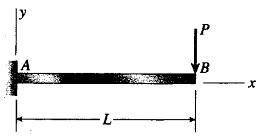
HW 15 Solutions, Prepared by Vijay Muralidharan & Pankaj Porwal.

Due on Dec. 10, 2002.

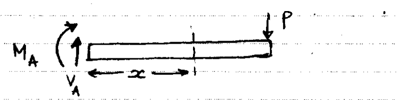
9.3/11

9.3-11 Derive the equation of the deflection curve for a cantilever beam AB supporting a load P at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)

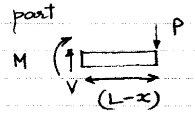


PROB. 9.3-11

Take a section at a distance x



take Right-part



BCs at A:

$v(0) = 0$ (deflection)

$v'(0) = 0$ (angle)

$\Sigma M_x = 0 \Rightarrow M + P(L-x) = 0$

$$\boxed{M = -P(L-x)}$$

we know that

$$EI\,v'' = M = -P(L-x)$$

so $v'' = \dfrac{-P}{EI}(L-x)$

integration $v' = -\dfrac{P}{EI}\left(Lx - \dfrac{x^2}{2}\right) + C_1$

One more integration

$$v = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

$v(0) = 0 \Rightarrow C_2 = 0$

$v'(0) = 0 \Rightarrow C_1 = 0$

so

$$v = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

and $\theta = v' = -\dfrac{P}{EI}\left(Lx - \dfrac{x^2}{2}\right)$

so $\delta_B = v(L) = -\dfrac{P}{EI}\left(\dfrac{L^3}{2} - \dfrac{L^3}{6}\right)$

$$= \frac{-PL^3}{3EI}$$

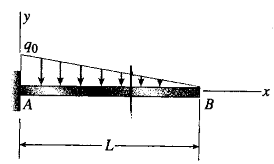
$$\theta_B = v'(L) = \frac{-P}{EI}\left(L^2 - \frac{L^2}{2}\right)$$

$$= \frac{-PL^2}{2EI}$$

θ_B is negative $\Rightarrow$ Clockwise rotation.

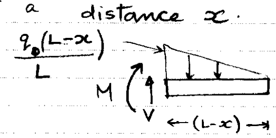
9.3/13

9.3-13 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure. Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



PROB. 9.3-13

Take right part of the FBD with section at a distance x.



$\Sigma M_x = 0 \Rightarrow M + \frac{1}{3}(L-x)\left[\frac{q_0}{2L}(L-x)(L-x)\right] = 0$

$$M = -\frac{q_0}{6L}(L-x)^3$$

$$EI\,v'' = M = -\frac{q_0}{6L}(L-x)^3$$

$$v'' = \frac{-q_0}{6EIL}(L-x)^3 \quad \text{— ①}$$

integrating ①

$$v' = \frac{q_0}{6(EIL)}\,\frac{(L-x)^4}{4} + C_1$$

at $x = 0$ $v'(0) = 0$

$$C_1 = \frac{-q_0}{24EI}L^3$$

$$v' = \frac{q_0}{24EIL}\left((L-x)^4 - L^4\right) \quad \text{— ②}$$

integrating ②

$$v = \frac{q_0}{24EIL}\left[-\frac{(L-x)^5}{5} - L^4 x\right] + C_2$$

at $x = 0$ $v(0) = 0$

$$C_2 = \frac{q_0}{24EIL}\,\frac{L^5}{5} = \frac{q_0}{120EI}L^4$$

$$v = \frac{q_0}{120EIL}\left[L^5 - (L-x)^5 - 5L^4 x\right]$$

at $x = L$

$$v_B = v(L) = \frac{q_0}{120EIL}\left(L^5 - 0 - 5L^5\right)$$

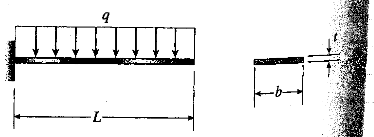
$$\boxed{v_B = -\frac{q_0 L^4}{30EI}}$$

$$\boxed{\theta_B = v'(L) = -\frac{q_0 L^3}{24EI}}$$

θ_B is negative $\Rightarrow$ Clockwise rotation.

9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 25$ μm and rectangular cross section of width $b = 15$ μm and thickness $t = 0.87$ μm. The total load on the beam is 44 μN.

