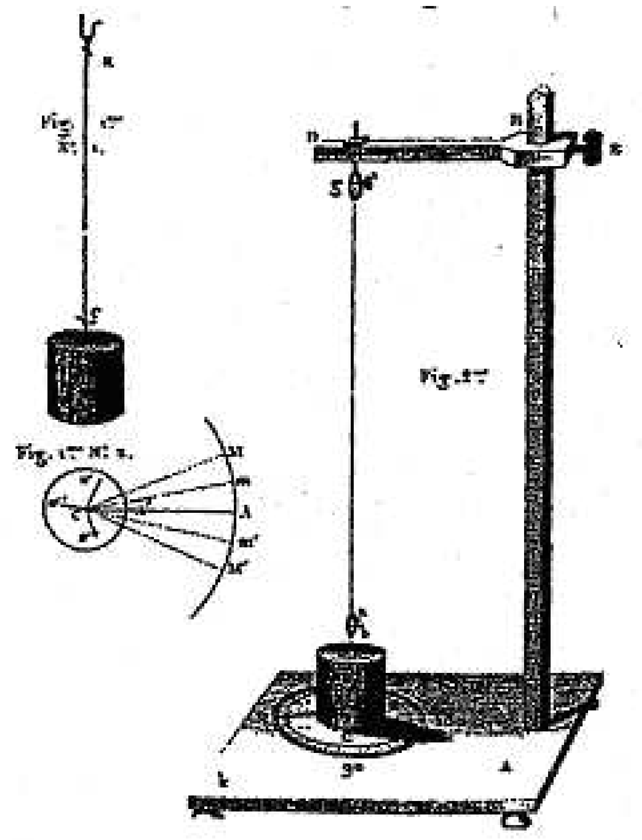
TAM 203 Lab Manual



Cornell University
Theoretical and Applied Mechanics

This manual has evolved over the years. Contributors in the past two decades include: Kenneth Bhalla, Jason Cortell, Drew Eisenberg, Jill Evensizer, Kwang Yul Kim, Richard Lance, Jamie Manos, Francis Moon, Dan Mittler, James Rice, Kevin Rompala, Andy Ruina, Bhaskar Viswanadham, and Alan Zehnder

Contents

TAM 203 Lab Introduction

Last Updated: January 22, 2008

PURPOSE

The laboratories in dynamics are designed to complement the lectures, text, and homework. They should help you gain a physical feel for some of the basic and derived concepts in dynamics: *force*, *velocity*, *acceleration*, *natural frequency*, *resonance*, *normal modes*, and *angular momentum*. You will also get exposure to equipment and computers which you may use in the future. Some mathematics from courses you have taken recently or are now taking will be used. We hope this will help you make the connection between mathematics and physical reality that is essential to much of engineering. The labs may come either before or after you cover the relevant material in lecture. Thus, they can be either a motivation for the lecture material or an application of what you have learned depending on the timing.

COURSE INFORMATION

There are four dynamics laboratories you will be performing during the semester:

1. One Degree-of-Freedom Oscillator
2. Two Degrees-of-Freedom Oscillator
3. Slider-Crank Mechanism
4. Gyroscopic Motion of a Rigid Body

Each of the four labs is taught for two or three weeks (depending on enrollment) in Thurston 101. You will be scheduled to attend lab during one of the weeks. The dates for your laboratory section will be posted outside Thurston 101 and on the course website. In general, you will have a lab once every two or three weeks, but be aware that this may vary due to exam and break schedules.

NOTE: See the Administrative Assistant in Kimball 212 if you have any problems with your lab schedule. You'll need to get his or her approval for any changes so that the lab sections do not become overly full. Turning in a course change form to the registrar is not enough.

LABORATORY ATTENDENCE

You are expected to attend the lab section you have signed up for. In the event of an **excused** absence you must make-up the lab. **All make-up labs must be arranged with your TA.** Your options for making-up labs are

1. Attend another of your lab TA's lab sections.

2. Attend another lab TA's lab section (requires permission from both lab TAs).

3. Attend the "Lab Make-Up Section" during the final week of the semester. Information regarding the date and time of this section will be given in lecture near the end of the semester.

If you show up for lab after it is under way, your lab instructor may ask you to leave and to perform the lab another time.

REQUIRED LABORATORY WORK

The laboratories will be done with physical equipment and some will also involve computer simulations. **It is essential that you read through the lab (especially the procedure section) before coming to lab.** It is not necessary that you understand all of the material perfectly before the lab period.

Prelab Questions

Each lab has prelab questions to be answered before you come to lab. These questions encourage you to review necessary theory and read through the laboratory procedure before attending the lab. Answers to prelab questions are due at the beginning of lab and will not be accepted for credit later.

Laboratory Notes

A rule of laboratory work is to keep a neat, complete record of what has been done, why it was done, how it was done, and what the result was.

The success or failure of an experiment in a research laboratory often depends critically upon the record made of the experiment. The outcome of a poorly documented experiment becomes a matter of personal recollection, which is not reliable enough to serve as a basis for further work (especially by someone else). You should take copious notes. If in doubt, **write it down.** One can ignore what is written, but one can not resurrect that which was never recorded. Similarly, **never** erase in your lab notes. If an erroneous reading was made, strike it out with a single line and record the new data. You may later decide that it was not in error.

All lab notes, signed by your lab TA and in their original form, must be stapled to the back of your final lab report.

Lab Report

Your laboratory report should be typed using a word processor. This report should communicate clearly and convincingly what was demonstrated or suggested by the lab work. Your TA is looking for evidence of thought and understanding on your part. Your logic and methods are as important as results or "correct" answers. It is essential that you provide information and calculations which indicate how you arrived at your conclusions. It is permissible (and a good idea if you want a very good grade) to discuss observations and material relevant to the lab which is not specifically asked about in the questions.

Each report must begin with a cover page containing the following (with appropriate substitutions for the words in quotes):

“NAME OF THE LAB”
TAM 203
By: “Your name and your signature (both partners if a joint report)”
Performed: “Date”
Performed with: “Name of person(s) with whom you performed the lab”
Discussed lab with: “Names of people with whom you discussed the lab, and nature of the discussions”
TA: “Lab TA’s name”
TA signed the data on page: “Page #”

It is a good idea to include an introduction, abstract, or overview of the laboratory work you performed as this will help communicate that you successfully grasp the purposes and goals of the lab. It also gives you an opportunity to review your laboratory work before answering specific questions asked in the manual. If you deviate from the procuedure specified in the manual you should also state how and why you did so here.

You should concisely answer the questions that are asked and number them as they are numbered in the lab manual. Include any necessary plots, data, or calculations (make sure to include the correct dimensional units). Your answers should be self-contained and presented in an orderly fashion (i.e. the reader of the report should not have to refer back to the questions that are asked, nor should he or she have to hunt through the report to find your answers). While many questions require that you perform calculations, written explanations of what you are doing and diagrams can be very helpful. Show all calculations that you perform in arriving at your answers. If you are performing repetitive calculations you need show only one sample calculation.

Finally, at the end of your lab report you may want to include any observations, mistakes you made, or suggestions you have in a concluding section.

When answering questions, percentage difference calculations can be used to quantify how well experimental results agree with theoretical or expected values. Rather than writing “the experimental results agree very well with the theoretical calculations,” this phrase can be changed to make a quantifiable statement; “the experimental results are within 5 percent of the theoretical calculations.” Percentage difference is calculated as:

100%× (Value being compared - Reference value) / (Reference value)

While formal error analysis can be used if it is necessary to make a point, your answers should include some discussion of the types and relative sizes of errors in your data.

All plots included with your lab report should be done on the computer using MATLAB (preferred) or Excel. Below are some guidelines for producing quality plots:

- All graphs should be titled and all axes labeled, with the appropriate units listed in parentheses.
- The independent variable should be placed on the horizontal axis.
- Numerical values on the axes should be set at reasonable intervals and scales chosen so that all of the data points can be displayed on the graphs.
- Curves should not be drawn between discrete data points unless the type of fitting used is explained and the equation of the curve given.
- On graphs with more than one curve a legend should be used to identify the curve. Data points can be enclosed by some symbol (i.e. circle, rectangle, etc.) to distinguish different data sets.

Figure 0.1 is an example of how your graphs should appear. The MATLAB code that produced the graph is given below:

```
t = linspace(0,10,1000);
x = 5*cos(2*t);
v = -10*sin(2*t);

figure(1); hold on;
plot(t,x,'b','LineWidth',2);
plot(t,v,'r--','LineWidth',2);
grid on;

plot_title = title('Plot of Position and Velocity vs. Time for Harmonic Oscillator');
x_axis_label = xlabel('Time (sec)');
plot_legend = legend('Position (m)','Velocity (m/s)');

hold off;

set(plot_title,'FontWeight','bold','FontSize',12);
set(x_axis_label,'FontWeight','bold','FontSize',12);
set(plot_legend,'FontWeight','bold','FontSize',12);
set(gca,'FontWeight','bold','FontSize',12);
```

For help with producing log-log and semi-log plots with MATLAB, type `help loglog`, `help semilogx`, or `help semilogy` in the main MATLAB window.

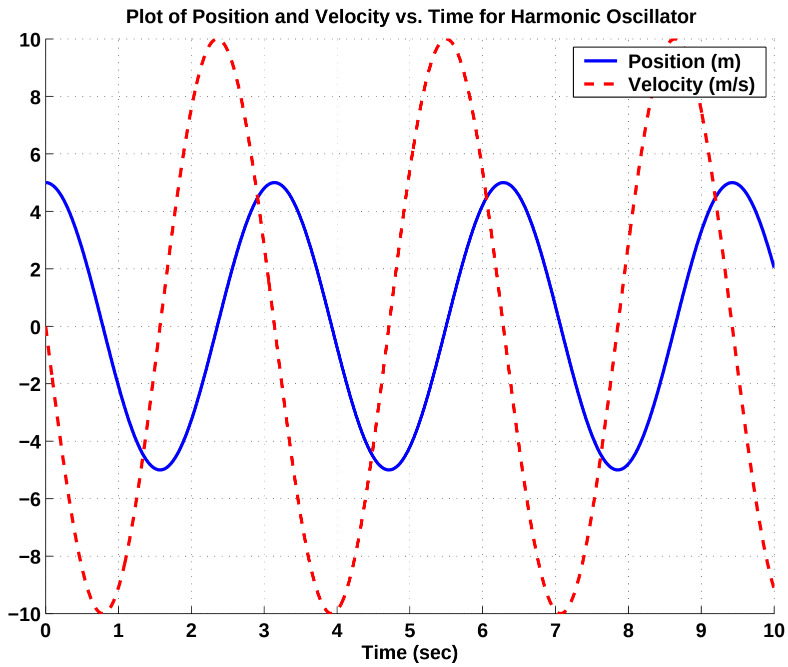


Figure 0.1: An example graph.

CREDIT AND GRADING

Lab reports are due at 10:00 AM one week from the day you performed the lab unless your TA specifies another time. Turn in reports in the boxes in the Don Conway room, Thurston 102. Put your report in the correct box corresponding to the TA in charge of your **laboratory** section. Reports placed in incorrect boxes might not be found.

Each laboratory is graded out of 15 points. The grade breakdown for each lab report will be determined by your lab TA. This grade will be given to your recitation TA.

ACADEMIC INTEGRITY

Your pre-lab answers and lab reports should be in your own words, based on your own understanding and your own calculations. You are encouraged to discuss the material with other students, friends, TAs, or even faculty. **Any help you receive from such discussions must be acknowledged on the cover of your lab report, including the name of the person or persons and the exact nature of the help.** Violations of this policy will be reported to the academic integrity board.

You may, however, do a joint report with your lab partners (turn in one report for your lab group). All partners get the same grade on the report but separate grades on pre-lab questions.

When you are done in the lab you must have your TA sign one of your data sheets. This sheet must include the name of your lab partners and the time and date the lab was performed.

The TA will not sign this sheet until your work station is clean and all equipment is accounted for. **No lab reports will be accepted without this signed sheet.**

Lab #1 - One Degree-of-Freedom Oscillator

Last Updated: June 13, 2007

INTRODUCTION

The mass-spring-dashpot is the prototype of all vibrating or oscillating systems. With varying degrees of approximation, car suspensions, violin strings, buildings responding to earthquakes, earthquake faults themselves, and vibrating machines are modeled as mass-spring-dashpot systems. This laboratory is aimed at demonstrating some of the basic concepts of the mass-spring-dashpot system. Additionally, the computer solution of the governing differential equations will be demonstrated with a computer simulation program. Phrases connected with some of the key ideas are: *natural frequency*, *resonance*, *forcing function*, and *frequency response*.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. Derive the equation of motion for a mass-spring-dashpot system with forcing term $f(t)$. Assume a constant linear spring constant k and linear damping constant c.

2. Solve the equation of motion you derived in #1 if the forcing term is given by $x_s(t) = A = constant$.

3. Repeat #2, this time numerically integrating the equation using *Matlab*. Choose $m = 1$, $k = 5$, $c = 0$ (undamped), and $A = 3$ and integrate over the time period $0 \leq t \leq 10$. Assume the mass starts from rest with an initial displacement of $x(0) = 1$ m. What is the period of the oscillation? Turn in a plot and an m-file of your code.

4. Define in your own words: *natural frequency*, *damping coefficient*, *critical damping coefficient*, *underdamped*, *overdamped*, *resonance*, *phase-shift*, and *amplitude ratio*.

5. Suppose you are measuring two sinusoidal waveforms of equal amplitude, $x_1(t)$ and $x_2(t)$, with a phase-difference of $\frac{\pi}{2}$. What would the shape of the curve be if you plotted $x_1(t)$ vs. $x_2(t)$ (i.e. if you plotted $x_1(t)$ on the y-axis and $x_2(t)$ on the x-axis)? What if the phase-difference were zero? What if it were π? If you have trouble visualizing the situation, try calculating a few points and plotting them.

