW 8.
Finding axis and angle from R, derivation of
$$\theta = \cos^{-1}\left[\frac{trace(R) - 1}{2}\right], \text{ and } \mathbf{n} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

see HW 9
2 approached to keeping track of sequences of rotations
i keep track of base vectors (left multiply)
ii keep track of components (right multiply)

Oct 17th: **Rotrations (cont'd)**
Euler angles
i 313, most common for tops etc.
ii 312, most common for aeronautics, robotics, etc
Building up Euler angles from three sequential rotations, using intermediate reference frames and associated rotation matrices.
Small rotations and $\frac{d}{dt}R$.
$\underline{\boldsymbol{\omega}}$ = the vector representation of the skew matrix $\dot{R}R^T$.

Oct 19: **Rigid Body Dynamics**
Evaluation of angular momentum $\underline{\mathbf{H}}$ with the rigid-body constraint $\Rightarrow$ moment of inertia $I$. Principal components. $\Rightarrow \underline{\mathbf{H}} = I\underline{\boldsymbol{\omega}}$.
Euler equations in body-fixed coordinates: $M_x = I_{xx}\dot{\underline{\boldsymbol{\omega}}}_x + I_{xx} - I_{yy})\underline{\boldsymbol{\omega}}_y\underline{\boldsymbol{\omega}}_z$ etc. comment on general uselessness of these equations because rotation is not easily determined by them.
Torque-free motion of an axisymmetric body. Free precession wobbling about near to rotation about the symmetry axes. Body cone and space cone.
Stability of rotation about a principal axis of a non-symmetric body.

Oct 24: **Quick intro to Lagrange's Eqns and Hamilton's principle (Guest lecture by R. Rand**
generalized (minimal) coordinates
Lagrange equations presented (not derived yet)
Pendulum: just the recipe of evaluating $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}$ with $\mathcal{L}(\theta, \dot{\theta}, t)$.
Two particles connected by string, one on a plane, one hanging under a hole in the plane.
Comparison of the mechanics involved with Newton vs Lagrange for these two examples.
Hamilton's principal and its use to derive pendulum eqn.

Oct 26: **Review and extension of $I, \underline{\boldsymbol{\omega}}$**
Contrasting $\dot{\underline{\mathbf{H}}} = \frac{d}{dt}\underline{\mathbf{H}}$ with $\dot{\underline{\mathbf{H}}} = \int \underline{\mathbf{r}} \times \underline{\mathbf{a}} dm$
Dyadic representation.
Frame dependent time differentiation.
Euler equations as first order ODEs.
Constructing R from $\dot{R} = [\underline{\boldsymbol{\omega}}]R$ ($[\underline{\boldsymbol{\omega}}]$ is the skew symmetric matrix associated with $\underline{\boldsymbol{\omega}}$).

Oct 31: **More on Rigid body dynamics**
Why are off-diagonal terms in $I$ called "the centripetal terms"?
Using $\underline{\boldsymbol{\omega}}$ and $R$ as state variables in ODEs for a rigid body. Redundancy of this method ($R$ has 9 numbers in it).
Numerical drift problems.
Finding Euler angle rates of change from $\underline{\boldsymbol{\omega}}$.
See homework 10.

Nv 2: **Circles on circles**
Gravity-forced precessing top in some detail (noting that the classic solution with the assumption that $\underline{\mathbf{H}}$ is parallel to the symmetry axes happens to give the exact answer even though the assumption is only an approximation.
See homework 11.

Nov 7: **Analytical mechanics**
Systems of particles in cartesian coordinates.
Principle of virtual work.
Decomposition of forces into applied forces and constraint forces (*assumed* to be workless.
Particle on a wire.
Particle on a surface.

Particles mutually held at fixed mutual distances (a rigid body).
Derivation of Angular momentum balance from linear momentum balance (using the questionable assumptions above).
Derivation of linear momentum balance from angular momentum balance (without using any questionable assumptions).

Nov 9: **Hamilton's principle.**
Derivation of Hamilton's principle from $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$.
Generalized (minimal coordinates) again, in more detail.
Expression of Hamilton's principle using generalized coordinates.

Nov 14: **Lagrange equations**
Hints for homework 10b (look at intersection of energy and angular momentum ellipsoids.).
Derivation of Lagrange equations from Hamilton's Principle.
Side story: informal introduction to calculus of variations.
Example: 1D motion of a particle.
Example: Simple pendulum (again)
See homework 12

Nov 16: **Miscellaneous confusions**
Extended discussion on how to calculate the rate of change of angular momentum, not respect to the center of mass, when an object and/or reference frame is moving.
Example, rolling cyllinder.
See homework 13
Example, coin rolling steadily in circles. What $\omega$ crossed with what $\underline{\mathbf{H}}$ gives the right expression for rate of change of angular momentum about what point?
Derivation of the equations of motion, using as generalized coordinates these variables $q_1 = \int \omega_1 dt$, $q_2 = \int \omega_2 dt$, $q_3 = \int \omega_3 dt$, and Lagrange equations (*NOT!*). Why doesn't this work?

