Your Name: SOLUTIONS

TA's name and Section time: TAs

T&AM 203 Final Exam

Tuesday May 13, 2008

Draft May 13, 2008

5 problems, 25^{+} points each, and 150.0 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are [boxed] in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/25
Problem 4:	/25
Problem 5:	/25

1) (25 points) Due to forcing from a motor (not shown) the support hinge C of a pendulum oscillates horizontally according to $x_C = A\cos(\lambda t)$. The pendulum is a uniform rod with mass m and length ℓ.

a)(20 pts) In terms of some or all of $A, \lambda, g, m, \ell, t, \theta$ and $\dot{\theta}$ find $\ddot{\theta}$.

b) (~~10~~ 5 pts) Assuming small angles, find the steady-state amplitude of forced oscillation (neglect the terms usually neglected in such situations).

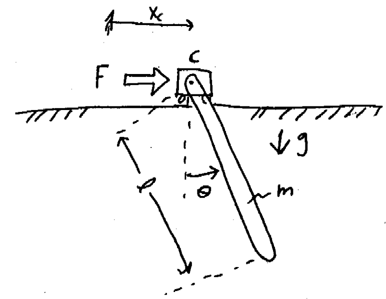


Figure 0.1: Pendulum with moving support.

Filename:Pendulum

A.)

AMB/C

$$\sum \underline{M}_C = \dot{\underline{H}}_C$$

$$\sum \underline{M}_C = \dot{\underline{H}}_G + \underline{r}_{G/C} \times m\underline{a}_G$$

$\dot{\underline{H}}_G = I_G \ddot{\theta}\,\hat{k}$, with $I_G = \frac{1}{12}m\ell^2$

$$\frac{\ell}{2}\hat{e}_r \times -mg\,\hat{\imath}$$

$$\downarrow$$

$$-\frac{\ell mg}{2}(\hat{e}_r \times \hat{\imath}) \quad (\hat{e}_r \times \hat{\imath} = -\sin\theta\,\hat{k})$$

$$\downarrow$$

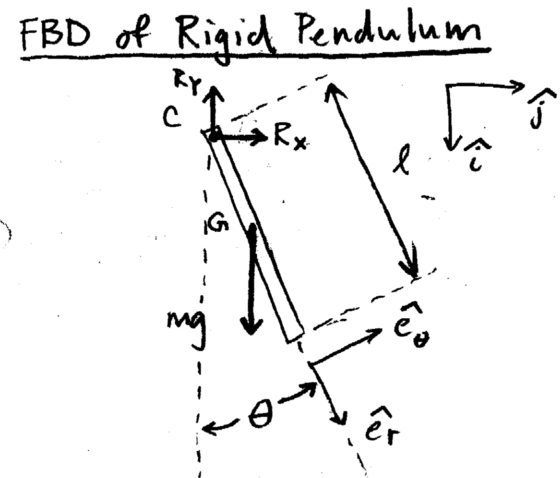
$$-\frac{\ell mg}{2}\sin\theta\,\hat{k}$$

$$\frac{\ell}{2}\hat{e}_r \times m\left[\underline{a}_C + \underline{a}_{G/C}\right]$$

$\underline{a}_C = -A\lambda^2\cos\lambda t\,\hat{\jmath}$, $\quad \underline{a}_{G/C} = -\frac{\ell}{2}\dot{\theta}^2\hat{e}_r + \frac{\ell}{2}\ddot{\theta}\,\hat{e}_\theta$

$$\downarrow$$

$$-\frac{\ell}{2}mA\lambda^2\cos\lambda t(\hat{e}_r \times \hat{\jmath}) + \frac{m\ell^2}{4}\ddot{\theta}\,\hat{k} \quad (\hat{e}_r \times \hat{\jmath} = \cos\theta\,\hat{k})$$



$$-\frac{\ell}{2}mg\sin\theta\,\hat{k} = \frac{1}{12}m\ell^2\ddot{\theta}\,\hat{k} - \frac{\ell}{2}mA\lambda^2\cos\lambda t\cos\theta\,\hat{k} + \frac{1}{4}m\ell^2\ddot{\theta}\,\hat{k}$$

$\{\cdot\}\cdot\hat{k}$ & solve for $\ddot{\theta}$

$$\longrightarrow \boxed{\ddot{\theta} = -\frac{3g}{2\ell}\sin\theta + \frac{3A\lambda^2}{2\ell}\cos\lambda t\cos\theta}$$