THE MASS-SPRING-DASHPOT SYSTEM

The picture in Figure 1.1a shows (crudely) the laboratory mass-spring-dashpot, or one degree-of-freedom oscillator. A mass is supported by a spring and is constrained to slide on a rod. In this lab you will record the vertical motion of the mass both with a fixed support and with the support oscillating vertically. Figure 1.1b shows an idealization of the laboratory apparatus. The spring is modeled as linear (the force it applies is proportional to its increase in length) with proportionality constant k. The damping is produced by a linear air dashpot. The force transmitted by a linear dashpot is proportional to the rate at which it is being stretched with proportionality constant c. The vertical displacement of the mass is $x(t)$ and the vertical displacement of the support is $x_s(t)$. Pictured in Figure 1.1c is a free

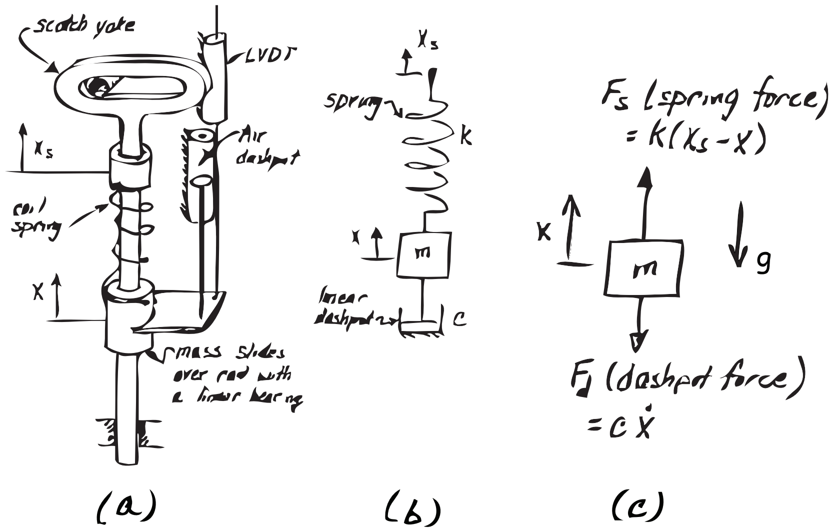


Figure 1.1: Three models of the mass-spring-dashpot system.

body diagram of the mass. Neglecting gravity (*Why can we neglect it?*), the mass has two forces acting on it in the $\hat{\mathbf{e}}_\mathbf{x}$-direction:

$$F_{sp}(t) = k\,(x_s - x) \tag{1.1a}$$

$$F_d(t) = c\dot{x} \tag{1.1b}$$

where $F_{sp}(t)$ is the linear spring force and $F_d(t)$ is the linear damping force. The system is a one degree-of-freedom system since a single coordinate is sufficient to describe the complete motion of the system. (The support displacement $x_s(t)$ does not count as a degree of freedom since it is specified by the motor position, over which we assume we have complete control.)

From Newton's second law the equation of motion for this system is

$$\left\{\sum \underline{\mathbf{F}}\right\} \cdot \hat{\mathbf{e}}_\mathbf{x} \Rightarrow -F_d + F_{sp} = m\ddot{x} \tag{1.2}$$

Assuming a linear spring (1.1a) and a linear dashpot (1.1b) this becomes

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \tag{1.3}$$

where $F_s(t) = kx_s(t)$ is the (presumably specified) excitation "forcing function". In this case the forcing function is the position of the end of the spring as a function of time multiplied by the spring stiffness.

The air dashpot provides resistance to motion by drawing air in and out of the cylinder through a small opening at the top of the cylinder. Due to the small, but nonzero viscosity of air, a pressure drop is created across the opening that is linearly proportional to the speed of the air flowing through. This produces linear damping. Nonlinearities are introduced due to the friction between the piston and the cylinder. Note also that the compressibility of the air in the dashpot introduces some springiness to the system in addition to the coil spring. The compressibility of the air may be thought of as a spring in series with the dashpot.

In the first part of this experiment you will attempt to determine the value of the viscous damping constant c by measuring the rate at which oscillations decay towards zero. In addition, the system response to both free vibration and "forced" motion will be observed experimentally and through computer simulation.

A REAL-WORLD EXAMPLE: THE LOUDSPEAKER

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one degree-of-freedom mass-spring-dashpot system. The one you will observe in this lab is typical (see Figure 1.2). It has a plastic cone supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure and (not visible from outside) a coil of wire attached to the point of the cone which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, it forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil, cone, and (in this case) LVDT core. The "spring" and "dashpot" effects in the system are due to the foam and cloth supporting the cone and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this by cupping your hands over the speaker (gently, without touching the moving parts) and observing amplitude or phase changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonant frequency) is essential in speaker system design. The equation of motion for the speaker is similar to that of the laboratory mass-spring-dashpot above, except the forcing function is electrical, rather than mechanical:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \tag{1.4}$$

where the forcing function $F(t) = ai(t)$, $i(t)$ is the electrical current flow through the coil in amps, and a is the electromechanical coupling coefficient, in Newtons per amp. In the second part of this experiment, the current flow through the speaker will be generated, controlled, and measured using a waveform generation and a data acquisition program. Using this data,

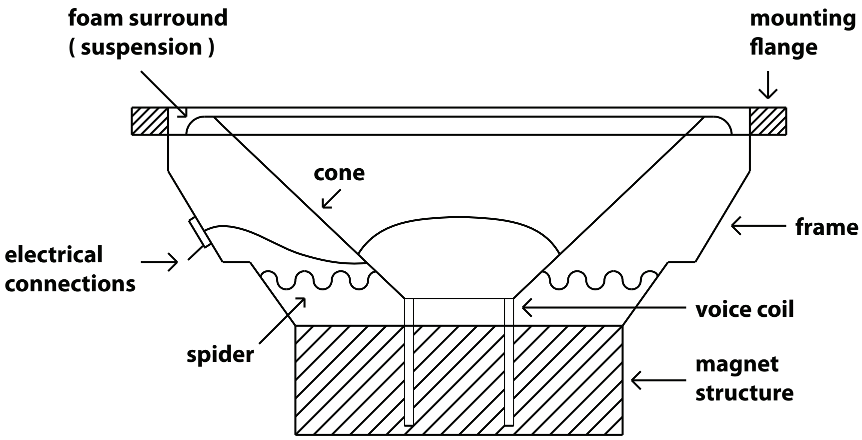


Figure 1.2: Cross-sectional view of a speaker.

the effective mass, damping coefficient, and spring constant for the speaker will be calculated.

SOLVING THE EQUATIONS OF MOTION

Our goal is to know the motion of the mass, $x(t)$, for a given forcing $F(t)$. The two most important cases are unforced or "free" motion, where $F(t) = 0$, and sinusoidal forcing, given by $F(t) = kx_s(t) = kF\cos\omega t$.

Recall that the differential equation governing the motion is given by

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \tag{1.5}$$

Before solving for the motion, we define new variables that will help streamline our analysis. First, we define the *natural frequency*, ω_n, as

$$\omega_n = \sqrt{\frac{k}{m}} \tag{1.6}$$

Secondly, we define a quantity known as the *critical damping constant*, c_{crit} as

$$c_{crit} = 2\sqrt{km} = 2m\omega_n \tag{1.7}$$

Finally, we define the *damping factor*, ζ, as

$$\zeta = \frac{c}{c_{crit}} = \frac{c}{2\sqrt{km}} \tag{1.8}$$

We see that the damping factor incorporates all 3 of the physical properties that define the system - the mass, the spring constant, and the damping constant. Thus we can think of the damping factor as an indicator of the overall damping of the system's response.

We can now rewrite the governing differential equation (1.4) in terms of these new variables, giving

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{F(t)}{m} \tag{1.9}$$

We will now solve (1.9) for the unforced case $F(t) = 0$. We assume an exponential solution of the form $x(t) = Ae^{\lambda t}$. Plugging this into (1.9), the amplitudes and exponential functions can be divided through, yielding the *characteristic equation*

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{1.10}$$

We then use the quadratic formula to solve for the λ's, giving

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \imath\omega_d \tag{1.11}$$

where ω_d is the *damped natural frequency* and is defined as

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{1.12}$$

From the definition of the damped natural frequency we see that our analysis will depend on the magnitude of ζ. We will concentrate in this lab on underdamped responses, where $\zeta < 1$. The two other cases are overdamped ($\zeta > 1$) and critically damped ($\zeta = 1$) responses.

Having solved for λ_1 and λ_2, we can now write the solution to (1.9) as

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{1.13}$$

where the amplitude, A, and phase, ϕ, are the unknowns. If this form of the solution appears unfamiliar to you, plug (1.13) into (1.9) and verify that it does indeed satisfy the governing differential equation. The two unknowns can be found from the given initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$.

THE LOGARITHMIC DECREMENT METHOD

The viscous damping constant, c, may be determined experimentally by measuring the rate of decay of unforced oscillations. The logarithmic decrement, which is the natural logarithm of the ratio of any two successive amplitudes, is used. The larger the damping, the greater will be the rate of decay of oscillations and the bigger the logarithmic decrement, D.

$$D = \ln\left(\frac{x_n}{x_{n+1}}\right) \tag{1.14}$$

where x_n and x_{n+1} are the heights of two successive peaks in the decaying oscillation pictured in Figure 1.3. To find a theoretical representation for the logarithmic decrement D, we look at the exponentially decaying envelope for the damped oscillation, which is given by

$$x_{envelope}(t) = x_0 e^{-\zeta\omega_n t} \tag{1.15}$$

Using this equation we now write the logarithmic decrement as

$$D = \ln\left(\frac{x_{envelope}(t)}{x_{envelope}(t + \tau_d)}\right) = \ln\left(\frac{x_0 e^{-\zeta\omega_n t}}{x_0 e^{-\zeta\omega_n (t+\tau_d)}}\right) = \ln\left(e^{\zeta\omega_n\tau_d}\right) = \zeta\omega_n\tau_d \tag{1.16}$$

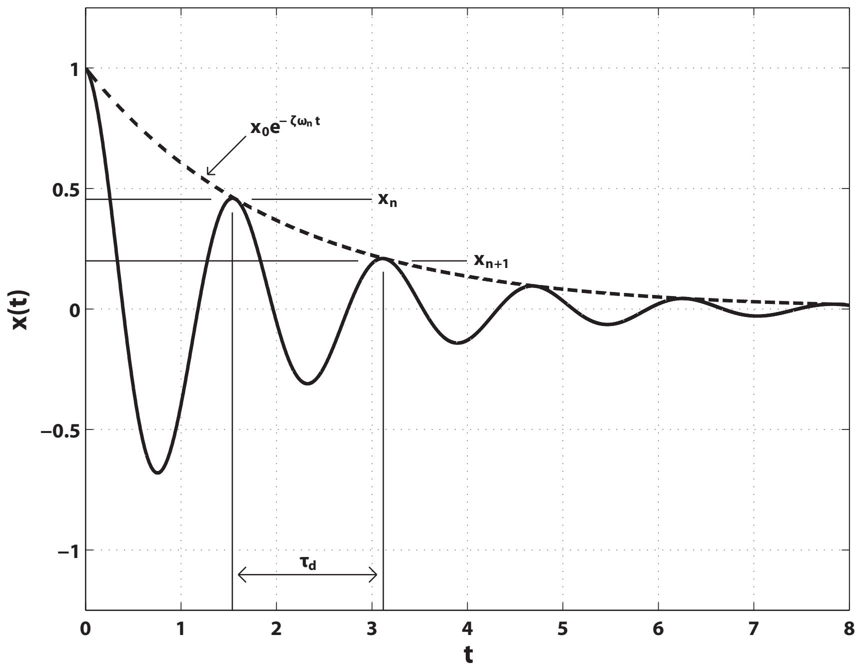


Figure 1.3: The logarithmic decrement method.

where τ_d is the period of the damped oscillation, i.e. $\tau_d = \frac{2\pi}{\omega_d}$. We simplify this expression by substituting in (1.8) for ζ and then solve for the damping constant c, yielding

$$c = \frac{2mD}{\tau_d} \tag{1.17}$$

We can also obtain an equation for k from (1.16) , yielding

$$k = \frac{c^2\left(1 + \frac{4\pi^2}{D^2}\right)}{4m} = \frac{4\pi^2 m}{\tau_d^2\left(1 - \zeta^2\right)} \tag{1.18}$$

Thus, using equations (1.17) and (1.18), we can find the damping constant c and spring constant k for the mass-spring-dashpot system from the experimentally obtained values for D and τ_d.

FORCED VIBRATIONS AND FREQUENCY RESPONSE

Next, we will solve (1.9) for the forced case. We assume the support is driven harmonically with its displacemnt given by $x_s(t) = F\cos\omega t$, where F is the displacement amplitude of the support and ω the natural frequency of its motion. Since the mass is coupled to the support via the spring, the force exerted by the support's motion on the mass is given by $F(t) = kx_s(t) = kF\cos\omega t$. The equation of motion (1.9) now becomes

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{kF}{m}\cos\omega t \tag{1.19}$$

From ordinary differential equation theory we can write the general solution to (1.19) as the sum of a complimentary (also referred to as the transient or homogeneous) solution $x_c(t)$ and a particular (or steady-state) solution $x_p(t)$.

$$x(t) = x_c(t) + x_p(t) \tag{1.20}$$

In this lab we will only be interested in the steady-state solution after the transient response dies out. Thus we take the general solution to be of the form

$$x_p(t) = A\cos(\omega t - \phi) \tag{1.21}$$

where the amplitude of oscillation of the mass position, A, and the phase of the displacement with respect to the exciting force, ϕ, are two unknowns. To solve for the unknowns we substitute (1.21) into (1.19) to get (after some trigonometric reductions)

$$\begin{aligned}&\left[2A\omega\omega_n\zeta\sin\phi + A\omega_n^2\cos\phi - A\omega^2\cos\phi - \frac{Fk}{m}\right]\cos\omega t\\ &+\left[-2A\omega\omega_n\zeta\cos\phi + A\omega_n^2\sin\phi - A\omega^2\sin\phi\right]\sin\omega t = 0\end{aligned} \tag{1.22}$$

While (1.22) looks intimidating, note that the coefficients of $\cos\omega t$ and $\sin\omega t$ are independent of t, i.e. they are constants. Therefore we can use the linear independence of $\cos\omega t$ and $\sin\omega t$ to claim that their respective coefficients must be identically equal to zero for (1.22) to hold. This gives us two equations and two unknowns, A and ϕ, to solve for. Solving for the unknowns yields

$$\tan\phi = \frac{2\omega\omega_n\zeta}{\omega_n^2 - \omega^2} \tag{1.23a}$$

$$A = \frac{\frac{Fk}{m}}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\omega^2\omega_n^2\zeta^2}} \tag{1.23b}$$

with the restriction $0 \leq \phi \leq \pi$.

RESONANCE

Resonance as defined by *Merriam-Webster* is *a vibration of large amplitude in a mechanical or electrical system caused by a relatively small periodic stimulus of the same or nearly the same period as the natural vibration period of the system.* From basic ODE theory we know that resonance occurs when we force the system with a frequency $\omega = \omega_n$. If the system has zero damping ($c = 0$) the response is unbounded, else we will see that the system's response amplitude is simply maximized.

