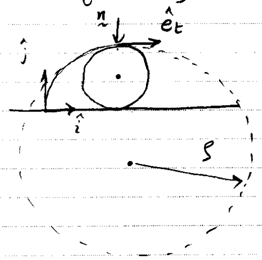
Solutions by Basant Sharma April 29 '02

① (particle on the top of a rolling wheel)



A particle on the top of a rolling wheel (with speed v and radius R) has position given by

$$\underset{\sim}{r}(t) = \left(vt - R\sin\left(\frac{vt}{R}\right)\right)\hat{i} + R\left(1 - \cos\left(\frac{vt}{R}\right)\right)\hat{j}$$

where at $t=0$ we have the origin. When the particle is at its highest point $vt = \pi R$, so

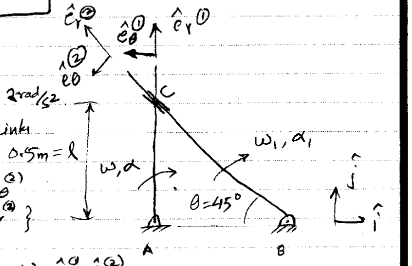
$$\underset{\sim}{v} = 2v\,\hat{i}\ , \quad \underset{\sim}{a} = -\frac{v^2}{R}\hat{j}$$

So $\hat{e}_t = \hat{i}\ ,\ \hat{e}_n = -\hat{j}$

$$\text{So}\quad \rho = \frac{|\underset{\sim}{v}|^2}{|\underset{\sim}{a}_n|} = \frac{(2v)^2}{(v^2/R)} = 4R$$

So $\boxed{\rho_{osculating} = 4R}$

② $\boxed{9.35}$



Given $\omega = 3\,\text{rad/s}$, $\alpha = 2\,\text{rad/s}^2$

Using kinematics of Links

$$\underset{\sim}{v}_C = \underset{\sim}{v}_C$$

$$\text{So}\ \{-l_{AC}\omega\,\hat{e}_\theta^{①} = -l_{BC}\omega_1\hat{e}_\theta^{②} + \dot{l}_{BC}\hat{e}_r^{②}\}$$

$$\{\}\cdot\hat{e}_r^{②} \Rightarrow \dot{l}_{BC} = -l_{AC}\omega\ \hat{e}_\theta^{①}\cdot\hat{e}_r^{②}$$

$$\Leftrightarrow \dot{l}_{BC} = -l_{AC}\omega\cdot\frac{1}{\sqrt{2}} = -\frac{3}{2\sqrt{2}}\,\text{m/s}$$

$$\{\}\cdot\hat{e}_\theta^{②} \Rightarrow -l_{BC}\omega_1 = -l_{AC}\omega\ \hat{e}_\theta^{①}\cdot\hat{e}_\theta^{②}$$

$$\Leftrightarrow \omega_1 = \frac{l_{AC}}{l_{BC}}\omega\cdot\frac{1}{\sqrt{2}} = \frac{\omega}{2} = \frac{3}{2}\,\text{rad/s}$$

$$\underset{\sim}{a}_C = \underset{\sim}{a}_C \Leftrightarrow -l_{AC}(-\omega)^2\hat{e}_r^{①} + l_{AC}(-\alpha)\hat{e}_\theta^{①} = (\ddot{l}_{BC} - l_{BC}(-\omega_1)^2)\hat{e}_r^{②} + (2\dot{l}_{BC}(-\omega_1) + l_{BC}(-\alpha_1))\hat{e}_\theta^{②}$$

$$\Leftrightarrow \{-l_{AC}\omega^2\hat{e}_r^{①} - l_{AC}\alpha\,\hat{e}_\theta^{①} = (\ddot{l}_{BC} - l_{BC}\omega_1^2)\hat{e}_r^{②} - (2\dot{l}_{BC}\omega_1 + l_{BC}\alpha_1)\hat{e}_\theta^{②}\}$$

$$\{\}\cdot\hat{e}_r^{②} \Rightarrow \ddot{l}_{BC} = l_{BC}\omega_1^2 - l_{AC}\omega^2\frac{1}{\sqrt{2}} - l_{AC}\alpha\frac{1}{\sqrt{2}}$$