B.) Assume $\theta \ll 1 \Rightarrow \sin\theta \approx \theta$ & $\cos\theta \approx 1$

$$\Rightarrow \quad \ddot{\theta} + \frac{3g}{2l}\theta = \frac{3A\lambda^2}{2l}\cos\lambda t$$

The solution is of the for

$$\theta(t) = C_1\cos\sqrt{\frac{3g}{2l}}\,t + C_2\sin\sqrt{\frac{3g}{2l}}\,t + x_p(t)$$

Assume $x_p(t) = C_3\cos\lambda t$. Then we have

$$-C_3\lambda^2\cos\lambda t + \frac{3g}{2l}C_3\cos\lambda t = \frac{3A\lambda^2}{2l}\cos\lambda t$$

$$\longrightarrow \quad -C_3\lambda^2 + \frac{3g}{2l}C_3 = \frac{3A\lambda^2}{2l} \longrightarrow \boxed{C_3 = \frac{\frac{3A\lambda^2}{2l}}{\frac{3g}{2l} - \lambda^2}}$$

Assuming $\lambda \neq \sqrt{\frac{3g}{2l}}$ & there is no resonance!

Assuming a "real" engineering system then there will exist (at least) a small amount of damping that will damp-out the homogeneous solution (think about Labs 1 & 2). Assuming this damping is not enough to seriously affect our answer $x_p(t)$ then the steady-state amplitude of forced oscillation is

$$\boxed{C_3 = \frac{3A\lambda^2/2l}{\frac{3g}{2l} - \lambda^2}}$$

2) (25 points) Two equal masses are held in place by three equal springs. They can only move horizontally.

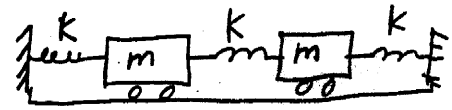


Figure 0.2: Two masses and three springs.

Filename:two masses

a) By any means you like find as many normal modes of this system as you can. Describe these normal modes in any precise way you like with words or equations (using variables you define).

b) For each of these modes find the period of oscillation.

only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\text{external}} = \sum a_i m_i = m_{\text{tot}} a_{\text{CM}}. \tag{9.56}$$

So the center-of-mass of a system (a system that may be deforming wildly) obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in 1,2, or 3 dimensions.

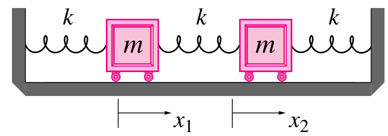


Figure 9.48: A two mass system. We define x_1 and x_2 so that the system is in equilibrium when $x_1 = x_2 = 0$.

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Normal modes

Systems with many moving parts often move in complicated ways. Consider the two mass system shown in *Fig.* 9.48. By drawing free body diagrams and writing linear momentum balance for the two masses we can write the equations of motion in matrix form (see *eqn.* (9.53)) as

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0$$

where

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

Example: **Complicated motion.**
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in *Fig.* 9.49a. Both masses move in a complicated way and not synchronously with each other.

On the other hand, all such systems, if started in just the right way, will move in a simple way.

Example: **Simple motion: a normal mode.**
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in *Fig.* 9.49b. Both masses move in a simple sine wave, synchronously and *in phase* with each other.

That this system has this simple motion is intuitively apparent. If both of the equal masses are displaced equal amounts both have the same restoring force. So both move equal amounts in the ensuing motion. And nothing disturbs this symmetry as time progresses. In fact the frequency of vibration is exactly that of a single spring and mass (with the same k and m).