Following the above definition, if we force the damped system (1.19) at a frequency $\omega = \omega_n$, the system's response's phase-lag (1.23a) and amplitude ratio become

$$\phi = \tan^{-1}\infty = \frac{\pi}{2} \tag{1.24a}$$

$$\frac{A}{F} = \frac{\frac{k}{m}}{2\omega_n^2\zeta} = \frac{1}{2\zeta} \tag{1.24b}$$

Thus when $\omega = \omega_n$ we should expect the response of the system to lag the forcing function by approximately 90 degrees and to have a finite amplitude dependent on the value of the damping factor. However, resonance in a damped mass-spring-dashpot system does not occur when the forcing frequency is exactly the undamped natural frequency ω_n. To find the resonance frequency, ω_r, we maximize the response's amplitude (1.23b) by differentiating with respect to the forcing frequency ω and setting it equal to zero.

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_r} = 0 \Rightarrow \omega_r = \omega_n\sqrt{1-2\zeta^2} \tag{1.25}$$

PHASE DIAGRAMS

While performing the lab we will need to graphically determine if we are forcing the mass-spring-dashpot system near its resonance frequency. In Figures 1.6 and 1.8 we see that the *Labview* software provides a graph of $x_s(t)$ vs. $x(t)$, i.e. it plots the position of the mass as a function of the position of the support. For different values of the forcing frequency ω this graph will have different qualitative behavior.

For a support whose position is given by $x_s(t) = F\cos\omega t$, the amplitude and phase of the resulting mass oscillation were found to be (1.23b) and (1.23a) respectively. Now lets assume that $\omega \ll \omega_n$. The amplitude and phase of the steady-state oscillation then become

$$\tan\phi = \frac{2\omega\omega_n\zeta}{\omega_n^2-\omega^2} \approx 0 \tag{1.26a}$$

$$A = \frac{\frac{Fk}{m}}{\sqrt{\left(\omega_n^2-\omega^2\right)^2 + 4\omega^2\omega_n^2\zeta^2}} \approx F \tag{1.26b}$$

From (1.21) we find that the position of the mass will be

$$x(t) \approx F\cos\omega t \tag{1.27}$$

We see that $x(t) = x_s(t)$. Therefore if we plotted $x_s(t)$ vs. $x(t)$ (parameterized by t) we would simply see a line through the origin with a slope equal to 1 (in other words the graph looks like $y = x$).

Now lets assume that we are forcing the system at resonance, i.e. $\omega \approx \omega_n$. In the previous section we found that the system's response would have amplitude and phase given by (1.24b) and (1.24a) respectively. Therefore the position of the mass is given by

$$x(t) \approx \frac{F}{2\zeta}\cos\left(\omega t - \frac{\pi}{2}\right) = \frac{F}{2\zeta}\sin\omega t \tag{1.28}$$

To see what the plot of $x_s(t)$ vs. $x(t)$ looks like, we note that from basic trigonometry we have

$$\frac{x^2(t)}{\left(\frac{F}{2\zeta}\right)^2} + \frac{x_s^2(t)}{F^2} = 1 \tag{1.29}$$

This equation represents (in general) an ellipse in the $x_s(t)$-$x(t)$ plane. For the special case of resonance the above equation reduces to

$$\cos^2 \omega t + \sin^2 \omega t = 1 \tag{1.30}$$

Thus the plot of $x_s(t)$ vs. $x(t)$ will appear to be nearly circular.

LABORATORY SET-UP

- **Mass-Spring-Dashpot System**
  The apparatus consists of a laboratory-model mass-spring-dashpot system with displacement transducers (Linear Variable Differential Transformers or LVDTs) for measuring $x(t)$ and $x_s(t)$. The output from the LVDTs is communicated to the computer via the data acquisition board. An electric motor and controller, acting through a scotch yoke, enable a sinusoidal forcing function to be applied to the system. Note that the controller dial readings are arbitrary; frequency and period data must be obtained from your computer plots.

- **Loudspeaker**
  The apparatus consists of a speaker on a stand with one LVDT to measure cone displacement. Waveforms are generated by the computer, amplified, and sent through a resistor to the speaker (approximating a current source). The computer is also used to measure current flow through the speaker and displacement of its cone (using the attached LVDT).

Please follow all safety precautions. Keep long hair and loose clothing well away from the electric motor, pulleys, and other moving parts.

- **Using the *LabView* Software**
  The four programs you will be using for part A of the lab are: *FreeAcq* (Figure 1.4) for making measurements of the unforced system; *FreeSim* (Figure 1.5) for simulation of the same; *ForcedAcq* (Figure 1.6) for measurements of the system with a sinusoidal forcing function; and *ForcedSim* (Figure 1.7) which may be used for simulation of the forced system. Although somewhat different in appearance and function, the programs share many key features.

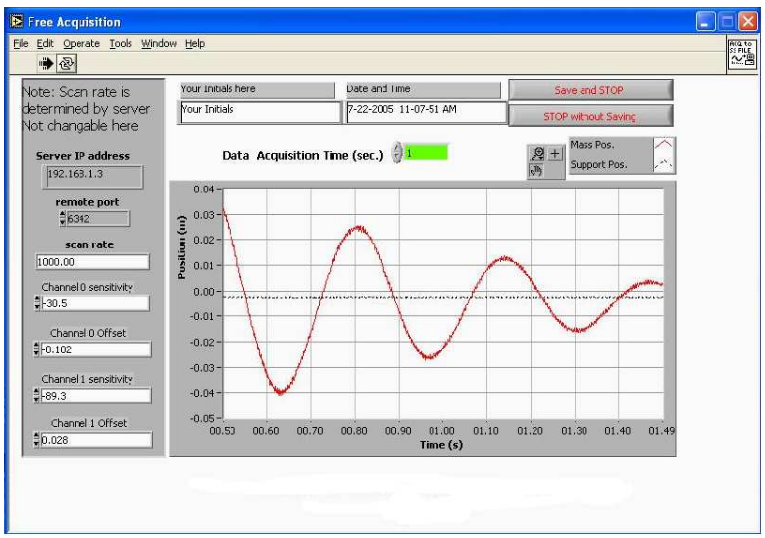


Figure 1.4: The *FreeAcq* program.

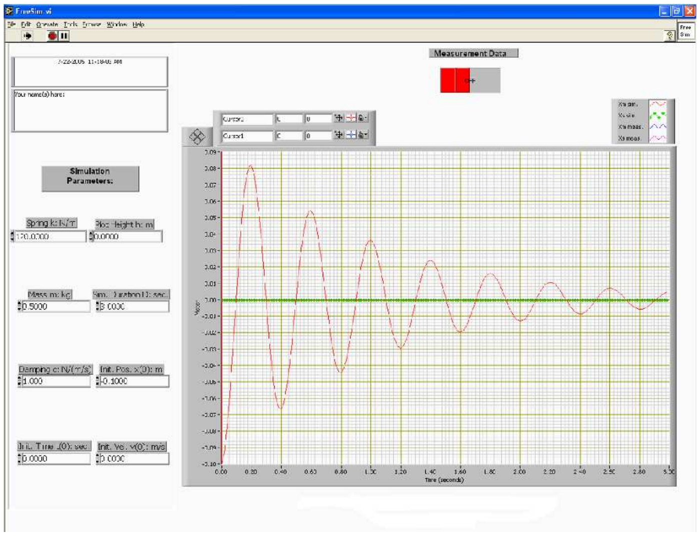


Figure 1.5: The *FreeSim* program.

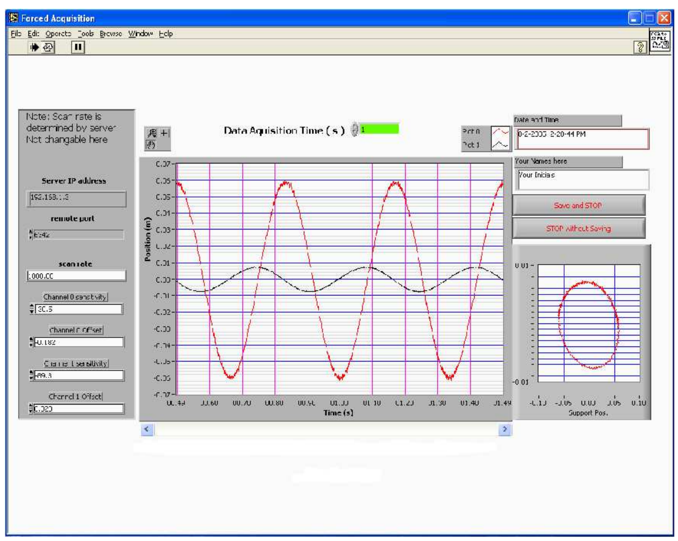


Figure 1.6: The *ForcedAcq* program.

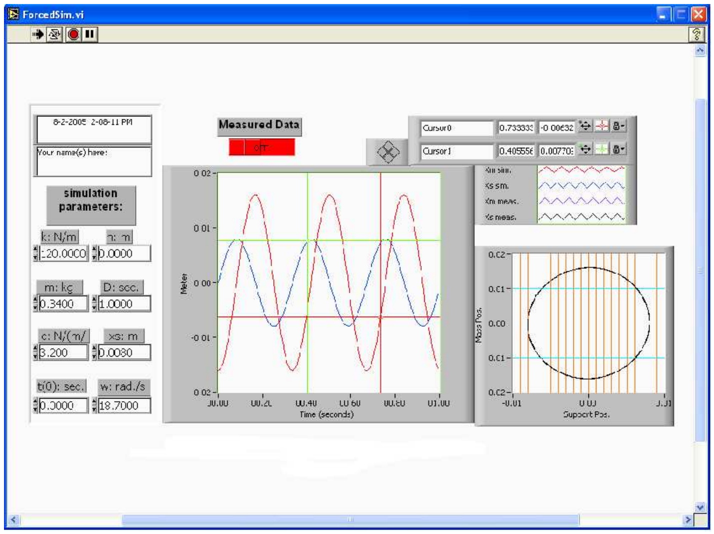


Figure 1.7: The *ForcedSim* program.

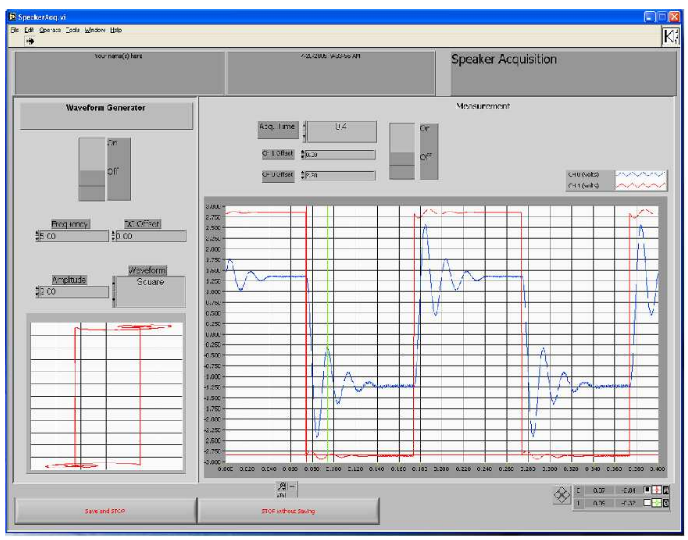


Figure 1.8: The *SpeakerAcq* program.

The data acquisition programs automatically convert the voltage output of the LVDTs to meters. To do this, they need a set of conversion factors, which are on a label on the mass-spring-dashpot base board. After starting a measurement program, make sure that the sensitivity and offset values on the left hand side of the window match the values listed on a small sheet of paper in front of the apparatus. Also please enter your name in the box provided.

To run the program, you must hit the white arrow in the top left of the screen. If this arrow is black, that means that the program is already running. For the data acquisition programs, a green box on top will define the amount of time the program will record the motion of the mass-spring after hitting the arrow. To reset the data acquisition, press on `STOP without Saving` and then press the white arrow to begin again.

After getting data pressing the `Save and STOP` button stores your current data on disk, for comparison later to the simulation. Any previous data is erased. The data file is only used by the simulation programs *FreeSim* and *ForcedSim*; it is not available to the data acquisition programs.

You may find it convenient to obtain numerical data from your plots using the cursors, rather than using a ruler. Two cursors are available, one indicated by a circle and one by a square. To use a cursor, use the mouse to drag it to the point you want to measure. The x and y values of the point you have chosen will appear above the graph, in the row corresponding to that cursor. For best accuracy, you should utilize every full cycle on the screen. For example, if three cycles are displayed, you should measure the time elapsed from the start of the first cycle to the end of the third, and then divide Δx by three to obtain the period. If your cursor has vanished off the screen, you can enter an on-screen position for it into the x and y display boxes, and it will reappear in the desired location. You can also move the cursors around using the little arrow "buttons" on the screen in the cursor control box. If a cursor turns a darker color, it is locked on to a data point, and will trace the curve point by point if the left or right arrow is pressed. Zoom and other features are available for the cursors and graphs; see the *LabView* manual for details.

PROCEDURE

- **Free Vibration, Mass-Spring-Dashpot**

  1. Here you will be recording the motion of the mass after it has been pulled down from equilibrium and then released. First start up the *FreeAcq* program. Start with a data acquisition time of 6 seconds. Give the system an initial position with zero initial velocity (i.e. pull down the mass and hold it still). Then press the white run arrow in the top left of the toolbar and immediately release the mass. Repeat this procedure until you have a nice oscillation over the three seconds. Please note that the zero position is somewhat arbitrary and will depend on the

position of the scotch yoke when the mass is released. Also, the zero level for the scotch yoke and for the mass may not agree exactly. Finally, you will need to take data long enough for the mass to stop oscillating in order to have a good zero reference. Save your best oscillation on disk by pressing the `Save and STOP` button.

2. Next we will simulate the free vibration of the mass-spring-dashpot system. Start the *FreeSim* program. To compare the simulation data to your saved experimental data, turn the `Measured Data` switch on. To change a simulation parameter type in the value you want and press `Enter`. The following parameters for the simulation can be changed:
   - k is the spring constant in Newtons/meter.
   - m is the mass in kilograms. **You need to include both the mass of the spring and the mass of the weight since springs in the real world are not massless.**
   - c is the damping coefficient in Newtons/meter/second.
   - $t(0)$ (in seconds) lets you adjust the relative starting point of the simulation plot. It allows you to move the plot horizontally, as necessary, making comparison with the measured data easier.
   - h (in meters) lets you adjust the simulation plot vertically.
   - D is the duration in seconds for the simulation to be run. Set it equal to the duration of the measured data set if you are comparing them.
   - $x(0)$ is the initial position of the mass in meters. To start the oscillations you need to initially pull down on the mass.
   - $v(0)$ is the initial velocity of the mass in meters/second. This should be zero.

   Add the measured data to the graph by pressing the `Measurement Data` switch above the graph. Using the cursors, measure the logarithmic decrement D of the measured data and the period of the damped oscillation τ_d. Check if c is constant by measuring the logarithmic decrement for several separate cycles. Determine k, the spring stiffness. Make a print-out of one of your curves.