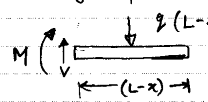
If the deflection at the end of the beam is 1.3 μm, what is the modulus of elasticity E_g of the gold alloy? (*Hint:* Use the formulas of Example 9-2.)



PROB. 9.3-6

consider a section at a distance x

consider right part as FBD



$$\Sigma M_x = 0 \Rightarrow M + \frac{q(L-x)^2}{2} = 0$$

$$\Rightarrow M = -\frac{q}{2}(L-x)^2$$

we know that

$$EI\,v'' = M = -\frac{q}{2}(L-x)^2$$

$$v'' = -\frac{q}{2EI}(L-x)^2 \quad -①$$

Integrating ①

$$v' = \frac{q}{2EI}\frac{(L-x)^3}{3} + c_1$$

at $x=0$ $v'(0)=0$

$$\Rightarrow 0 = \frac{q}{6EI}(L^3) + c_1 = 0$$

$$c_1 = -\frac{qL^3}{6EI}$$

so $$v' = \frac{q}{6EI}\left((L-x)^3 - L^3\right) \quad -②$$

integrating ②

$$v = \frac{q}{6EI}\left(-\frac{(L-x)^4}{4} - L^3x\right) + c_2$$

at $x=0$ $v(0)=0$

$$c_2 = \frac{q}{6EI}L^4$$

$$v = \frac{q}{24EI}\left(L^4 - (L-x)^4 - 4L^3x\right)$$

at $x=L$

$$v(L) = \frac{q}{24EI}\left(L^4 - 4L^4\right)$$

$$v(L) = -\frac{qL^4}{8EI} = -\delta_B$$

$$\Rightarrow E = \frac{qL^4}{8I\delta_B}$$

$$I = \frac{bt^3}{12} = \frac{(15\,\mu m)(0.87\,\mu m)^3}{12}$$

$$= .82\ (\mu m)^4$$

$$E = \frac{(qL)L^3}{8I\delta_B} = \frac{44\,\mu N\,(25\,\mu m)^3}{8 * .82\ (\mu m)^4 * 1.3\,\mu m}$$

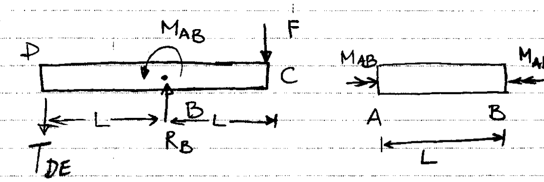
$$= 80.32\ GPa.$$

PROBLEM STATEMENT ON NEXT PAGE.

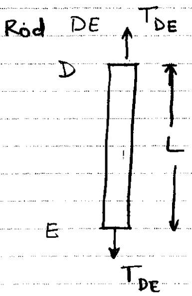
Sythesis HW solution

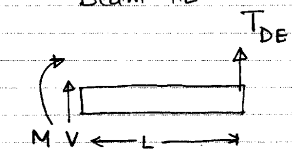
FBD's

Rigid Membes DBC shaft AB

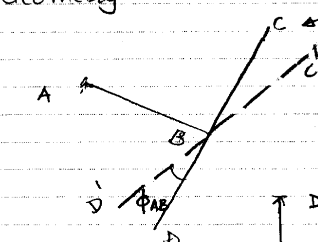


Rod DE Beam HE





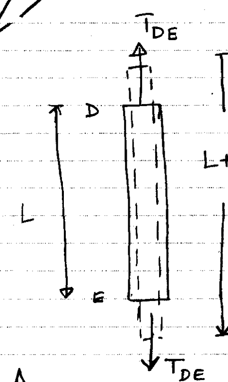
Geometry $\delta_c = cc' = DD'$

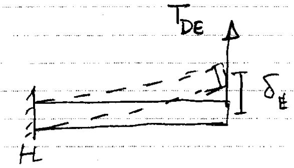


Rotation of DBC due to twist ϕ_{AB}.