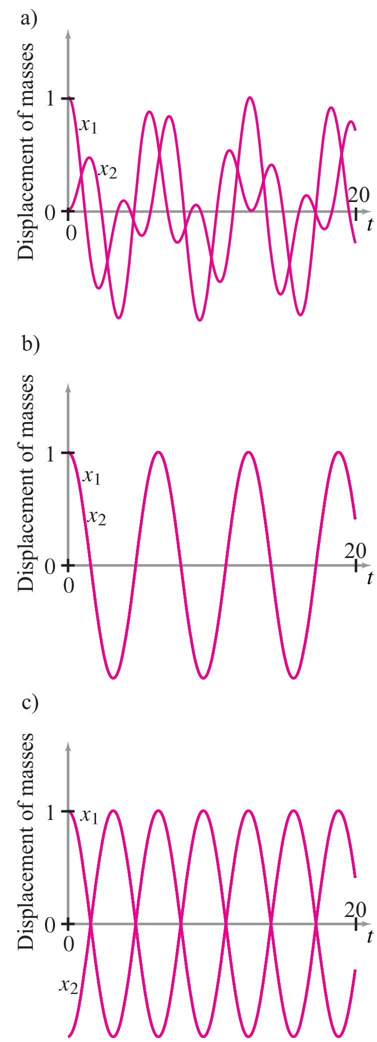

Figure 9.49: Motions of the masses from *Fig.* 9.48 for three different initial conditions, all released from rest ($v_1 = v_2 = 0$)
a) $x_1 = 1,\ x_2 = \ \ 0$,
b) $x_1 = 1,\ x_2 = \ \ 1$, and
c) $x_1 = 1,\ x_2 = -1$.

Filename:tfig-simplenormalmode

A given system can have more than one such simple motion.

Example: **Another normal mode.**
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in *Fig.* 9.49c. Both masses move in a simple sine wave, synchronously and exactly *out of phase* with each other. Being exactly out of phase is actually a form of being exactly in phase, but with a negative amplitude.

This motion is also intuitive. Each mass has restoring force of $3k\Delta x$. One k from a spring at the end and $2k$ because each mass experiences a spring with half the length (and thus twice the stiffness) in the middle (because the middle of the middle spring doesn't move in this symmetric motion).

The system above is about the simplest for demonstration of *normal mode* vibrations. But more complicated elastic systems always have such simple normal mode vibrations.

All elastic systems with mass have *normal mode* vibrations in which all masses

- have simple harmonic motion
- with the same frequency as all the other masses, and
- exactly in (or out) of phase with all of the other masses

Thus the first and second normal modes from *Fig.* 9.49b,c can be written as

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_1 t \\ \cos\lambda_1 t \end{bmatrix}}_{\text{First normal mode}} \qquad \text{and} \qquad \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_2 t \\ -\cos\lambda_2 t \end{bmatrix}}_{\text{Second normal mode}}$$

where, by the physical reasoning in the examples we know that $\lambda_1 = \sqrt{k/m}$ and $\lambda_2 = \sqrt{3k/m}$. We could equally well have used the sine function instead of cosine.

Superposition of normal modes

Note that the governing equation (*eqn.* (9.4)) is 'linear' in that the sum of any two solutions is a solution. If we add the two solutions from *Fig.* 9.49b,c we have a solution. And if divide that sum by two we get a solution. And not just any solution, but the solution in *Fig.* 9.49a. The top curve is the sum of the bottom two divided by two (The curves for $x_1(t)$ and $x_2(t)$ need to be added separately).

For more complicated systems it is not so easy to guess the normal modes. Most any initial condition will result in a complicated motion. Nonetheless the concept of normal modes applies to any system governed by the system of equations (*eqn.* (9.4)):

$$[M]\ddot{\mathbf{x}} + [K]\mathbf{x} = 0.$$

Any collection of springs and masses connected any which way has normal mode vibrations. And because elastic solids are the continuum equivalent of a collection of springs and masses, the concept applies to all elastic structures. Here are the basic facts

- An elastic system with n degrees of freedom has n independent normal modes.
- In each normal mode i all the points move with the same angular frequency λ_i and exactly in phase.
- Any motion of the system is a superposition of normal modes (a sum of motions each of which is a normal mode).

Example: **Musical instruments**
The pitch of a bell is determined by that normal mode of the bell that has the lowest natural frequency. Similarly for violin and piano strings, marimba keys, kettle drums and the air-column in a tuba.

A recipe for finding the normal modes of more complex systems is given in box 9.7 on page 494.

Normal modes and single-degree-of-freedom systems

Any complex elastic system has simple normal mode motions. And all motions of the system can be represented as a superposition of normal modes. Hence sometimes we can think of every system as if it *is* a single degree of freedom system. For example, if a complex elastic system is forced, it will resononate if the frequency of forcing matches any of its normal mode (or natural) frequencies.

9.7 *THEORY*
The math of, and how to find, normal modes

Consider a system of n masses and springs whose motions are governed by ***eqn.*** (9.4)

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0,$$

where $\boldsymbol{x} = \boldsymbol{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]'$. For definiteness we are just thinking of masses in a line, but the concepts are actually more general.