3. Simulate unforced motion by inputting the values of m, k, and c that you just determined into the *FreeSim* program. Obtain $x(0)$ from your measured data. Compare your simulation with your measured data. If agreement is not good, adjust k and c until you have good agreement. Make a print-out.

- **Forced Vibration, Mass-Spring-Dashpot**

1. We will now be recording the motion of the mass as it undergoes sinusoidal forcing. Start up the *ForcedAcq* program. Set the acquisition time to 30 seconds, start the data acquisition, and turn on the motor. Two graphs will be displayed. The left one contains two plots. One is a plot of the mass position $x(t)$ vs. time and the second one is a plot of the spring support position (forcing) $x_s(t)$ vs. time. The right graph plots $x(t)$ vs. $x_s(t)$ and helps show phase relationships.

2. For at least five different forcing speeds get nice plots of several cycles of motion. The forcing speeds should include:
   - The slowest speed for which the motor runs smoothly.
   - A very fast speed.
   - Resonance.
   - A speed just slower than resonance.
   - A speed just faster than resonance.

   Make sure to save each data set to disk in order to analyze them in the *ForcedSim* program (print-outs are not necessary). Set the data acquisition time to 10 seconds and run the program in order to find the desired speed. Then hit `STOP without SAVING`. Reduce the data acquisition time to 1 second and then run the program again. Then hit `SAVE and STOP`.

3. Next we will simulate the forced vibration of the mass-spring-dashpot system. Open the *ForcedSim* program. Turn on the measured data switch to view your saved data. To change the current measured data set you must close and then re-open the *ForcedSim* program. Once experimental data is loaded, make your necessary measurements using the computer cursors. **You will need to make sufficient measurements in order to make plots of amplitude ratio, $\frac{x(t)}{x_s(t)}$, and phase shift, ϕ, vs. forcing frequency ω. In particular, you should measure the period, forcing function amplitude, mass motion amplitude, and phase-lag between the forcing function and the resulting mass motion. You need to do this for each of your five speeds.** You can then enter your calculated experimental values into the *ForcedSim* program and see how well the experimental data follows theory. The parameters in the *ForcedSim* program include most of the ones described for the *FreeSim* program, plus the following:
   - $x_s(t)$ is the amplitude (in meters) of the motion of the spring support, which is moved up and down by the motor and scotch yoke. This motion supplies the forcing of the system.
   - ω is the angular velocity of the spring support motion in radians per second.

You may also want to save the data to a USB storage device or write it to a CD for later analysis. To do this just copy the text files of the desired data onto your storage device.

- **Vibration of a Speaker**

1. You will now experimentally measure the "free"-response of the loudspeaker. Open the folder named *speaker* on the desktop and then open the *Speaker* program. Set the waveform generator controls as follows in order to obtain a maximum-amplitude square wave at about 5 Hz. Set the `Waveform` control to `Square`, the `Frequency` control to 5, and the `Amplitude` control to 2. Leave the `DC Offset`

control set to 0. Note that when changing waveforms or frequencies, you must wait a few seconds for the computer to equilibrate and display correct data. Set the data acquisition time control to about 0.4 seconds. The displacement of the cone (channel 0) is displayed as a blue line and the current flow through the speaker coil (channel 1) is displayed as a red line. You want to see a square wave with "ringing" wiggles after each shift in level that gradually damp out. Use a low enough frequency for the square wave so that the "ringing" damps out completely before the square wave changes levels again. **Be careful not to shake the table during the experiment as small vibrations can cause errors.** This is the step-response of the speaker, which is approximately equivalent (in this lab) to the "free"-response you obtained earlier for the mass-spring-dashpot system. When you have a good display of the "ringing" turn off the data acquisition.

2. Measure the logarithmic decrement D. Try to measure the coordinates of at least three successive peaks of the blue curve yielding at the minimum two values for the logarithmic decrement. **Remember to measure amplitudes relative to the equilibrium level (the level your exponential decay curve ends up at), not to the zero level of the plot.** Measure the period of the damped oscillation.

3. Next you will force the loudspeaker at its resonance frequency in order to experimentally determine the mass m and spring constant k of the loudspeaker. Set the `Waveform` control to `Sine` and the `Amplitude` control to 2. Leave the `DC Offset` control set to 0. Set the data acquisition time to 0.1 seconds. The `CH 0 Offset` and `CH 1 Offset` controls may be used to adjust the plots vertically if necessary. Turn on the waveform generator and data acquisition switches and adjust the `Frequency` control value until you observe resonance of the speaker cone. **To change the frequency you must press `STOP without Saving`, enter the desired frequency and then start the program again in order to observe the new frequency.** Neither the spring constant k nor the mass m of the speaker is easily measured at resonance. However, you can derive the approximate mass and spring constant by observing what happens when the mass is changed a known amount. Measure the mass of the rubber weight and then carefully press it onto the LVDT shaft. The best way is to spread the weight open, position it, and release it. Find the new resonant frequency.

LAB REPORT QUESTIONS

Please answer the following questions concerning the mass-spring-dashpot part of the lab within your lab report:

1. What is the spring constant k and damping constant c for your mass-spring-dashpot setup? Indicate the measured data and formulas you used to calculate these values. Is the damping constant c really constant? What does this say about the air dashpot acting linearly?

2. Compare your experimental data to the simulated data for unforced motions. Comment on any similarities or differences of interest. How did adjusting k and c to better fit your data change the simulation graph? Please attach print-outs from before and after you adjust k and c to better fit your data.

3. Make a plot of the amplitude ratio (peak mass displacement divided by peak forcing displacement) versus forcing frequency ω.

4. Make a plot of the phase-angle ϕ between $x(t)$ and $x_s(t)$ versus the forcing frequency ω.

5. For a typical value of damping constant c that you measured, what is the percent difference between the natural frequency ω_n and the damped natural frequency ω_d? Does the addition of a dashpot to a mass-spring system increase or decrease its oscillation frequency?

6. Discuss the plots from questions #3 and #4. Do they look like what you expect based on textbook solutions to the damped one degree-of-freedom oscillator? Relate the phase-angle plot to the $x(t)$ vs. $x_s(t)$ plots. Why do the ellipses change shape and rotate as you go through resonance? State your observations about the behavior of the mass as the forcing frequency is varied in words without using numbers, angles, graphs, or equations.

Please answer the following questions concerning the loudspeaker part of the lab within your lab report:

1. Calculate k and m for the speaker, using the resonant frequencies and mass you measured in lab.

2. Calculate c, the damping coefficient, for the speaker. Is the speaker overdamped or underdamped? How linear was the speaker damping?

3. Find another real-world vibrating system which could be reasonably modeled as a mass-spring-dashpot. Give the system a "push" and observe its response. Try applying a forcing function of various frequencies, and look for resonance.

(a) Describe how you modeled your vibrating system as a mass-spring-dashpot. That is, what does the mass represent, what is the spring, and what is the dashpot? Be as specific as possible.

(b) Is this system typically overdamped? Underdamped? If applicable, what was the resonant frequency (approximately)?

(c) In what ways does the system you found most significantly differ from an ideal linear mass-spring-dashpot system?

CALCULATIONS & NOTES

Lab #2 - Two Degrees-of-Freedom Oscillator

Last Updated: February 14, 2008

INTRODUCTION

The system illustrated in Figure 2.1 has two degrees-of-freedom. This means that two is the minimum number of coordinates necessary to uniquely specify the position of the system. The purpose of this laboratory is to introduce you to some of the properties of linear vibrating systems with two or more degrees-of-freedom. You have already seen a one degree-of-freedom vibrating system (the mass-spring-dashpot system) and should have some familiarity with the ideas of *natural frequency* and *resonance*. These ideas still apply to an undamped linear system with two or more degrees-of-freedom.

The new idea for many degrees-of-freedom systems is the concept of *modes* (also called *normal modes*). Each *mode shape* has its own natural frequency and will resonate if forced at that frequency. The number of modes a system has is equal to the number of degrees-of-freedom. Thus the system above has two modes and two natural frequencies.

The primary goals of this laboratory are for you to learn the concept of normal modes in a two degrees-of-freedom system – the simplest system which exhibits such modes. You will learn this by experimentation and calculation.

PRE-LAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. Are the number of degrees of freedom of a system and the number of its normal modes related? Explain.
2. How can a normal mode be recognized physically?
3. What do you expect to happen when you drive a system at one of its natural frequencies?
4. Draw a free body diagram and derive the equations of motion for a three degrees-of-freedom system, with three different masses and four equal springs. Put them in matrix form. (See the derivation for a two degrees-of-freedom system in the lab manual. Your result should resemble equation (2.5).) Substitute in the normal mode solution (2.7) to get an eigenvalue problem similar to (2.9).
5. Using MATLAB, find the eigenvalues and eigenvectors of the following matrix and print the results (HINT: Type `help eig` for assistance).

$$[A] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{2.1}$$

NORMAL MODES

The concept of normal modes can be expressed mathematically in the following way. Say the position of an n-degree of freedom system can be described by the n numbers x_1, x_2, x_3 $\ldots x_n$ (this is, in fact, the definition of an n degrees-of-freedom system). Since the system is dynamic, each of these variables is a function of time $x_1(t)$, $x_2(t)$ etc. A motion of the system corresponds to a specified list of these functions. In general these functions of time can be quite complicated. However, for linear undamped systems there turns out to be many solutions that are, in some sense, simple. In fact, there are n such simple solutions called *normal mode vibrations*. A fortunate and often used fact is that every possible solution of the system can be written as a sum of these solutions. (In the language of linear algebra one can say that the normal mode solutions *span* the space of all solutions.) A normal mode solution for a five degrees-of-freedom system looks like

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \underline{\mathbf{v}}\,(A\cos\omega t + B\sin\omega t) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} (A\cos\omega t + B\sin\omega t) \tag{2.2}$$

A normal mode vibration is characterized by a mode shape $\underline{\mathbf{v}}$ and an angular frequency ω (the "natural frequency" for the given mode shape). The mode shape $\underline{\mathbf{v}}$ is a list of constants (v_1, v_2, $\ldots$) that determine the relative amplitude of motion for each degree-of-freedom of the system. The constants A and B determine the amplitude and phase of the vibration. Note that in a normal mode vibration each point moves exactly as in simple harmonic motion. All points are moving with the same angular frequency ω and are exactly in-phase or exactly out-of-phase, depending on the signs of the appropriate elements of $\underline{\mathbf{v}}$.

The general motion of an n degrees-of-freedom undamped linear vibrating system can be written as the sum of normal mode solutions.

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^{n} (A_i \cos\omega_i t + B_i \sin\omega_i t)\,\underline{\mathbf{v_i}} \tag{2.3}$$

The system is characterized by its natural frequencies ω_i and mode shapes $\underline{\mathbf{v_i}}$. The constants A_i and B_i are determined by the initial conditions and specify the amplitude and phase of the i-th normal mode. The mathematics involved in the discussion above is very similar to the mathematics for a set of first-order differential equations. (The governing equations for an n degrees-of-freedom vibrating system can, in fact, be written as a set of $2n$ first order equations.)

DERIVING AND SOLVING THE EQUATIONS OF MOTION

We will now derive the equations of motion for the two degrees-of-freedom air track experiment. The variables and physical setup are shown in Figures 2.1 and 2.3. We will draw the free-body diagram for each mass and work out its equation of motion. To help get the signs right, assume that the displacements, velocities, and accelerations of the masses are all

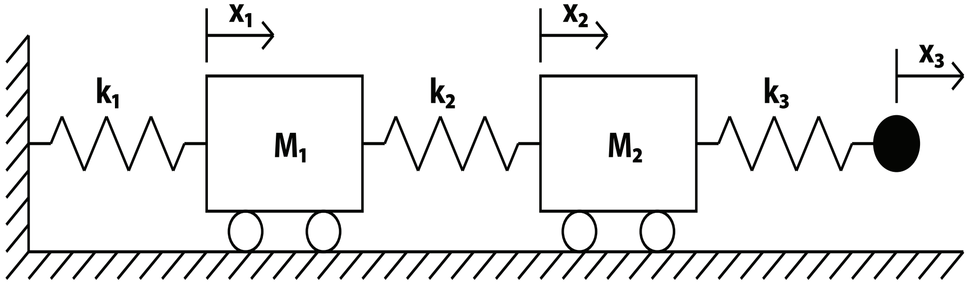


Figure 2.1: Illustration of a coupled mass-spring system.

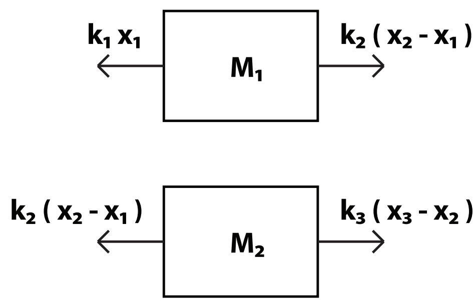


Figure 2.2: The free-body diagrams for masses m_1 and m_2.

positive (i.e. to the right) with $x_1 < x_2 < x_3$. This puts all of the springs into tension relative to their equilibrium condition. The equations of motion (assuming equal spring constants) are

$$k\,(x_2 - x_1) - kx_1 = m_1\ddot{x}_1 \tag{2.4a}$$

$$k\,(x_3 - x_2) - k\,(x_2 - x_1) = m_2\ddot{x}_2 \tag{2.4b}$$

We can rewrite this in matrix form as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{2k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{kx_3(t)}{m_2} \end{bmatrix} \tag{2.5}$$

or as

$$\ddot{\underline{\mathbf{x}}} = [A]\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t) \tag{2.6}$$

We now take a lead from ODE theory and propose a solution to (2.6) (assuming $\underline{\mathbf{f}}(t) = \underline{\mathbf{0}}$, i.e. no external forcing) of the form

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{v}}e^{\alpha t} \tag{2.7}$$

Substituting (2.7) into (2.6) and canceling out the exponentials yields

$$\alpha^2\underline{\mathbf{v}} = [A]\,\underline{\mathbf{v}} \tag{2.8}$$

Equation (2.8) is in the form of an eigenvalue problem from linear algebra. The values of α we need to find to complete our solution (2.7) are really the square roots of the eigenvalues of the matrix $[A]$. Furthermore the vector $\underline{\mathbf{v_i}}$ is the eigenvector associated with eigenvalue λ_i. Rearranging this equation we get

$$\left([A] - \alpha^2\,[I]\right)\underline{\mathbf{v}} = \underline{\mathbf{0}} \tag{2.9}$$

We would like to find a solution to this equation that doesn't involving setting $\underline{\mathbf{v}} = \underline{\mathbf{0}}$ since this is the trivial solution where neither mass is moving and the entire system is at rest. From basic linear algebra theory this requires the matrix $([A] - \alpha^2\,[I])$ be singular, i.e. non-invertible. Stated mathematically, we require the determinant of this matrix to be equal to 0.

$$|[A] - \lambda\,[I]| = \lambda^2 + \left(\frac{2k}{m_2} + \frac{2k}{m_1}\right)\lambda + \frac{3k^2}{m_1 m_2} = 0 \tag{2.10}$$

where we have substituted $\lambda = \alpha^2$. To simplify our calculations we now make the assumption that $m_1 = m_2 = 1$. This reduces the equation to

$$\lambda^2 + 4k\lambda + 3k^2 = 0 \tag{2.11}$$

with solutions $\lambda_1 = -k$ and $\lambda_2 = -3k$. The two eigenvectors associated with λ_1 and λ_2 are found by substituting each eigenvalue back into equation (2.8) and solving for $\underline{\mathbf{v}}$, giving us

$$\underline{\mathbf{v_1}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{\mathbf{v_2}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2.12}$$

Therefore we can write the solution to (2.6) as

$$\underline{\mathbf{x}}(t) = c_1\underline{\mathbf{v_1}}e^{\sqrt{-k}t} + c_2\underline{\mathbf{v_2}}e^{\sqrt{-3k}t} \tag{2.13}$$

Using Euler's identity we can rewrite this in terms of trigonometric functions as

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{v_1}}\left(A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)\right) + \underline{\mathbf{v_2}}\left(A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t)\right) \tag{2.14}$$

where $\omega_1 = \sqrt{k}$ and $\omega_2 = \sqrt{3k}$. The coefficients A_i, B_i (for $i = 1, 2$) are the four unknowns to be determined by initial conditions (recall that our system is comprised of 2 second-order ODEs, thus the 4 required initial conditions).