Nov 21: **Vibrations**
Final project, HW 14 assigned.
Assume a holonomic conservative system.
Examples: pendulum, inverted pendulum, pt mass on a saddle.
Assume an equilib position with positive definite potential energy near by.
Lagrange equations $\Rightarrow b_{kj}\ddot{q}_j + a_{kj}q_j = 0$.
Quote math fact about generalize eigenvectors.
normal modes.
example with a line of masses connected with springs.

Nov 28: **Collisions**
What is a collision.
Strong rigidity and Weak rigidity (personal jargon that is useful).
Impulse and angular impulse.
Frictionless collisions

Nov 30: **Variable mass system**
Start with extended answer to question about rate of change of angular velocity and what it has to do with angular velocity itself.
Collection of particles.
Careful derivation of the rocket equation.
The danger of using $\frac{d}{dt}(m\mathbf{v}) = \underline{\mathbf{0}}$ (it often gives wrong answers.
Falling and lifting chain problems (wrong or incomplete in essentially every text book or classical mechanics book).

# III. Homework problems:

(rough figures for some problems at the end)

All homework problems are discussed in lecture, often several times as students work on them.

**1)** **Simple pendulum review and diagnostic.** Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{\ell}\theta = 0$ as many ways as you can without looking anything up in books.

**2)** **Ballistics, linear drag** (not a very good model of drag).
Set up necessary equations and solve as appropriate using analytical, numerical, or combined methods. The goal is to answer questions (a) and (b) as best as possible. First think about the questions and guess your answer.

- **a)** What is the best angle to maximize the range with a fixed initial velocity, gravity, and drag constant?
- **b)** For a given angle, what happens to the shape of the trajectory in the limit $v \to \infty$. (The answer is interesting and simple.)
- **c)** Find a closed form solution for the trajectory, given all constants and initial conditions.
- **d)** For some $c, \underline{\mathbf{v}}_0, g, \theta$ find and the solution both by numerical integration. By experiment, parameter values and a time interval so the plot is visibly different than a parabola and the projectile returns to something near the launch height.
- **e)** Plot the two trajectories above on the same plot at the same scale. Note the degree of agreement or disagreement.
- **f)** Try to answer questions (a) and (b) as well as you can.

**3)** **What means "rate of change of angular momentum"?**
Consider a moving particle P. Consider also a point C with position which changes as time progresses in an arbitrary manner (relative to a Newtonian frame $\mathcal{F}$ that has an origin 0). For which of these definitions does $\dot{\underline{\mathbf{H}}}_{/\mathrm{C}} = \underline{\mathbf{M}}_{\mathrm{C}}$, i) in general, ii) for some special cases concerning the motions of P and C that you name.

- **a)** $\underline{\mathbf{H}}_{/C} = \underline{\mathbf{r}}_{\mathrm{P/C'}} \times \underline{\mathbf{v}}_{\mathrm{P/0}}$, where C' is a point fixed in $\mathcal{F}$ that instantaneously coincides with C.
- **b)** $\underline{\mathbf{H}}_{/C} = \underline{\mathbf{r}}_{\mathrm{P/C}} \times \underline{\mathbf{v}}_{\mathrm{P/0}}$.
- **c)** $\underline{\mathbf{H}}_{/C} = \underline{\mathbf{r}}_{\mathrm{P/C}} \times \underline{\mathbf{v}}_{\mathrm{P/C}}$.

**4)** **Mass hanging from spring.** Consider a point mass hanging from a zero-rest-length linear spring in a constant gravitational field.

- **a)** Set up equations. Set up for numerical solution. Plot 2D projection of 3D trajectories.
- **b)** By playing around with initial conditions, find the most wild motion you can find. make a plot.
- **c)** Using analytical methods justify your answer to part (b).

**5)** **Pendulum as constrained particle**
Consider a particle in the plane subject to the constraint that $x^2 + y^2 = \ell^2$. Assume the constraint force is radial. Find, for various initial conditions the motion of the particle with and without gravity. (Use Cartesian coordinates and constraint equations, no polar coordinates allowed). Animate the solution.

**6)** **Braking stability**
Consider the stearing stability of a car going straight ahead with either the front brakes locked or the rear brakes locked. The stearing is locked. For simplicity assume that the center of mass is at ground height between the front and back wheels. Assume that the locked wheels act the same as a single dragging point on the centerline of the car midway between the wheels.

- **a)** Develop the equations of motion.
- **b)** Set them up for computer solution.
- **c)** For some reasonable parameters and initial conditions find the motion and make informative plots that answer the question about steering stability. Note, in this problem where there is no steady state solution you have to make up a reasonable definition of steering stability.

**d)** See what analytical results you can get about the steering stability (as dependent on the car geometry, mass distribution, the coefficient of friction and the car speed).

**7) Double pendulum**

Consider the double pendulum made of two uniform bars each with mass $m$ and length $\ell$.

**a)** Set up governing equations by either on-the-fly method or using generalized coordinates and judicious AMB equations.