The basic approach is, and this the real approach used by the professionals, to *guess* that there are normal mode solutions and see if they exist. A normal mode solution, all masses moving sinusoidally and synchronously, would look like this

$$\boldsymbol{x} = \begin{bmatrix} V_1 \cos\lambda t \\ V_2 \cos\lambda t \\ \vdots \end{bmatrix} = \boldsymbol{V}\cos\lambda t.$$

Upper case bold $\boldsymbol{V}$ (to distinguish it from lower case velocity) is a list of constants $[V_1, V_2, ...]'$. We could have used sin just as well as cos for our guess. Now we plug our guess into the governing equations to see if it is a good guess:

$$\begin{aligned} [M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} &= 0 \\ [M]\frac{d^2}{dt^2}\{\boldsymbol{V}\cos\lambda t\} + [K]\{\boldsymbol{V}\cos\lambda t\} &= 0 \\ -\lambda^2[M]\boldsymbol{V}\cos\lambda t + [K]\boldsymbol{V}\cos\lambda t &= 0 \\ \left\{-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V}\right\}\cos\lambda t &= 0. \end{aligned}$$

This equation has to hold true for all t therefor the constant column vector inside the brackets $\{\}$ must be zero:

$$-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V} = 0$$
$$\left[-\lambda^2[M] + [K]\right]\boldsymbol{V} = 0$$

At this point the reasoning depends on knowing some linear algebra. We'll just pretend that you do. If you don't, trust us and hold on to these facts until you learn better what they are about in a math class. The matrix $[M]$ is invertible, in fact the inverse of $[M]$ is $[M]$ with the diagonal elements replaced by their reciprocals. So we can multiply through by $[M]^{-1}$ to get:

$$[M]^{-1}[K]\boldsymbol{V} = \lambda^2\boldsymbol{V},$$

where we used that $[M]^{-1}[M] = [I] =$ the identity matrix, and that $[I]\boldsymbol{V} = \boldsymbol{V}$. Defining the product $[B] = [M]^{-1}[K]$ and substituting we get the classic eigenvalue problem:

$$[B]\boldsymbol{V} = \lambda^2\boldsymbol{V}. \tag{9.57}$$

There is a lot to know about ***eqn.*** (9.57). Its a famous equation. ***eqn.*** (9.57) says that $\boldsymbol{V}$ is a vector that, when multiplied by $[B]$ gives itself back again, multiplied by a constant. For the special vector $\boldsymbol{V}$, being multiplied by the matrix $[B]$ is equivalent to being multiplied by the scalar λ^2.

Because $[B]$ is positive semi-definite (if you don't know what that means, let it go) and symmetric a bunch of things follow. In particular, Given $[B]$ there are n linear independent and mutually orthogonal *eigen vectors* $\boldsymbol{V}^1, \boldsymbol{V}^2, ...\boldsymbol{V}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. Each eigen vector $\boldsymbol{V}_i$ has an associated eigen value λ_i^2.

In the case of our vibration problem the eigen vectors are called *modes* or *eigen modes* or *mode shapes* or *normal modes.* The word 'normal' is because of modes being 'normal' (orthogonal) to each other.

Recipe for finding normal modes

Given the matrices $[M]$ and $[K]$ proceed as follows.

- Calculate $[B] = [M]^{-1}K$
- Use a math computer program to find the eigenvalues and eigenvectors of $[B]$, call these $\boldsymbol{V}^i$ and λ_i^2. Usually this is a single command, like:

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eig(B)
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- For each i between 1 and n write each normal mode as $\boldsymbol{x}(t) = \boldsymbol{V}\cos\lambda_i t$ or as $\boldsymbol{x}(t) = \boldsymbol{V}\sin\lambda_i t$

For example, if

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

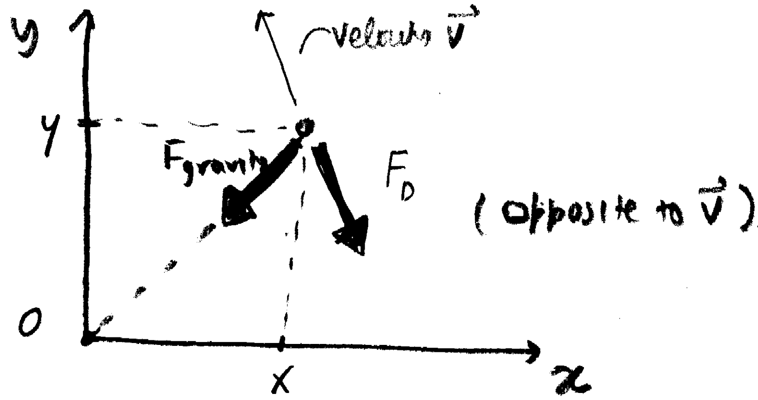
then, for any values of k and m, the computer will return for the eigen values and eigenvectors of $[B] = [M]^{-1}[K]$:

$$\boldsymbol{V}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{with } \lambda_1^2 = k/m \text{ and } \boldsymbol{V}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{with } \lambda_2^2 = 3k/m$$

3) (25 points) 2D problem. A small satellite is moving around the earth. It is not close to the earth's surface so you should use the inverse square law of attraction. There is a small atmospheric drag on the satellite which opposes its motion: $F_d = cv^2$. Assume you are given the gravity constant g, the earth's radius R, the satellite mass m, the position (x, y) and velocity $(\dot{x}, \dot{y})$ and the drag constant c. Assume the origin of the xy coordinate system is at the center of the earth and that the coordinate system may be treated as a Newtonian frame.

Find $\ddot{x}$.

#3



• $F_{gravity}$

NOTE! gravity force = mg when R=r.

$$= -\left(g\frac{R^2 m}{r^2}\right)\frac{\vec{r}}{|\vec{r}|} = -g\frac{R^2 m}{r^3}\vec{r}$$

with $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ $\quad |\vec{r}| = r = \sqrt{x^2+y^2}$

$$= \frac{-gR^2 m}{(x^2+y^2)^{3/2}}(x\hat{\imath} + y\hat{\jmath})$$

• F_{drag} $= -cv^2\frac{\vec{V}}{|\vec{V}|} = -cv\vec{V}$

note $|\vec{v}| \neq \dot{(|\vec{r}|)}$!

with $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ $\quad |\vec{V}| = v = \sqrt{\dot{x}^2+\dot{y}^2}$

$$= -c\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})$$

• By LMB $\quad m\vec{a} = \sum \vec{F}_i$

$$m\ddot{x}\hat{\imath} + m\ddot{y}\hat{\jmath} = -g\frac{R^2 m}{(x^2+y^2)^{3/2}}(x\hat{\imath} + y\hat{\jmath}) - c\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})$$

$\{\ \}\cdot\hat{\imath}$ and dividing by m

$$\boxed{\ddot{x} = -\frac{gR^2 x}{(x^2+y^2)^{3/2}} - \frac{c}{m}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}$$

4) (25 points) A uniform disk with mass m and radius R is released from rest and rolls down a slope γ as accelerated by gravity g.

a) In terms of some or all of the variables given, what is the total angle of rotation of the disk after time t?

b) In terms of some or all of m, R, g, γ and t how big must be the friction coefficient to prevent sliding?

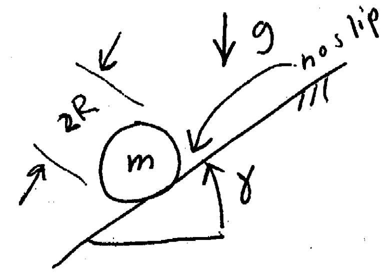


Figure 0.3: Disk rolls down a hill.

Filename:disk

LMB:

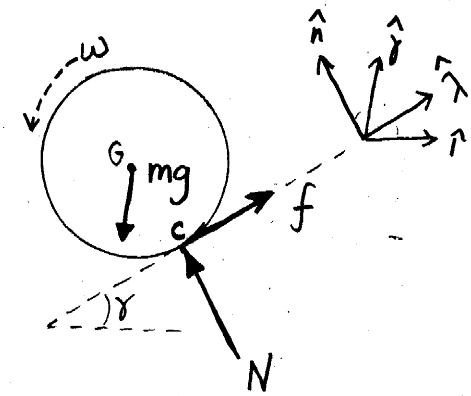
$$N\hat{n} + f\hat{\lambda} - mg\hat{\jmath} = m\vec{a} \quad ①$$

Dot with $\hat{n}$:

$$N - mg(\hat{\jmath}\cdot\hat{n}) = m(\vec{a}\cdot\hat{n}) \quad ②$$

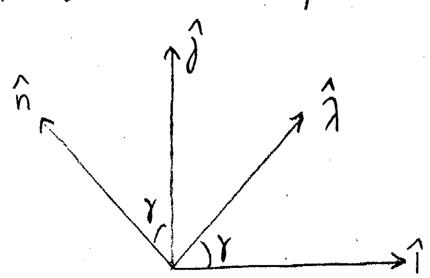
Dot with $\hat{\lambda}$:

$$f - mg\hat{\jmath}\cdot\hat{\lambda} = m(\vec{a}\cdot\hat{\lambda}) \quad ③$$



Take positive ω to be ccw.