Physically, equation (2.14) tells us that the motion of each mass can be written as a linear combination of a high-frequency and a low-frequency harmonic oscillation. These are the *normal mode oscillations*. To get a better idea of the physical significance of the mode shapes $\underline{\mathbf{v_i}}$, let us perform a simple initial value problem (IVP).

First we will assume that we initially displace both m_1 and m_2 by a positive distance x_0 (placing them in-phase with one another) and release them from rest. Plugging $t = 0$ into (2.14) the first initial condition yields

$$\underline{\mathbf{x}}(0) = A_1\underline{\mathbf{v_1}} + A_2\underline{\mathbf{v_2}} = \underline{\mathbf{x_0}} \tag{2.15}$$

Differentiating (2.14) with respect to t, the second initial condition (initially at rest) yields

$$\underline{\dot{\mathbf{x}}}(0) = B_1\omega_1\underline{\mathbf{v}_1} + B_2\omega_2\underline{\mathbf{v}_2} = \underline{\mathbf{0}} \tag{2.16}$$

Equations (2.15) and (2.16) give us four equations (since $\mathbf{v_1}$ and $\mathbf{v_2}$ are both 2 by 1 vectors) involving four unknowns. Solving for the unknowns gives us $A_1 = x_0$ and $A_2 = B_1 = B_2 = 0$. The solution to the IVP is then

$$\underline{\mathbf{x}}(t) = x_0\underline{\mathbf{v}_1}\cos\omega_1 t \tag{2.17}$$

We see that by setting the system to be initially in-phase, the resulting motion consists only of the first normal mode. Since x_0 was chosen arbitrarily, we could easily just assume that $x_0 = 1$. Thus at $t = 0$ we have

$$\underline{\mathbf{x}}(0) = \underline{\mathbf{v}_1} \tag{2.18}$$

We see from (2.18) that it is the eigenvector associated with the normal mode that tells us the necessary initial displacements in order to excite that normal mode when starting from rest. Furthermore, since any scalar multiple of an eigenvector still satisfies the eigenvalue equation (2.9), we do not need to worry about what units we take the eigenvector to be in (i.e. if the eigenvector tells us to move each mass by 1, we can move them 1 cm or 1 inch).

If we were to perform another IVP with initial displacements $x_1(0) = x_0$ and $x_2(0) = -x_0$ we would see that the solution would consist of only the second normal mode. Thus we can conclude that for the two degrees-of-freedom system the first normal mode represents in-phase motion while the second normal mode represents out-of-phase motion. One interesting result of our analysis is that the normal mode corresponding to in-phase motion has a lower natural frequency than the out-of-phase normal mode. *Why do you think that is?*

Fortunately for us we won't need to perform all this linear algebra during the lab. We can easily compute the eigenvalues and eigenvectors of the matrix $[A]$ using computer software. Each lab station computer has a numerical analysis program called *SciLab* installed on it and by following the directions given in the lab set-up you will be able to calculate the necessary values easily.

LABORATORY SET-UP

- **Air Track**
  The lab set-up consist of an air-track hooked up to the lab's air system, four or more air track gliders, four plug-in springs, a mechanical oscillator (for external forcing), a photogate timer, and a digital stopwatch. Please note that there are two somewhat incompatible styles of glider which should only be used on the appropriate air tracks. Each glider has a label listing its mass (including spring) and the air tracks on which it will work. **You should make sure to remeasure the masses of the gliders and springs at the start of your lab.**

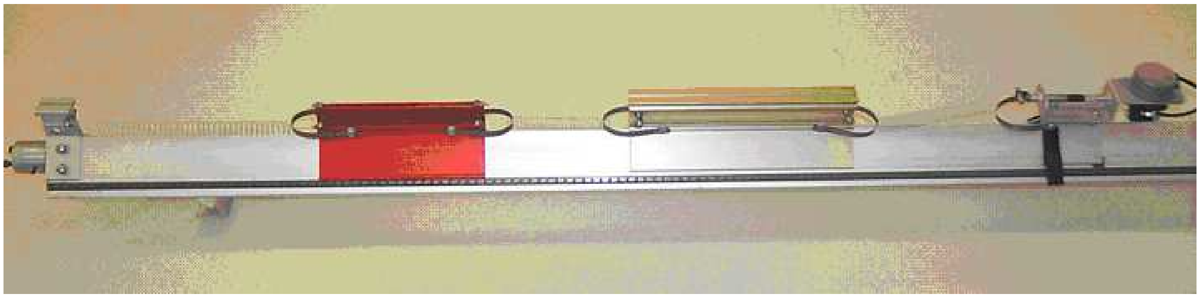


Figure 2.3: The laboratory set-up you will be working with.

- **Using the *SciLab* Software**

  1. Open *SciLab* by clicking on its icon located on the desktop of your computer. This program is a freeware program similar to *MatLab* and should look quite similar.
  2. To find the eigenvalues and eigenvectors of a matrix you must use the function `spec()` as shown below. Send the matrix $[A]$ as the function parameter and the program will return the eigenvalues along the diagonal of a square matrix and the eigenvectors as the columns of the second returned matrix.

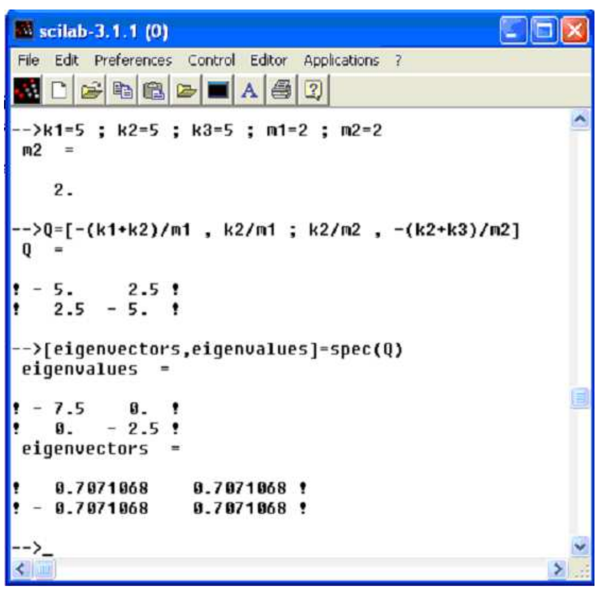


Figure 2.4: Screenshot of *Scilab* in use.

  3. For the lab, however, you must find $[A]$ for a three degrees-of-freedom system. This example is for the two degrees-of-freedom simulation. If you wish to try this function on *MatLab*, everything is the same except for the function name. In *MatLab* you must enter `eig(A)` to find the eigenvalues and eigenvectors.

PROCEDURE

1. Play with the air track, gliders, and timer. Adjust the mechanical oscillator left or right so that each spring, at equilibrium, has a total length of about 20 cm (the oscillator is attached with a Velcro strap). Have the TA turn on the main air supply, if it is not already on, and turn on the valve at the end of the air track.

2. Find the spring constant for your springs. Attach a small weight (40 to 50 grams) to one end of the spring and hold the other end solidly against the tabletop. Pull the weight down a few centimeters and release it, and then measure the period of oscillation. Use $\omega = \sqrt{\frac{k}{m}}$ to find k. Remember to include part of the spring as well as the plug mass in m – half is a good approximation in this case. Check several springs to determine the variability in k.

3. Choose two gliders of different sizes and calculate the eigenvalues and eigenvectors for the two normal modes. The eigenvalues are the squares of the natural frequencies of the normal modes (in radians/sec.), and the eigenvectors describe the relative amplitudes of the mass motions. You may do the calculations by hand or use *SciLab* on the computer. Weigh the gliders if necessary; remember to include the mass of the plug-in springs.

4. The system is set into a normal mode oscillation by applying the appropriate initial conditions. First, place the system in equilibrium. One simple method is to turn the air track on and off repeatedly until the gliders stop moving. With the air off, displace m_1 an arbitrary distance d (normally 1 or 2 centimeters) and displace m_2 a distance $d(\frac{v_2}{v_1})$. For example, if $\frac{v_2}{v_1} = -2$ and you move m_1 2 cm to the right, you should move m_2 4 cm to the left. (NOTE: The variables v_1 and v_2 represent the first and second elements of an eigenvector $\underline{\mathbf{v}}$, not the eigenvectors themselves.) Turn on the air track valve abruptly. The system should oscillate in a normal mode.

   Find the angular frequency of oscillation (radians per second) corresponding to each normal mode and verify that they are approximately equal to the natural frequencies calculated. Note the phase difference between the two masses at each normal mode. The angular frequency of the masses is found by timing a number of oscillations (i.e. 10) and then converting the resulting period to ω. Digital stopwatches are available at the air track.

5. Use some arbitrary initial conditions and set the system into a non-normal mode oscillation. Observe the motion. (It should be difficult to see that it is the sum of normal mode vibrations.)

6. Attempt to obtain normal mode vibrations by driving the system at each natural frequency. The frequency of oscillation is obtained by timing the motion of the driving rod connected to the motor, using either a stopwatch or a photogate timer. With the air off, set the driving frequency to one of the natural frequencies you have calculated. Does the system resonate when you turn the air on? Be patient. Start the system from rest every time you change the motor speed. Time the frequency at resonance

and compare it to the natural frequencies. Observe, as best you can, the relative phase between the scotch yoke and the masses at resonance.

7. Set up the air track with three (approximately) equal masses and four (approximately) equal springs. Adjust the mechanical oscillator to give an equilibrium spring length of about 20 cm. Verify by observation that $[1 \quad -1.414 \quad 1]^T$ is approximately a normal mode for this system.

8. Find another normal mode for this system by observation. Find another still. Are there any more? Use *SciLab* to find the normal modes and natural frequencies.

9. Using *SciLab*, find the normal modes and natural frequencies for a system with three unequal masses and four equal springs, and test them on the air track (free vibration only).

LAB REPORT QUESTIONS

1. List your values of k for the springs and a sample calculation. What is the average value of k, and what was the largest variation from the average (in percent)?

2. Did you obtain normal mode oscillations using initial conditions based on your eigenvectors? How could you tell?

3. How close were your experimental frequencies to those calculated? How does this experiment deviate from theory?

4. In what way did the block motions look like normal mode vibrations when you forced the system? In what ways did they not look like normal mode vibrations? Consider three cases:

   (a) forcing frequency = a natural frequency

   (b) forcing frequency close to a natural frequency (Was the amplitude of the oscillations constant in this case? If not, how did it vary?)

   (c) forcing frequency far from a natural frequency.

5. Write down the equations of motion for the system with two equal masses and three equal, massless, linear springs, as derived previously in the pre-lab. Assume x_3 (see Figure 2.1) is a given function of time: i.e. $x_3 = \sin t$.

6. Derive the equations again, this time with x_3 fixed at zero but with a known force F acting on m_2 in addition to the two spring forces.

7. Suppose that $F = k \sin t$. Show that the systems in #5 and in #6 are mathematically equivalent.

8. If the spring forces were given by the equation $F_{sp} = kx^2$ and the force F in #6 was given by $F = k \sin^2 t$, would the two systems still be equivalent?

9. How many normal modes are there in the three equal mass system? What are they and how did you recognize them as normal modes? How many were you able to find experimentally? How do they compare with those you calculated?

10. What were your calculated normal modes and natural frequencies for the system with three unequal masses? Did normal mode oscillations occur with these ratios and frequencies on the air track?