$\delta_D = DD'$

extension of DE



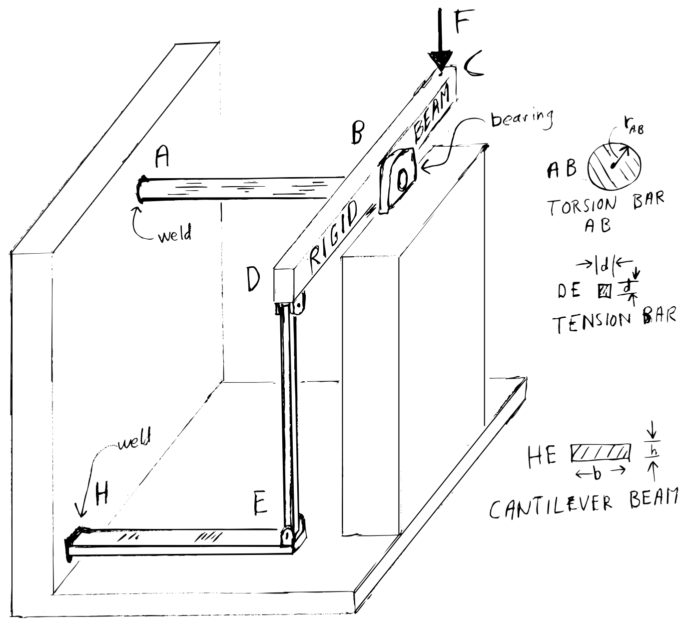


deflection of point E due to bending

T&AM 202 Synthesis HW question

due Tuesday Dec 10, 2002

This version last edited December 5, 2002.



Structure and geometry description:
Torsion bar AB is welded to a rigid support at its left end at A and is supported by a bearing at B. Rigid beam DBC is welded to, and rotates with, the right end of the bar AB. The load F is applied to point C on DBC. The width of this beam can be neglected when considering the length of AB. Tension rod DE hangs from a pin joint at D and pulls up on a pin joint at E. Cantilever beam HE is welded to the wall at H and has its end pulled up by the pin at E.

The load F tries to rotate the beam DBC clockwise (as viewed from the right). This motion is resisted by the torsional stiffness of rod AB and would also be resisted by the bending stiffness of HE but for the compliance of tension rod DE which diminishes this resistance.

Assume linear elastic behavior throughout. The structure is stress-free when there is no load ($F = 0$).

Given:
$L_{AB} = L_{HE} = L_{DB} = L_{BC} = L_{DE} = 0.5\,\mathrm{m}$
$r_{AB} = 2\,\mathrm{cm}$, $G_{AB} = 80$ GPa,
$d = 2\,\mathrm{mm}$, $E_{DE} = 200$ GPa,
$b = 2\,\mathrm{cm}$, $h = .5\,\mathrm{cm}$, $E_{HE} = 200$ GPa
$F = 1000\,\mathrm{N}$.

- **a)** What is the deflection of point C?
- **b)** What is the maximum shear stress on any surface in bar DE?
- **c)** What is the maximum tension stress in bar HE?
- **d)** What is the maximum tension stress on any surface in bar AB?

Note - 1. The member DBC is a rigid members so there will be no deforma-tion of member DBC.

2. Deflection of point D in members CBD will be same as the sum of deflection of HE at E due to bending and extension of members DE due to tension.

So

$$DD' = \phi_{AB} \cdot L \quad (i)$$

[Assuming small displacement. if displacement is large then we can consider

$$\frac{DD'}{L} = \sin(\phi_{AB})$$]

also $DD' = \delta_E + \delta_{DE}$ —(ii)

from (i) and (ii)

$$\phi_{AB} L = \delta_E + \delta_{DE} \quad \text{— } 1$$

strength of material result

A) Twist of AB

$$\phi_{AB} = \frac{M_{AB} L_{AB}}{G_{AB} J_{AB}} \quad \text{— } 2$$

$$J_{AB} = \frac{\pi r_{AB}^4}{2} \; ; \quad L_{AB} = L$$

B) Sketch of DE

$$\delta_{DE} = \frac{T_{DE} L_{DE}}{E_{DE} A_{DE}} \quad \text{— } 3$$

$$A_{DE} = d^2 \; ; \quad L_{DE} = L$$

C) Deflection of point E in beam HE

$$\delta_E = \frac{T_{DE} L_{HE}^3}{3 E_{HE} I_{HE}} \quad \text{— } 4$$

$$I_{HE} = \frac{bh^3}{12} \; ; \quad L_{HE} = L$$

(from problem 9.3/11)

FBDs and Mechanics.