**b)** Set up for computer solution and animation.

**c)** Check the equations and solutions for special cases that you think you understand.

**d)** Given that $\theta_1(t=0) = \pi/2$ and $\theta_2(t=0) = \pi/2$ run the equations for long enough time to see complicated behaviour. Animate the solution on your screen. Plot the trajectory of the end point of the second bar.

**8) Relation between axis-angle and rotation matrix.** Given the formula $\underline{\mathbf{r}} = [R]\underline{\mathbf{r}} = \cos\theta\ \underline{\mathbf{r}} + (1-\cos\theta)(\mathbf{n}\cdot\underline{\mathbf{r}})\mathbf{n} + \sin\theta\mathbf{n}\times\underline{\mathbf{r}}$ derived in lecture, write the rotation matrix in terms of the components of $\mathbf{n}$ and $\theta$.

**9) Rotation of a book.** Draw a book. Now imagine

**a)** Rotating it first 90° about the $z$ axis and then 90° about the $x$ axis.

**b)** Rotating it first 90° about the $x$ axis and then 90° about the $z$ axis.

**c)** Draw the book before and after for each of these two cases.

**d)** Find $R$ and $R^T$ for each of these cases.

**e)** Find $\mathbf{n}$ and $\theta$ for each of these cases, draw the vector $\mathbf{n}$ on the pictures of the book for both of these cases.

**10) Free motion of a rigid body**

Given that $[I^G] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$, a perfectly general rigid body in principal co-ordinates

**a)** For the special case that $A = B$ and $C = 2A$ find the motion very near spinning about the symmetry axis. Animate a disk moving like that from your simulation. Draw a 2D projection of the path of two particles on the boundary of the disk. Note that for small angles these are almost circular paths tilted slightly one with respect to the other. Can you think of a simple explanation for this?

**b)** Now consider a more general case $C > B > A$. Set up for numerical solution and animation the motion of such a body. For pictoral purposes you can represent the body as an ellipsoid, a box, or a jumping jack, as you find convenient or fun. Check in simulation that for rotation nearly about the 1 and 3 axes that the rotation is stable. Check that rotation about the 2 axis is not stable. The instability should show in animation, and in a plot of the position of some point on the nominal axis of rotation. Look at the motion for initial conditions in $\omega$ of $[01\epsilon]$ and see if you can qualitatively predict the motion without looking at your computer animations (Hint: angular momentum and energy are both conserved but the motion $\omega = [010]$ is unstable.

**11) Steady precession of a rolling disk.** A disk or radius $r$ rolls in steady circles on a ground track with radius $R$. The key is the ground contact condition that the velocity of the point on the disk which touches the ground is zero.

**a)** For a given disk, find all solutions. There is a 2 parameter family of them.

**b)** Assume that, instead of rolling, the disk slides with no friction (and the ground contact point need not have zero velocity). Find all steady solutions where the ground contact traces a circle. There is a 2 parameter family of these.

**c)** Find all solutions common to the two cases above.

**d)** Take a coin or flat disk and roll it on a flat ground. Look at it as it shutters. Which of the solutions above do you think the disk actually tracks? Why?

**12) Bead on parabolic wire**

For a frictionless point-mass bead sliding on a rigid wire on the curve $y = cx^2$ with gravity in the $-y$ direction, find the equation of motion.

**a)** Derive the equations of motion using Lagrange equations. Use, say the projection of the position on the $x$ axis as the generalized coordinate.

**b)** Derive the equations of motion using Newton's laws. First write $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$ with an unknown constraint force orthogonal to the wire. Then dot both sides with a vector tangent to the wire. You should get the same answer as for part (a) with a very similar amount of algebra.

**13) Rolling cylinder**

**a)** Consider a cylinder with center of mass at its center rolling down a ramp. Find the equations of motion using $I^G$ (the moment of inertia about the center of mass) and $I^C$ (the moment of inertia about the ground contact point.

**b)** Do the problem above, but with a cylinder whose center of mass is off-center. Why don't the two methods give the same answer for this problem, whereas they do for (a)?

**14) FINAL PROJECT (no final exam)** Rolling disk. Must give final demonstration of derivations and functioning software (30 min per student). Do these parts in any order that pleases you. Acknowledge clearly the source of all help that you got. Some helpful books will be on reserve for you (Greenwood, Goldstein, Landau& Lifshitz, Routh, Pars.

**a)** Derive the equations of motion of a rolling disk at least 2 different ways (not just steady precession, but general rolling motion). Some options: Newton-Euler with Euler angles, N-E with rotation matrices, Lagrange with Lagrange multipliers, viscous contact with the limit of viscosity going to zero, etc.

**b)** Set up the equations you get by both means for simulation.

**c)** Check that for some fairly arbitrary initial condition that both solutions above give the same motion.

**d)** See that at least one set of your equations reduces to the steady precession equations that you previously derived by other means.

**e)** Animate one of your solutions (showing a disk, the plane, and the path made by the contact point.).

**f)** Check that Energy is conserved in your simulation.

**g)** Show, in your simulations, that fast rolling is stable and slow rolling is not stable.

**h)** Show by analysis that fast rolling is stable and slow rolling is not stable.

Figures for some of the problems.