CALCULATIONS & NOTES

SOLVING THE EQUATIONS OF MOTION VIA A CHANGE OF BASIS

So far we have discussed how normal modes are the simplest oscillatory functions from which *all* motions of the two degrees-of-freedom system can be thought to be comprised of. Mathematically, the normal modes y_1 and y_2 satisfy the equations of motion for simple harmonic oscillators with natural frequencies ω_1 and ω_2 respectively.

$$\ddot{y}_1 + \omega_1^2 y_1 = 0 \tag{2.19a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = 0 \tag{2.19b}$$

Since the equations of motion for the normal modes are simple in terms of the y_1, y_2 coordinates, it would be nice if we could find some transformation between the physical coordinates x_1, x_2 and these new variables, i.e. $\underline{\mathbf{x}} = f(\underline{\mathbf{y}})$, so that we can solve the problem in terms of the easier coordinates and then transform back into the original ones. We can accomplish this mathematically by performing a change-of-basis from the original basis into the *eigenbasis* of $[A]$. We define our new normal mode coordinates by

$$\underline{\mathbf{x}} = [S]\,\underline{\mathbf{y}} \tag{2.20}$$

where the change-of-basis matrix $[S]$ is defined as

$$[S] = \begin{bmatrix} \underline{\mathbf{v_1}} & \underline{\mathbf{v_2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.21}$$

Plugging this change of variables into (2.6) we get the new equation

$$[S]\,\ddot{\underline{\mathbf{y}}} = [A]\,[S]\,\underline{\mathbf{y}} + \underline{\mathbf{f}}\,(t) \tag{2.22}$$

Left-multiplying both sides by $[S^{-1}]$ gives us

$$\ddot{\underline{\mathbf{y}}} = \left[S^{-1}\right][A]\,[S]\,\underline{\mathbf{y}} + \left[S^{-1}\right]\underline{\mathbf{f}}\,(t) = [\Lambda]\,\underline{\mathbf{y}} + \tilde{\underline{\mathbf{f}}}(t) \tag{2.23}$$

where

$$[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & -3k \end{bmatrix} \tag{2.24}$$

Looking at the unforced case, $\tilde{\underline{\mathbf{f}}}(t) = \underline{\mathbf{0}}$, we see from (2.23) that in the new normal mode coordinates we now have two uncoupled second-order ODEs,

$$\ddot{y}_1 + k y_1 = 0 \tag{2.25a}$$

$$\ddot{y}_2 + 3k y_2 = 0 \tag{2.25b}$$

the solutions of which are

$$y_1 = A_1 \cos\sqrt{k}t + B_1 \sin\sqrt{k}t \tag{2.26a}$$

$$y_2 = A_2 \cos\sqrt{3k}t + B_2 \sin\sqrt{3k}t \tag{2.26b}$$

Using (2.20) we can now transform back into the original x_1, x_2 coordinates giving

$$\begin{aligned}\underline{\mathbf{x}} &= [S]\,\underline{\mathbf{y}} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} = \\ &\underline{\mathbf{v_1}}\left(A_1 \cos\left(\omega_1 t\right) + B_1 \sin\left(\omega_1 t\right)\right) + \underline{\mathbf{v_2}}\left(A_2 \cos\left(\omega_2 t\right) + B_2 \sin\left(\omega_2 t\right)\right)\end{aligned} \tag{2.27}$$

where we have substituted $\omega_1 = \sqrt{k}$ and $\omega_2 = \sqrt{3k}$. This is the same result we found before in (2.14), so you might not think much was gained by performing this change-of-basis. However, the real advantage of this method appears when we consider the forced case.

FORCED TWO-DEGREE-OF-FREEDOM SYSTEM

We now reconsider equation (2.23) when $\underline{\tilde{\mathbf{f}}}(t) \neq \underline{\mathbf{0}}$.

$$\ddot{\underline{\mathbf{y}}} = [\Lambda]\,\underline{\mathbf{y}} + \underline{\tilde{\mathbf{f}}}(t) \tag{2.28}$$

The two resulting equations are

$$\ddot{y}_1 + \omega_1^2 y_1 = \frac{k x_3}{2m_1} \tag{2.29a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = -\frac{k x_3}{2m_1} \tag{2.29b}$$

where $x_3(t) = F\cos\omega t$ and ω is the forcing frequency. Solving both of these non-homogeneous second-order ODEs yields

$$y_1(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t - \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_1^2}\right)\cos\omega t \tag{2.30a}$$

$$y_2(t) = A_2 \cos\omega_2 t + B_2 \sin\omega_2 t + \frac{Fk}{2m_1}\left(\frac{1}{\omega^2 - \omega_2^2}\right)\cos\omega t \tag{2.30b}$$

Once again we use (2.20) to transform back into the original coordinates to get

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x_c}}(t) + \frac{Fk}{2m_1}\begin{bmatrix} \frac{1}{\omega^2-\omega_2^2} - \frac{1}{\omega^2-\omega_1^2} \\ -\frac{1}{\omega^2-\omega_2^2} - \frac{1}{\omega^2-\omega_1^2} \end{bmatrix}\cos\omega t \tag{2.31}$$

where we have suppressed the homogeneous (or complementary) part of the solution. We note that the particular solution becomes unbounded as the forcing frequency approaches either $\omega = \omega_1$ or $\omega = \omega_2$. In other words, *resonance* occurs when we force the two degrees-of-freedom system at one of the normal modes' natural frequencies. (Obviously the oscillations you will observe in the lab will not be unbounded as the lab set-up is not entirely frictionless.)

We now rewrite the particular solution as

$$\underline{\mathbf{x_p}}\,(t) = \frac{F}{2}\begin{bmatrix} \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 1} \\ -\frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 3} - \frac{1}{\left(\frac{\omega}{\omega_1}\right)^2 - 1} \end{bmatrix}\cos\omega t \tag{2.32}$$

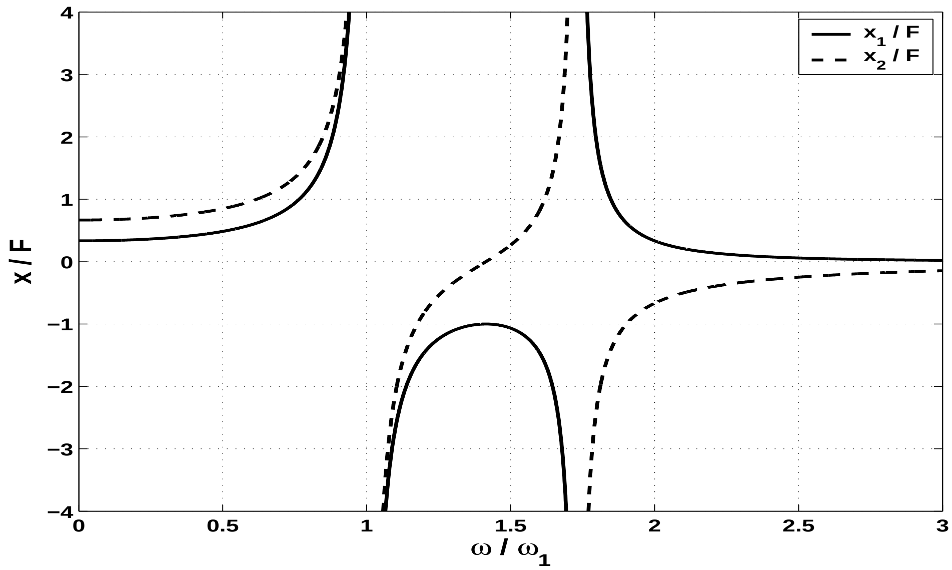


Figure 2.5: Plot of the response amplitude to forcing amplitude ratio for the forced two degrees-of-freedom system.

where we have written it in terms of the ratio of the forcing frequency to the smaller normal mode frequency ω_1. Figure 2.5 graphically shows how the amplitudes of the particular (or steady-state) solutions change as the forcing frequency ω is varied.

The plot graphically illustrates what we found earlier – that when the forcing frequency is near the natural frequency of a normal mode, that mode resonates. As $\omega \rightarrow \omega_1$ the two masses move in-phase and when $\omega \rightarrow \omega_2$ the masses move out-of-phase.

Lab #3 - Slider-Crank Lab

Last Updated: June 13, 2007

INTRODUCTION

In this laboratory we will investigate the kinematics of some simple mechanisms used to convert rotary motion into oscillating linear motion and vice-versa. The first of these is the slider-crank - a mechanism widely used in engines to convert the linear thrust of the pistons into useful rotary motion. In this lab you will measure the acceleration of the piston of a lawn mower engine at various speeds. The results exemplify a simple relation between speed and acceleration for kinematically restricted motions, which you will discover. An adjustable slider-crank apparatus and a computer simulation will show you some effects of changing the proportions of the slider-crank mechanism on piston velocity and acceleration. Other linkages and cam mechanisms may also be used for linear-rotary motion conversion and some of these will be included in the lab.

Because $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$, knowledge of the acceleration permits analysis of the forces which occur in an engine or other machine. Knowledge of these forces is crucial if one is to choose the right material, proportions, and operating conditions for a new design.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What data will you collect from the lawn-mower engine and what will you simulate on the computer?
2. Which parameter(s) can be varied on the adjustable slider-crank? Which are fixed?
3. Derive the equation relating the piston displacement to the crankshaft speed, ω, time, t, connecting rod length, L, and crank radius R. (Hint: Use basic trigonometry).

SLIDER-CRANK KINEMATICS & INTERNAL COMBUSTION ENGINES

Figure 3.1 shows a sketch of the slider-crank mechanism. The point A is on the piston, line AB (with length L) is the connecting rod, line BC (with length R) is the crank, and point C is on the crankshaft. In an engine, a mixture of gasoline and air in the cylinder is ignited in an exothermic (heat producing) reaction. As a result, the pressure in the cylinder rises, forcing the piston out. The force transmitted through the connecting rod has a moment about the center of the crankshaft, causing the shaft to rotate. An exhaust valve releases the gas pressure once the piston is extended. Inertia of machinery (often a flywheel) connected to the crankshaft (as well as forcing from other pistons in multi-cylinder engines) forces the piston back up the cylinder. In a standard "four-cycle" engine the crankshaft makes another full revolution before another ignition (to bring in fresh air and compress it before ignition).

Though the piston is only being forced one fourth of the time, the crankshaft rotates at a more or less constant rate.

In this experiment the crankshaft is driven by an electric motor. The piston is driven by this crankshaft rotation at a more or less constant rate. The same motion results as when the combustion process takes place. As the crankshaft rotates the piston moves in the positive and negative x direction. The basic measurements in this lab are the position and velocity of the piston in the x direction (which happens to be vertical in the laboratory). These measurements can be compared to those calculated by hand (if you are energetic) or to the results of a computer simulation. The simulation and the adjustable crank will allow you to see some of the effects of varying the ratio of connecting rod length L to crank length R.

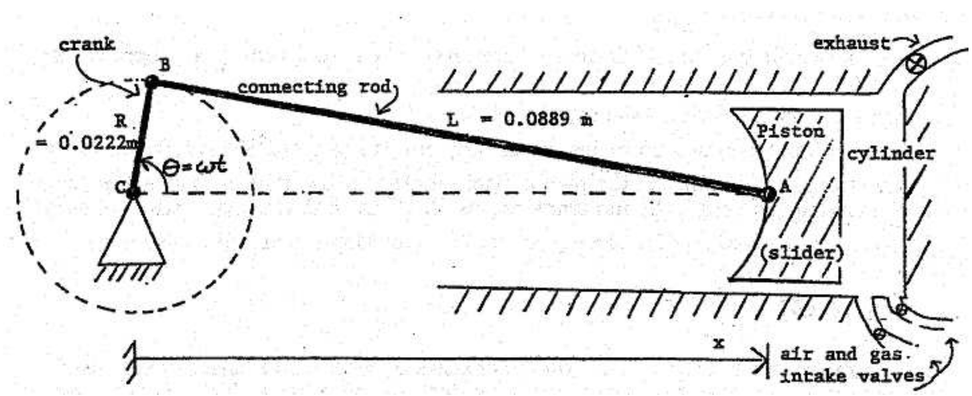


Figure 3.1: A diagram of the slider-crank system.

LABORATORY SET-UP

A stripped-down lawn mower engine is driven by a variable-speed electric motor. Sensors are installed on the engine's piston to measure displacement and velocity. A data acquisition program is used to measure, analyze, and record the piston data. Look at the engine and see how its various parts fit together. It may help to look at Figure 3.1 and at the various demonstration slider-cranks present in the dynamics laboratory. Identify the piston, connecting rod, and crankshaft (the connecting rod won't be visible at your lab set-up, but you can see it in the demonstration slider-cranks). The cylinder head has been removed, exposing the top of the piston and allowing sensors to be attached.

The speed and direction of the electric motor are controlled by a knob and switch on the motor controller. The numbers on the speed controller are arbitrary; do not write them down as r.p.m. or radians per second (instead obtain angular velocity information from the data acquisition program). *Does the direction of motor rotation affect the slider-crank kinematics?*

The larger cylindrical metal device attached to the piston is a Linear Variable Differential Transformer (or LVDT) for measuring $x(t)$. An LVDT is an electrical transformer which produces an output voltage proportional to the linear displacement of the ferromagnetic core of the transformer. The LVDT can be used for static as well as dynamic experiments. The smaller cylindrical device attached to the top of the piston measures velocity. The velocity transducer consists of a coil of wire in which a magnetic core moves linearly. The transducer generates a voltage proportional to the velocity of the core.

The displacement and velocity data are collected and plotted on computer using the *LabView* software, which also measures and displays the peak values (see Figure 3.2). Acceleration is calculated by the computer through numerical differentiation of the velocity data. The computer also measures and displays the angular frequency by timing successive crossings of the zero line and converting to radians per second. A simple simulation program lets you compare your data to theoretical values and look at the effects of different slider-crank geometries.

Please follow safety precautions. The electric motor driving the lawn mower engine is powerful enough to cause serious injury if you get in its way. Keep long hair and loose clothing well away from the belt and pulleys at the back of the engine. If you need to touch the pulley, piston, or LVDT for some reason, check first that the electric motor power is off and that the speed control is set to zero. Make sure your lab partner knows what you are doing.

Using the LabView software

1. Open up the *Engrd203Lab* account and then open the folder *Crank* on the desktop. Open the program *Crank*. As soon as the program is running, it will ask you to move the piston to the top of its travel. Press `Ready` after you have done this and wait until the next pop-up comes before moving the piston again. Then once prompted move the piston to the bottom of its travel and press `Ready` again and allow the computer a few seconds to calibrate. This calibration procedure allows the computer to convert the output of the LVDT (in volts) into displacement (in meters). Do this carefully. It may help to rock the pulley back and forth slightly as you try to home in on the highest (or lowest) piston position. If you make a mistake, you can redo the procedure by clicking on the `SET-UP` button. The *Crank* program has a box for the initials of your lab group. Click on the box with the mouse, type, and then press the `Enter` key, not the `Return` key. Your initials will then appear on your plots, making it easier to identify them as they emerge from the laser printer.

2. When the data acquisition "switch" on the screen is turned on, the computer acquires and displays a new set of data every ten seconds or so. Allow ten or twenty seconds for the data plot to stabilize after changing the motor speed. If you have a plot that you want to keep, turn the data acquisition off. Also turn the motor off promptly when you are not acquiring data to save wear and tear on the lab set-ups and on the nerves of other students.

   The legend and scale factors for the plots are displayed in the top right corner. Multiply the y-axis reading (between -1 and 1) by the appropriate scale factor to obtain the actual measured value, in the units given in the legend. For example, if the velocity plot has a y-value of 0.5 at a particular time, and a scale factor of 4 m/s, the measured velocity at that time would then be 2 m/s.

3. **Before printing, check that data acquisition is off. Otherwise, one plot can take 20 minutes or more.** Also, be sure your initials are on the graph so you can distinguish it from other lab groupsŠ graphs. To print, pull down the `File` menu and select `Print`. Each new graph takes a minute or two, so only print one out if you really need it. However, you can get a copy for your lab partner in just a few additional seconds by setting `Number of Copies` equal to two. You can continue working while plots are being printed.

4. The `SAVE` button stores your data on the hard disk, but the file created this way can only be used by the simulation program (*CrankSim2*).