Need those dot products:



$\hat{\jmath}\cdot\hat{n} = \cos\gamma$

$\hat{\jmath}\cdot\hat{\lambda} = \sin\gamma$

Define:

$\vec{a}\cdot\hat{n} \equiv a_n$

$\vec{a}\cdot\hat{\lambda} = a_\lambda$

Apply to ②, ③:

$$N - mg\cos\gamma = m a_n^{\nearrow 0} = 0 \quad ④ \quad \text{(no acceleration} \perp \text{ to slope)}$$

$$f - mg\sin\gamma = m a_\lambda \quad ⑤$$

AMB: Take about point C, so we don't need knowledge of $\vec{N}, \vec{f}$

$$\Sigma\vec{M}_c = \dot{\vec{H}}_c = \vec{r}_{G/c}\times m\vec{a}_G + I^G\dot{\omega}\hat{k}$$

The only moment about C is the weight:

$$\Sigma\vec{M}_c = \vec{r}_{G/c}\times(-mg\hat{\jmath})$$

$$= (R\hat{n})\times(-mg\hat{\jmath}) = -Rmg\underbrace{(\hat{n}\times\hat{\jmath})}_{=-\sin\gamma\hat{k}}$$

$$= Rmg\sin\gamma\,\hat{k} \quad ⑥$$

$$\dot{\vec{H}}_c = (R\hat{n})\times(ma_\lambda\hat{\lambda}) + \left(\tfrac{1}{2}mR^2\right)\dot{\omega}\hat{k}$$

$$= Rma_\lambda\underbrace{(\hat{n}\times\hat{\lambda})}_{-\hat{k}} + \tfrac{1}{2}mR^2\dot{\omega}\hat{k}$$

$$= -Rma_\lambda\hat{k} + \tfrac{1}{2}mR^2\dot{\omega}\hat{k} \quad ⑦$$

Now, employ no-slip rolling condition:

$-R\dot{\omega} = a_\lambda$ (note: negative sign because an acceleration in the positive direction $\hat{\lambda}$ means CW rolling, which I defined as negative. A positive rotation (CCW) of the disc implies a linear acceleration down the slope, in the negative $\hat{\lambda}$ direction)

Apply to equation ⑦:

$$\dot{\vec{H}}_c = -Rm(-R\dot{\omega})\hat{k} + \frac{1}{2}mR^2\dot{\omega}\hat{k} = \frac{3}{2}mR^2\dot{\omega}\hat{k} \quad ⑧$$

Equate ⑥, ⑧:

$$Rmg\sin\gamma\,\hat{k} = \frac{3}{2}mR^2\dot{\omega}\hat{k}$$

Dot with $\hat{k}$,

$$Rmg\sin\gamma = \frac{3}{2}mR^2\dot{\omega} \rightarrow \dot{\omega} = \frac{2g\sin\gamma}{3R} \quad \text{(a constant)}$$

$\dot{\omega} = \ddot{\theta} = \text{const} \rightarrow \theta = \frac{\dot{\omega}t^2}{2} + \omega_0 t + \theta_0$ (ω_0 is initial ang. velocity, θ_0 initial ang. position) starts from rest at $\theta_0 = 0$,

$$\boxed{\theta(t) = \frac{g\sin\gamma}{3R}t^2}$$

Note that only AMB was needed. A lot of incorrect solutions used eqns ④, ⑤, setting $f = \mu N = \mu mg\cos\gamma$. This is only true when the disc is on the verge of sliding. Using $f = \mu N$ in general is incorrect.

b) Since we are looking for the μ at which the disc slides, we can use the condition that $f = \mu N = \mu mg\cos\gamma$. Then eqn ⑤ gives:

$$\mu mg\cos\gamma - mg\sin\gamma = ma_\lambda = m(-R\dot{\omega}) = -mR\left(\frac{2g\sin\gamma}{3R}\right) = -\frac{2mg\sin\gamma}{3}$$