Consider FBD of DBC

$$\Sigma M_B = 0 \Rightarrow M_{AB} + T_{DE} L - F \cdot L = 0 \quad 5$$

From eq 1

$$\phi_{AB} \cdot L = \delta_E + \delta_{DE}$$

use eqⁿ 2 3 and 4

$$\frac{M_{AB} L}{G_{AB} J_{AB}} \cdot L = \frac{T_{DE} L}{E_{DE} d^2} + \frac{T_{DE} L^3}{3 E_{HE} \left(\frac{bh^3}{12}\right)}$$

and from eq 5

$$T_{DE} = (FL - M_{AB}) \frac{1}{L}$$

$$\Rightarrow \frac{M_{AB} L^2}{G_{AB}\left(\frac{\pi r_{AB}^4}{2}\right)} = \left[\frac{L}{E_{DE} d^2} + \frac{L^3}{3 E_{HE}\left(\frac{bh^3}{12}\right)}\right] * (FL - M_{AB}) \frac{1}{L}$$

$$\Rightarrow M_{AB}\left[\frac{L^2}{G_{AB}\left(\frac{\pi r_{AB}^4}{2}\right)} + \frac{1}{L}\left\{\frac{L}{E_{DE} d^2} + \frac{L^3}{3E_{HE}\left(\frac{bh^3}{12}\right)}\right\}\right] = F\left\{\frac{L}{E_{DE} d^2} + \frac{L^3}{3E_{HE}\left(\frac{bh^3}{12}\right)}\right\}$$

$$\Rightarrow M_{AB} = \frac{(1000\,N)\left\{\frac{0.5\,m}{(200\,GPa)(2\,mm)^2} + \frac{(.5\,m)^3}{3\,(200\,GPa)\left(\frac{(2\,cm)(.5\,cm)^3}{12}\right)}\right\}}{\frac{(.5\,m)^2}{(80\,GPa)\left(\frac{\pi}{2}(2\,cm)^4\right)} + \frac{1}{(.5\,m)}\left\{\frac{.5\,m}{(200\,GPa)(2\,mm)^2} + \frac{(.5\,m)^3}{3\,(200\,GPa)\left(\frac{(2\,cm)(.5\,cm)^3}{12}\right)}\right\}}$$

$$\Rightarrow \boxed{M_{AB} \approx 496.91\ N\cdot m}$$

use eqⁿ 5

$$496.91\ Nm - 1000\,N * .5\,m + T_{DE} * .5\,m = 0$$

$$\Rightarrow T_{DE} = \left(\frac{500 - 496.91}{.5}\right) N \approx 6.18\ N$$

a) deflection of pt C

as $L_{DB} = L_{BC}$

$\Rightarrow \delta_C = \delta_D$ (see fig on pg 6)

from eq^n (i)

$$\delta_C = \delta_D = DD' = \phi_{AB} L$$

$$= \frac{M_{AB} L^2}{G_{Al} J_{AB}}.$$

$$= \frac{(496.91 \text{ N·m})(.5\text{ m})^2}{(80 \text{ GPa})\left(\frac{\pi}{2}(2\text{cm})^4\right)}$$

$$= 6.18 \text{ mm}$$

(b) Max shear stress on any surface in DE. DE is under pure tension and the max shear stress will be at an angle 45° from the longitudinal axis and its value will be

$$\tau_{max} = \frac{\sigma_x}{2}$$

$$\sigma_x = \frac{T_{DE}}{A_{DE}}$$

$$\tau_{max} = \frac{1}{2} \frac{6.18 \text{ N}}{(2\times10^{-3}\text{m})^2}$$

$$= 772.25 \text{ KPa}$$

c) Max Tension stress in beam HE (bending of beam solution)

$$\sigma_{max} = \frac{M_{max} y_{max}}{I}$$

$$M_{max} = T_{DE} * L$$

$$y_{max} = \frac{h}{2}$$

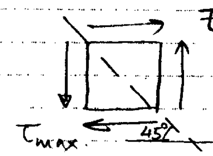
$$\sigma_{max} = \frac{T_{DE} * L \; h}{I * 2}$$

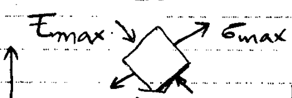
$$= \frac{(6.18 \text{ N})(0.5\text{m})(0.5\text{ cm})}{\frac{(2\text{cm})(.5\text{cm})^3}{12} \cdot 2}$$

$$= 37.068 \text{ MPa}.$$

d) Max tension stress in bar AB.