5. To exit from the program, click the "close" box in the top right corner of the window. To leave *LabView* completely, at any time, pull down the `File` menu and select `Quit`. If the program tells you that "Quitting now will stop all active VIs" select `OK`.

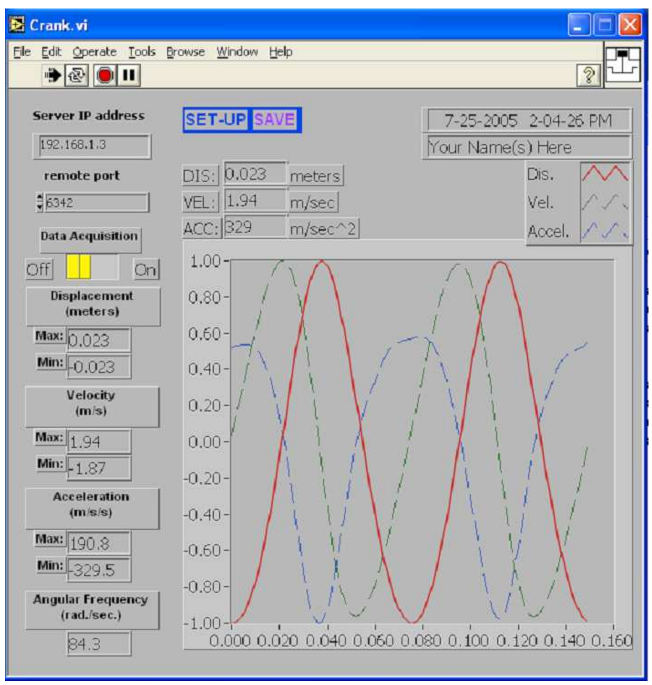


Figure 3.2: Using the *LabView Crank* program.

PROCEDURE

You will record and analyze $x(t)$, $v(t)$, and $a(t)$ while spinning the lawn mower engine at various speeds.

1. Check that the electric motor power switch is off, the speed control knob is at zero, and the data acquisition is on. Twist the pulley back and forth by hand and look at the resulting plot of piston position, velocity, and acceleration. *If the piston moves upwards, in what direction does the plotted curve move?* You will need to wait several seconds for the data to be displayed.

2. Put a penny on top of the piston, turn on the motor, and adjust the motor speed so that the penny just barely starts to bounce on top of the piston. You should be able to hear a faint clinking sound. Wait until you have a good graph of the data and then turn off first the data acquisition and then the motor. Record the angular velocity and the minimum and maximum values for the displacement, the velocity, and the acceleration. Check that the displacement plot makes sense, given that the crank length is known to be 0.0223 m.

3. Remove the penny and repeat the procedure above for at least four additional speeds. Try to get as wide a variety of speeds as possible. At very slow speeds the motor does not turn smoothly and the data is drowned out by noise. When using very high speeds, try to acquire data quickly, turn off the data acquisition "switch", and shut the motor off immediately. Record your data in a table (including the penny data). It is good practice to make at least a rough plot of your data as you go along so you will know what additional data points are needed while you are still in the lab.

You will now simulate the slider-crank mechanism on the computer. The *CrankSim2* program (Figure 3.3) will be used to compare the theoretical values for displacement, velocity, and acceleration with the values measured above. The effects of changing the crank length R, connecting rod length L, and angular velocity ω of the crankshaft may also be observed.

1. To start up the simulation program double-click on *CrankSim2* in the *Crank Lab* folder. If you want to compare your simulation to your most recently saved data, turn the measured-data "switch" on; otherwise, turn it off to eliminate the clutter of all the extra graphs. Described below are the parameters you can change in the simulation:

   - R is the crank length in meters.
   - L is the connecting rod length in meters.
   - ω is the angular velocity in radians per second.

   As with the data acquisition program, the maximum and minimum values are displayed. These are the simulation maxima and minima. Note that the displacement shown is the value x in Figure 3.1 minus the connecting rod length L. This makes it more

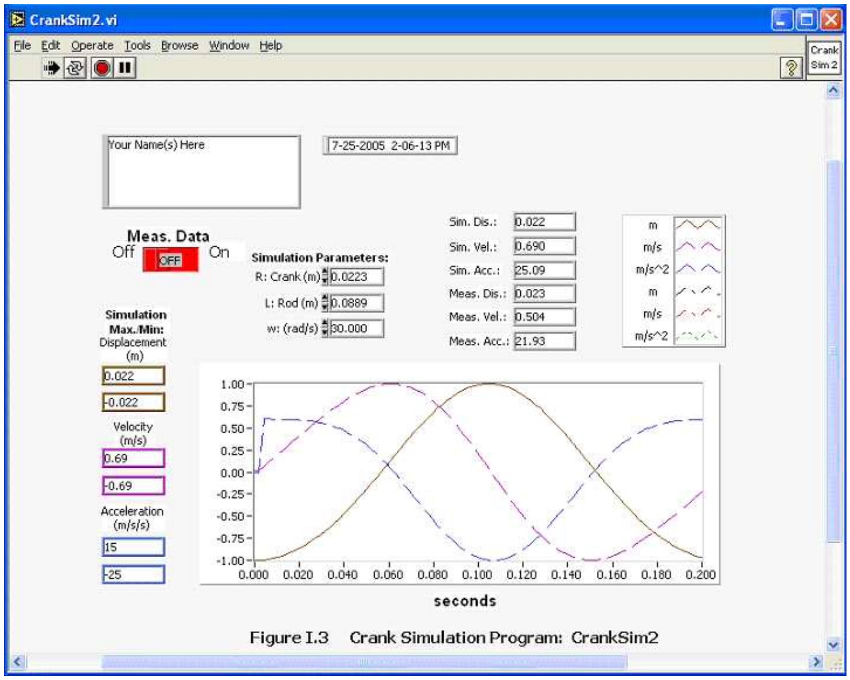


Figure 3.3: Using the *LabView CrankSim2* program.

easily comparable to the measured data. The $x = 0$ point is thus defined to be halfway between the piston's top and bottom positions instead of at the center of the crankshaft.

2. Set up the simulation with the crank length and connecting rod length of the lawn mower engine. Enter the angular velocity used in the data you saved previously and turn the `Measured Data` switch on. Adjust the simulation curve up or down for best alignment and compare the two sets of plots. You will need to print your data.

3. Switch off the measured-data curve. Now simulate slider-cranks with different geometries by varying the crank length R and the connecting rod length L. Observe and record velocities and accelerations when L is much greater than R (e.g., L of 10 m, with R of 0.0223 m); when R is increased, but still much smaller than L (e.g., L of 10 m and R of 0.223 m); when L is decreased, but still much larger than R (e.g., L of 1 m and R of 0.0223 m); when R and L equal the values for the lawn mower engine (L of 0.089 m and R of 0.0223 m); and when L is only slightly greater than R (e.g., L of 0.0224 m and R of 0.0223 m). *What happens physically when R is greater than L?* Make one or two print-outs if necessary to support your observations and conclusions.

Next you will work with the adjustable slider-crank. This device allows you to adjust its ratio from zero to slightly more than one, using an adjustment knob which changes the effective crank length. A handle is located underneath to rotate the apparatus by hand. **Please be gentle with it!** Large forces can be generated with even a small input torque when the ratio is close to 1. If you see things bending, back off. When turning the hand crank, do it slowly. You can also push and pull on the masses at the end of the "piston" to look at the way it converts linear to rotary motion. Be sure you can identify the crank, connecting rod, and piston on the adjustable crank apparatus as first appearances may be misleading. HereŠs a hint: the long thin rod with a weight on each end is the piston. Compare the shapes of the curves you saw in the simulation above to what you observe and feel with the adjustable crank.

The slider-crank is just one of many devices that have been invented to convert linear to rotational motion or vice-versa. The scotch yoke, the cam, and the four-bar linkage are some others.

1. Look over the scotch yoke mechanism, which is driven by an electric motor and gearbox. Try it at different speeds and (with the motor off) push and pull on its various parts. Rotate the pulley by hand while watching the motion of the rod. Take measurements or make a drawing if you wish. Be prepared to find a kinematical equation relating disk rotation to yoke displacement and think about the advantages and disadvantages of the scotch yoke relative to the slider-crank.

2. Cam-and-follower mechanisms are a particularly versatile way to convert rotary to linear motion because you can select the type of motion you want by changing the shape of the cam. For example, cams are used in an internal combustion engine to open and close the intake and exhaust valves. Cam shapes are chosen to optimize fuel economy, power, and emission control. The cam in this lab is a simple eccentric disk. Try out the cam mechanism by turning it with your hand. Feel the output from the follower as the cam is rotated and then try rotating the cam by pushing and pulling on the follower. As with the scotch yoke, be prepared to relate the angle of cam rotation to follower displacement and think about the mechanismŠs advantages and disadvantages.

LAB REPORT QUESTIONS

1. Plot peak piston acceleration vs. crankshaft angular velocity on linear and log-log paper. From these graphs find an appropriate equation relating the two variables. Does this equation make sense? Explain.

2. How does the peak piston velocity depend on the angular velocity of the crankshaft? Plot your experimental data and find an approximate formula relating the two variables. Does this equation make sense? Explain.

3. Examine your plot comparing the measured data and the corresponding simulation data. What explanations can you give of the similarities or differences in the graphs?

4. From your experimental data, what is the crankshaft angular velocity for which an ant standing on the top of the piston would start to need sticky feet in order to not lose contact with the piston? Explain.

5. Using your simulation data, how does the length of the connecting rod, relative to the crank length, affect the shape of the displacement, velocity, and acceleration curves?

6. The lawn mower engine piston weighs 0.175 kg. Using your simulation data, what are the maximum velocity, acceleration, and force on the piston, approximately, for a connecting rod length only slightly longer than the crank length? For a connecting rod length extremely long compared to the crank length? For the connecting rod length actually used in the engine? Use the same crank angular velocity and length in each case.

7. Argue for or against the following points. Back up your arguments with either real or simulated data and/or any other appropriate analysis and logic.

   (a) For all slider-cranks the peak velocity occurs at the midpoint of the stroke.

   (b) There is an optimum $\frac{L}{R}$ ratio for a lawn mower engine (Clearly state what is being optimized).

8. For the scotch yoke, work out the equation relating rotation of the pulley to linear motion of the rod.

9. Why is the slider-crank, and not a scotch yoke, used in an engine? Also, what special advantages does the scotch yoke have in some applications?

10. How does the cam-follower mechanism you saw in lab compare kinematically to the scotch yoke? What reasons might a designer have for choosing one over the other?

CALCULATIONS & NOTES

Lab #4 - Gyroscopic Motion of a Rigid Body

Last Updated: June 13, 2007

INTRODUCTION

Gyroscope is a word used to describe a rigid body, usually with symmetry about an axis, that has a comparatively large angular velocity of spin, $\dot{\psi}$, about its spin axis. Some examples are a flywheel, symmetric top, football, navigational gyroscopes, and the Earth. The gyroscope differs in some significant ways from the linear one and two degrees-of-freedom systems with which you have experimented so far. The governing equations are 3-dimensional equations of motion and thus mathematical analysis of the gyroscope involves use of 3-dimensional geometry. The governing equations for the general motion of a gyroscope are non-linear. Non-linear equations are in general hard (or impossible) to solve. In this laboratory you will experiment with some simple motions of a simple gyroscope. The purpose of the lab is for you to learn the relation between torque, angular momentum, and rate of change of angular momentum. You will learn this relation qualitatively by moving and feeling the gyroscope with your hands and quantitatively by experiments on the precession of the spin axis.

PRELAB QUESTIONS

Read through the laboratory instructions and then answer the following questions:

1. What is a gyroscope?
2. Where is the fixed point of the lab gyroscope?
3. How will moments (torques) be applied to the lab gyroscope?

THE GYROSCOPE

Our experiment uses a rotating sphere mounted on an air bearing (see Figure 4.2) so that the center of the sphere remains fixed in space (at least relative to the laboratory room). This is called a *gyroscope with one fixed point*.

As the gyroscope rotates about its spin axis it is basically stable. That is, the spin axis remains fixed in space and resists any externally applied force that would tend to alter its direction. As you should see in the experiment, the larger the spin rate the larger the moment needed to change the direction of the spin axis. When a moment is applied to a gyroscope, the spin axis will itself rotate about a new axis which is perpendicular to both the spin axis and to the axis of the applied moment. This motion of the spin axis is called *precession*.

DYNAMICS OF THE SYMMETRIC TOP

We will now use 3-dimensional rigid-body dynamics to determine the equations of motion for a symmetric top under the influence of gravity. This is a famous mechanics problem equivalent to our experimental set-up. Our analysis requires us to first define 2 different coordinate frames (see Figure 4.1). The $\left\{\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\right\}$ coordinate system remains fixed in space (in an inertial frame) while the $\{\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{e_3}}\}$ coordinate system is semi-fixed to the rotating rigid-body (in a rotating non-inertial frame). That is it's allowed to only rotate about the $\hat{\mathbf{e_1}}$ and $\hat{\mathbf{e_2}}$ axes (in other words the rotating frame does not spin with the body about its spin axis). Furthermore, the semi-fixed coordinate axis is chosen to be a *principal coordinate axis* of the rigid body. This will simplify our analysis by diagonalizing the inertia tensor. Using

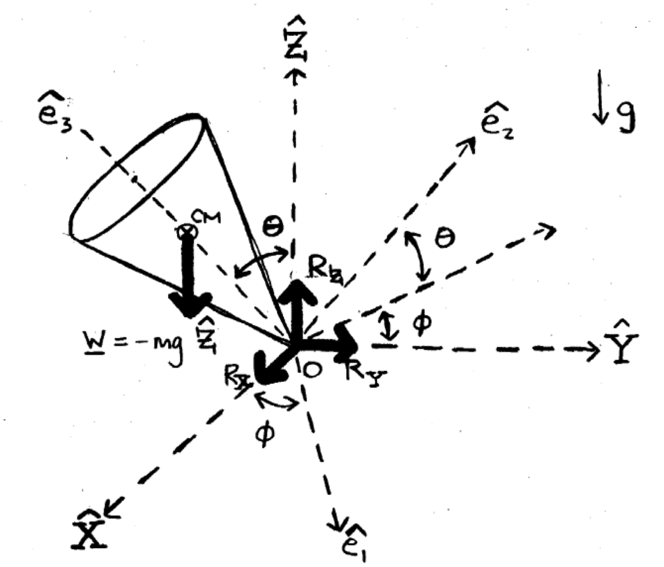


Figure 4.1: A free-body diagram of the symmetric top including both coordinate frames.