$\div mg$

$$\mu\cos\gamma - \sin\gamma = -\frac{2}{3}\sin\gamma \rightarrow \mu\cos\gamma = \frac{1}{3}\sin\gamma \rightarrow \mu = \frac{\tan\gamma}{3} \quad \text{(for verge of sliding)}$$

If $\mu < \frac{\tan\gamma}{3}$, the disc will roll **AND** slide

Thus, $\boxed{\mu \geq \frac{\tan\gamma}{3} \text{ to prevent sliding.}}$

5) (25 points) A car has mass m and moment of inertia I^G about its center of mass. Dimensions are as shown. The suspension is stiff so the car tipping can be neglected. The wheels are light (negligible mass). The rear brake has coefficient of friction μ. Gravity g points down.

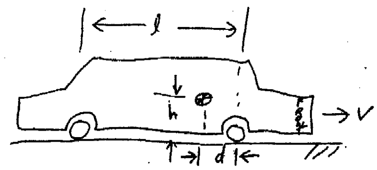


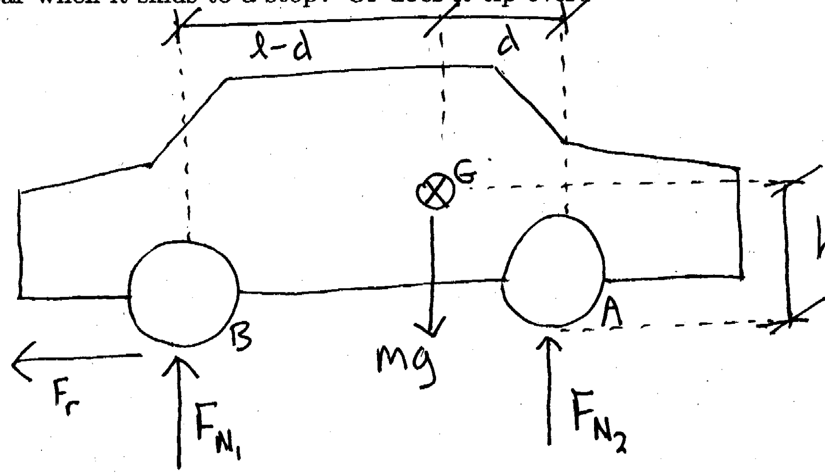
Figure 0.4: Car with rear wheel brakes.

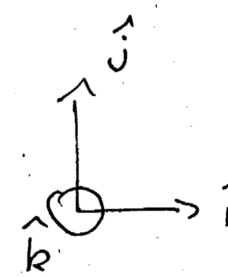
Filename:car

a) (20 pts) In terms of some or all of the variables given find the deceleration of the car when the rear wheel skids.

b) (10 pts) A super new rubber is discovered that has arbitrarily large friction coefficient. In the limit of $\mu \to \infty$ what is the stopping distance of this car when it skids to a stop? Or does it tip over?

a) FBD





Brute Force Method:

Rear Wheel Skid $\rightarrow$ $F_r = \mu F_{N_1}$ (1)

LMB

$\sum \vec{F} = m\vec{a}$

$$F_r(-\hat{i}) + F_{N_1}\hat{j} + F_{N_2}\hat{j} + mg(-\hat{j}) = m\vec{a}_{cm}$$

LMB$\cdot\hat{j}$ $\rightsquigarrow$ $F_{N_1} + F_{N_2} - mg = m a_{cm}^y$; Stiff suspension $\rightarrow a_{cm}^y = 0$;

$\longrightarrow F_{N_1} + F_{N_2} = mg$ (2)

LMB$\cdot\hat{i}$ $\rightsquigarrow$ $-F_r = m a_{cm}^x$; $F_r = \mu F_{N_1}$ from (1)

$\longrightarrow -\mu F_{N_1} = m a_{cm}^x$ (3)