From torsion theory we know that max Shear stress in a shaft with a circular section occurs at the outer radius. Consider a stress element





and max tension stress will occur at a 45° angle.

$\sigma_{max} = \tau_{max}$ at a 45' section

$$\sigma_{max} = \tau_{max} = \frac{M_{AB}\, r_{AB}}{J} = \frac{2 M_{AB}}{\pi r_{AB}^3}$$

$$= \frac{2(496.91)}{\pi (2\times10^{-2}\text{m})^3}$$

$$= 39.53 \text{ MPa}$$

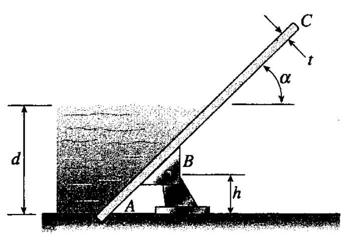
5.6-20

5.6-20 Water pressure acts against an inclined panel *ABC* that serves as a barrier (see figure). The panel is pivoted at point *B*, which is height *h* above the base, and presses against the base at *A* when the water level is not too high (note that the panel will rotate about the pin at *B* if the depth *d* of the water exceeds a certain maximum depth d_{max}). The panel has thickness *t* and is inclined at an angle α to the horizontal. The allowable bending stress in the panel is σ_{allow}.

Derive the following formula for the minimum allowable thickness of the panel:

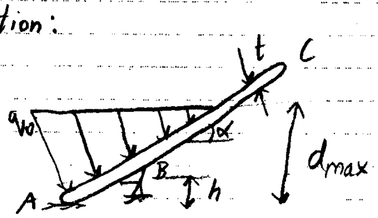
$$t_{min} = \sqrt{\frac{8\gamma h^3}{\sigma_{allow}(\sin^2\alpha)}}$$

(*Note:* To aid in deriving the formula, observe that the maximum stress in the panel occurs when the depth of the water reaches the maximum depth d_{max}. Also, consider only the effects of bending in the panel, disregard the weight of the panel itself, and let γ be the weight density of water.)



PROB. 5.6-20

Solution:



d_{max}: Maximum depth of water

(There is no reaction at end A when the water depth equals d_{max})

b = width of panel perpendicular to plane of figure

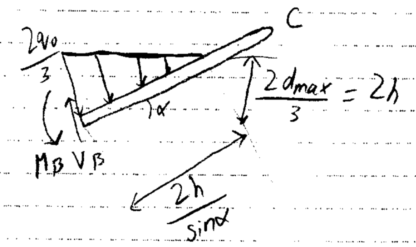
q_0 = Max. intensity of distributed load on panel

$= \gamma d_{max} b$

Resultant of the triangular load acts through the centroid of triangle. This must also act through B.

$$\therefore \boxed{h = \frac{d_{max}}{3}}$$

FBD of part BC of panel:



Maximum bending moment occurs at section B.

$$M_{max} = M_B = \frac{1}{2}\left(\frac{2q_0}{3}\right)\left(\frac{2h}{\sin\alpha}\right) \times \frac{1}{3}\left(\frac{2h}{\sin\alpha}\right)$$

$$= \frac{4q_0 h^2}{9\sin^2\alpha}$$

$$q_0 = \gamma d_{max} b = \gamma(3h)(b)$$

$$\therefore M_{max} = \frac{4\gamma b h^3}{3\sin^2\alpha} \longrightarrow (1)$$

Also, $M_{max} = (\sigma_{allow})\left(\frac{bt^2}{6}\right) \longrightarrow (2)$

From (1) & (2), we get (by equating M_{max})

$$\sigma_{allow}\left(\frac{bt^2}{6}\right) = \frac{4\gamma b h^3}{3\sin^2\alpha}$$

$$\Rightarrow \boxed{t_{min} = \sqrt{\frac{8\gamma h^3}{\sigma_{allow}\sin^2\alpha}}}$$

Hence, the result.