the aforementioned coordinate definitions, the frame rotation vector $\underline{\mathbf{\Omega}}$ is

$$\underline{\mathbf{\Omega}} = \dot{\phi}\hat{\mathbf{Z}} + \dot{\theta}\hat{\mathbf{e_1}} = \dot{\theta}\hat{\mathbf{e_1}} + \dot{\phi}\sin\theta\hat{\mathbf{e_2}} + \dot{\phi}\cos\theta\hat{\mathbf{e_3}} \tag{4.1}$$

while the body rotation vector $\underline{\omega}$ is

$$\underline{\omega} = \underline{\mathbf{\Omega}} + \dot{\psi}\hat{\mathbf{e_3}} = \dot{\theta}\hat{\mathbf{e_1}} + \dot{\phi}\sin\theta\hat{\mathbf{e_2}} + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\hat{\mathbf{e_3}} \tag{4.2}$$

The angular momentum of the top about the fixed origin, $\underline{\mathbf{H}}_{\mathbf{o}}$, in the rotating coordinate frame, is

$$\underline{\mathbf{H}}_{\mathbf{o}} = [I_o]\underline{\omega} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = I\omega_1\hat{\mathbf{e_1}} + I\omega_2\hat{\mathbf{e_2}} + I_{zz}\omega_3\hat{\mathbf{e_3}} \tag{4.3}$$

where $I_{xx} = I_{yy} = I$ due to the symmetry of the rigid body. Differentiating with respect to time, we find the time rate of change of the angular momentum to be

$$\dot{\underline{\mathbf{H}}}_{\mathbf{o}} = I\dot{\omega}_1\hat{\mathbf{e_1}} + I\dot{\omega}_2\hat{\mathbf{e_2}} + I_{zz}\dot{\omega}_3\hat{\mathbf{e_3}} + \underline{\mathbf{\Omega}} \times \underline{\mathbf{H}}_{\mathbf{o}} \tag{4.4}$$

where the final term arises due to the use of a rotating coordinate frame. Performing the required vector cross-product we get

$$\underline{\mathbf{\Omega}} \times \underline{\mathbf{H}}_{\mathbf{o}} = \begin{vmatrix} \hat{\mathbf{e_1}} & \hat{\mathbf{e_2}} & \hat{\mathbf{e_3}} \\ \omega_1 & \omega_2 & \Omega_3 \\ I\omega_1 & I\omega_2 & I_{zz}\omega_3 \end{vmatrix} = \left(I_{zz}\omega_2\omega_3 - I\omega_2\Omega_3\right)\hat{\mathbf{e_1}} + \left(I\omega_1\Omega_3 - I_{zz}\omega_1\omega_3\right)\hat{\mathbf{e_2}} + 0\hat{\mathbf{e_3}} \tag{4.5}$$

Using Figure 4.1 we find the total applied torque to be

$$\sum \underline{\mathbf{M}}_{\mathbf{o}} = \underline{\mathbf{r}}_{\mathbf{cm}} \times \underline{\mathbf{W}} = h\hat{\mathbf{e_3}} \times -mg\hat{\mathbf{Z}} = hmg\sin\theta\hat{\mathbf{e_1}} \tag{4.6}$$

We now use angular momentum balance about the fixed origin - $\sum \underline{\mathbf{M}}_{\mathbf{o}} = \dot{\underline{\mathbf{H}}}_{\mathbf{o}}$. Substituting (4.4), (4.5), and (4.6) into the angular momentum balance and "dotting" with all 3 rotating unit vectors, we end up with 3 separate equations:

$$I\dot{\omega_1} + I_{zz}\omega_2\omega_3 - I\omega_2\Omega_3 = hmg\sin\theta \tag{4.7a}$$

$$I\dot{\omega_2} + I\omega_1\Omega_3 - I_{zz}\omega_1\omega_3 = 0 \tag{4.7b}$$

$$I\dot{\omega_3} = 0 \tag{4.7c}$$

Equation (4.7c) says that $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ is constant. Physically, we interpret this as saying the "total spin" of the rigid body about the $\hat{\mathbf{e_3}}$-axis is constant.

We simplify the analysis of the two remaining equations by restricting ourselves to "steady-precession". Steady-precession occurs when we restrict the kinematics to constant spin rate $\dot{\psi}_{\mathbf{o}}$, constant precession $\dot{\phi}_{\mathbf{o}}$, and constant pitch $\theta_{\mathbf{o}}$. With these restrictions, (4.7b) is trivially satisfied and we are left with one equation

$$\dot{\phi}_{\mathbf{o}}\sin\theta_o\left[I_{zz}\left(\dot{\phi}_{\mathbf{o}}\cos\theta_o + \dot{\psi}_{\mathbf{o}}\right) - I\dot{\phi}_{\mathbf{o}}\cos\theta_o\right] = hmg\sin\theta_o \tag{4.8}$$

There are 3 constants in (4.8), two of which can be independently fixed in order to solve for the third. **In this lab you will set the spin rate $\dot{\psi}_{\mathbf{o}}$ and the pitch angle $\theta_{\mathbf{o}}$ and find the resulting precession speed $\dot{\phi}_{\mathbf{o}}$ for several different applied torques.**

Taking a look at the special case of $\theta_o = \frac{\pi}{2}$, equation (4.8) reduces to

$$I_{zz}\dot{\phi}_{\mathbf{o}}\dot{\psi}_{\mathbf{o}} = hmg \tag{4.9}$$

Thus for a gyroscope (or rotor) whose spin axis is orthogonal to the applied torque we find that the product of the moment of inertia, spin rate, and precession rate is equal to the applied torque.

LABORATORY SET-UP

Our lab gyroscope is a steel ball on an air bearing (see Figure 4.2). On one side of the ball a rod is mounted for reference and for touching. This side of the ball has also been bored out so that the rod side is lighter and the center of mass can be adjusted to either side of the center of the sphere by sliding a balance weight in or out. The balance weight is black, with reflective tape, to make rotation rate measurements easier. The sphere is supported in a spherical cup into which high pressure air is supplied so that the sphere is actually supported by a thin layer of air (similar to the air track).

To experimentally measure the spin rate $\dot{\psi}$ of the gyroscope you will use a tachometer (measures in rotations per minute, or rpm). To measure the precession rate $\dot{\phi}$ you will use a stop-watch. Finally, the metric scale will be used to measure the torques you will be applying to the gyroscope.

As a final example of the gyroscopic effect you will play around with a bicycle wheel and rotating platform for hands-on experience and a demonstration of the conservation of angular momentum.

PROCEDURE

1. Turn on the air source.

2. Place the black balance weight on the rod so that if the sphere is released with no spin the rod does not tend to fall down or pop upright from a horizontal position. Note that this is easier said than done, so try to get it as close to motionless as possible. *Where is the center of mass of the system (sphere, rod, and disk) after the gyroscope is balanced? What effect does gravity have on the motion of the balanced gyroscope? If you don't perfectly balance the gyroscope it will result in an error in the calculation of what quantity?*

3. Without spinning the ball, point the rod in some particular direction (up, or towards the door, for example). Carefully release the rod and watch it for several seconds. *Does it keep pointing in the same direction?* Touch the rod lightly with a small strip of paper. *How much force is required to change the orientation of the rod? In which direction does the rod move?* Rotate the table underneath the air bearing. *Does the rod move?*

4. Get the ball spinning and repeat step #4. One good way to do this is to roll the rod between your hands. Stop any wobbling motion by holding the tip lightly and briefly. Avoid touching the ball itself. **Do not allow the rod to touch the base and do not jar the ball while it is spinning.** *What is the effect of spin on the gyroscope motion? Why are navigation gyroscopes set spinning?*

5. While the ball is spinning, apply forces to the end of the rod using one of the pieces of Teflon on a string. The ball should continue to rotate freely as you apply the force because of the low friction of the Teflon. Gently move the end of the rod (keep the rod from touching the bearing cup, or the rod may spin wildly). *What is the relationship between the force you are applying and the velocity of the tip of the rod (estimated magnitude and direction)?* Remember that tension is always in the direction of the string.

6. For a more quantitative look at the motion of a gyroscope:

   (a) Add another weight to the rod so that the gyroscope is no longer balanced. Record its mass and position on the rod for use in calculations later (see Figure 4.2).

   (b) Get the ball spinning, but not wobbling, and point the rod towards one of the three support screws on the air bearing platform. With the rod horizontal, simultaneously release the rod and start the handheld digital stopwatch. The spinning ball and rod will begin to precess in a horizontal plane. Depending upon the precession rate you may want to stop the timer after one full revolution, or after only one-third or two-thirds of a revolution.

(c) Halfway through the timing interval use the optical tachometer to measure the spin rate of the ball (this gives an average). The light beam from the tachometer should be aimed at the reflective tape on the black balance weight. The tachometer measures the rate of the pulses of light returning from the tape, and displays the result in r.p.m. Hold the tachometer at a distance of 10 cm or so. For higher accuracy, try to follow the precession of the rod with the tachometer. This may require practice and patience. If you find it more convenient, measure the spin rate at the start of the precession period and again at the end, and then find the average.

(d) Repeat the procedure for at least two additional spin rates. Try to use a wide range of spin rates; e.g., 200, 400, and 600 r.p.m.

7. Remove the weight and repeat step #6 with at least two more weights for a total of at least three different weights and three different spin rates per weight. The spin rates need not be the same as the ones you used before, but they should cover a similarly wide range of r.p.m.

8. Turn off the air source and clean up your lab station.

9. Hold the bicycle wheel while someone else gets it spinning. Twist it different ways. Hold your hands level and turn your body in a circle. *How do the forces you apply depend on the direction you twist the axle and on the rotation speed and sense?*

10. Repeat #9 while standing on the rotatable platform.

LAB REPORT QUESTIONS

1. Answer all of the questions given in the procedure above using full self-contained sentences.

2. Suppose that the rod on one spinning air gyroscope is pointed north, at an angle of 42.5 degrees from the horizontal (i.e. along the earth's axis of rotation). A second air gyroscope is pointed east, with its rod horizontal. Assume that the ball is perfectly balanced and that air friction is negligible. How does the orientation of each spinning gyroscope change over a period of several hours?

3. Use your recorded data from parts 6 and 7 of the lab procedure for the following questions.

   (a) Plot the precessional period τ vs. the spin rate $\dot{\psi}$ for your different applied torques. Make sure to use a different color and/or symbol for each data point.

   (b) From your plot derive the relationship between the precessional period τ and the spin rate $\dot{\psi}$?

   (c) For a fixed torque show that the product of the precessional rate $\dot{\phi}$ and the spin rate $\dot{\psi}$ is a constant.

   (d) The torque should be proportional to the product of the spin rate and the precession rate. Find the constant of proportionality and plot the relationship between torque and the product of spin rate and precession rate (i.e. M_o vs. $\dot{\psi}\dot{\phi}$).

   (e) You have now found a simple formula relating torque, spin rate and precession rate. What is the meaning of the numerical constant in the formula?

4. Explain in words why when you stand on the platform with a spinning bicycle wheel and proceed to rotate the wheel, the platform begins to rotate.

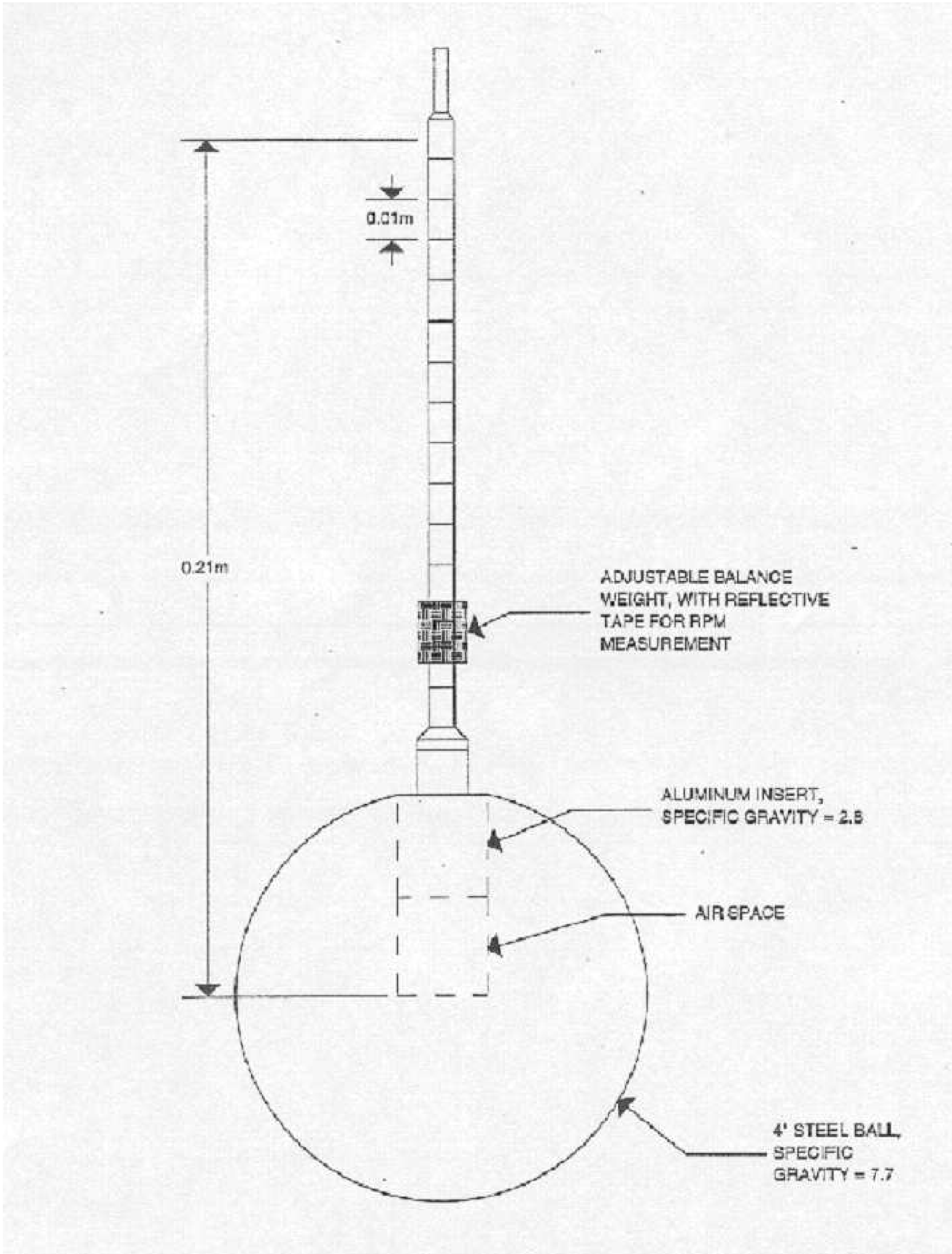


Figure 4.2: A diagram of the lab gyroscope.

CALCULATIONS & NOTES