AMB a/b B

$$\sum \vec{M}_{/B} = \dot{\vec{H}}_{/B} = \vec{r}_{cm/B} \times m\vec{a}_{cm} + I\dot{\omega}\hat{k} \; ; \text{ no tipping} \rightsquigarrow \dot{\omega} = 0$$

$$\sum \vec{M}_{/B} = \vec{r}_{cm/B} \times m a_{cm}^{x} \hat{i}$$

$$\{ l \cdot F_{N_2} \hat{k} - mg(l-d)\hat{k} = [(l-d)\hat{i} + h\hat{j}] \times m a_{cm}^{x} \hat{i} \} \cdot \hat{k}$$

$$l F_{N_2} - mgl + mgd = -hma_{cm}^{x} \quad (\underline{\underline{4}})$$

Solving 3 equations, 3 unknowns:

$$F_{N_1} + F_{N_2} = mg \rightarrow F_{N_2} = mg - F_{N_1}$$

$$-\mu F_{N_1} = m a_{cm}^{x} \rightarrow F_{N_1} = \frac{-m}{\mu} a_{cm}^{x}$$

$$-hma_{cm}^{x} = lF_{N_2} - mgl + mgd$$

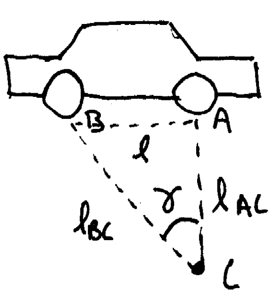
$$\rightarrow -hma_{cm}^{x} = l(mg - F_{N_1}) - mgl + mgd$$

$$-hma_{cm}^{x} = lmg - l\left(\frac{-m}{\mu} a_{cm}^{x}\right) - mgl + mgd$$

$$-hma_{cm}^{x} = \frac{lm}{\mu} a_{cm}^{x} + mgd$$

$$\boxed{a_{cm}^{x} = \frac{-gd}{h + \frac{l}{\mu}}}$$

Quick Solution



AMB a/b C

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C} = \vec{r}_{cm/C} \times m\vec{a}_{cm}$$

$$\{ mgd\hat{k} = (l_{AC} + h) m a_{cm}^{x} (-\hat{k}) \} \cdot \hat{k}$$

$$mgd = -m a_{cm}^{x} \left(h + \frac{l}{\mu}\right)$$

$$a_{cm}^{x} = \frac{-gd}{h + \frac{l}{\mu}}$$

$$\frac{F_{N_1}}{\mu F_{N_1}} = \frac{l_{AC}}{l}$$

$$\rightarrow l_{AC} = \frac{l}{\mu}$$

b) Brute Force Method:

$$F_{N_1} = \frac{-m}{\mu} a^x_{cm} \text{ From } (\underline{3}) \longrightarrow F_{N_1} = \frac{mgd}{\mu h + l} \; ; \; F_{N_1} > 0$$

Since $F_{N_1} > 0$, the car does not tip over.

$$\lim_{\mu \to \infty} a^x_{cm} = \lim_{\mu \to \infty} \frac{-gd}{h + \frac{l}{\mu}} = \frac{-gd}{h}$$

$$\rightarrow a^x_{cm} = \frac{-gd}{h}$$

$$V^x_{cm} = \frac{-gd}{h} t + V \; ; \; V^x_{cm} = 0 \rightarrow t = \frac{Vh}{gd}$$

$$X = \frac{-gd}{2h} t^2 + Vt + X_0 \rightarrow \Delta x = \frac{-gd}{2h}\left(\frac{Vh}{gd}\right)^2 + V\left(\frac{Vh}{gd}\right)$$

$$\Delta x = -\frac{1}{2}\frac{V^2 h}{gd} + \frac{V^2 h}{gd}$$

$$\boxed{\Delta x = \frac{1}{2}\frac{V^2 h}{gd}}$$

Quick Solution:

AMB a/b A

$$\sum \vec{M}_{/A} = \dot{\vec{H}}_{/A} = \vec{r}_{cm/A} \times m\vec{a}_{cm}$$

$$\{ -F_{N_1} l \hat{k} + mgd\hat{k} = (-d\hat{i} + h\hat{j}) \times m a^x_{cm} \hat{i} \} \cdot \hat{k}$$

$$mgd - F_{N_1} l = -hm a^x_{cm}$$

$$F_{N_1} = \frac{-hma^x_{cm} - mgd}{-l}$$

$$F_{N_1} = \frac{-hmgd}{l(h + \frac{l}{\mu})} + \frac{mgd}{l}\frac{(h + \frac{l}{\mu})}{(h + \frac{l}{\mu})} = \frac{\frac{mgdl}{\mu}}{l(h + \frac{l}{\mu})} = \frac{mgd}{\mu h + l}$$

→ Found F_{N_1}, the rest follows as in brute Force Method.