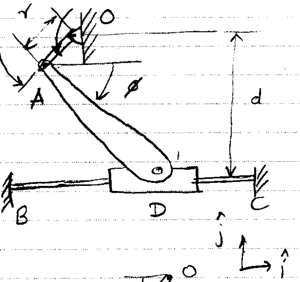
$$= \left(\frac{1}{\sqrt{2}}\frac{9}{4} - \frac{1}{2\sqrt{2}}\cdot 9 - \frac{1}{2\sqrt{2}}\cdot 2\right)\frac{m}{s^2} \approx -2.298\ \text{m/s}^2$$

$$\text{Finally}\ \{\}\cdot\hat{e}_\theta^{②} \Rightarrow l_{BC}\alpha_1 = -2\dot{l}_{BC}\omega_1 + l_{AC}\omega^2\left(\frac{-1}{\sqrt{2}}\right) + l_{AC}\alpha\frac{1}{\sqrt{2}}$$

$$= \left[-2\left(-\frac{3}{2\sqrt{2}}\right)\frac{3}{2} + \frac{1}{2}(3)^2\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\cdot 2\cdot\frac{1}{\sqrt{2}}\right]\frac{m}{s^2}$$

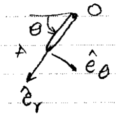
$$\text{So}\quad \alpha_1 = \frac{\sqrt{2}}{2\sqrt{2}}(9 - 9 + 2]\ \text{rad/s}^2\ \boxed{= 1\ \text{rad/s}^2}$$

③ $\boxed{9.41}$



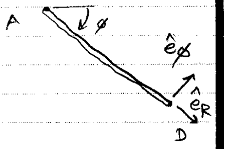
Given $r = 50\,\text{mm}$
$\dot{\theta} = \omega = 30\,\text{rad/s}$
$l_{AD} = 250\,\text{mm}$
$d = 150\,\text{mm}$

Using kinematics of mechanisms,



$$\underset{\sim}{v}_D = \cancelto{0}{\underset{\sim}{v}_O} + \underset{\sim}{v}_{A/O} + \underset{\sim}{v}_{D/A}$$

$$= r\dot{\theta}\,\hat{e}_\theta + l_{AD}(-\dot{\phi})\,\hat{e}_\phi$$



$$\underset{\sim}{a}_D = \cancelto{0}{\underset{\sim}{a}_{/O}} + \underset{\sim}{a}_{A/O} + \underset{\sim}{a}_{D/A}$$

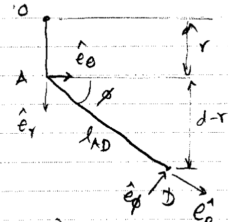
$$= (-r\dot{\theta}^2\hat{e}_r + r\cancelto{0}{\ddot{\theta}}\hat{e}_\theta) + (-l_{AD}(-\dot{\phi})^2\hat{e}_R - l_{AD}\ddot{\phi}\,\hat{e}_\phi)$$

$$\text{So}\quad \underset{\sim}{v}_D = r\dot{\theta}\,\hat{e}_\theta - l_{AD}\dot{\phi}\,\hat{e}_\phi \quad \text{———} (1)$$

$$\underset{\sim}{a}_D = -r\dot{\theta}^2\hat{e}_r - l_{AD}(\dot{\phi}^2\hat{e}_R + \ddot{\phi}\,\hat{e}_\phi) \quad \text{—} (2)$$

at the instant of interest $\theta = \frac{\pi}{2}$

$$\text{So}\quad \phi = \sin^{-1}\left(\frac{d-r}{l_{AD}}\right) \quad \text{———} (3)$$



Due to the constraint of slider at D

$$\underset{\sim}{v}_D\cdot\hat{j} = 0\ ,\quad \underset{\sim}{a}_D\cdot\hat{j} = 0 \quad \text{—} (4)$$

$$\left.\begin{matrix}(4)\\(1)\end{matrix}\right] \Rightarrow r\dot{\theta}\cancelto{0}{(\hat{e}_\theta\cdot\hat{j})} - l_{AD}\dot{\phi}(\hat{e}_\phi\cdot\hat{j}) = 0$$

$$\text{or}\quad \dot{\phi} = 0\ (\text{Since } \cos\phi \neq 0,\ l_{AD} \neq 0)$$

Similarly

$$\left.\begin{matrix}(2)\\(4)\end{matrix}\right] \Rightarrow -r\dot{\theta}^2\hat{e}_r\cdot\hat{j} - l_{AD}(\cancelto{0}{\dot{\phi}^2}\hat{e}_R\cdot\hat{j} + \ddot{\phi}\,\hat{e}_\phi\cdot\hat{j}) = 0$$

$$\text{or}\quad -r\dot{\theta}^2(-1) - l_{AD}(\ddot{\phi}\cos\phi) = 0$$

$$\Rightarrow \quad \ddot{\phi} = \frac{r\dot{\theta}^2}{l_{AD}\cos\phi}$$

$$\text{using (3)}\quad \boxed{\ddot{\phi} = \frac{r\dot{\theta}^2}{\sqrt{(l_{AD}^2 - (d-r)^2)}} \approx 196.4\ \text{rad/s}^2}$$

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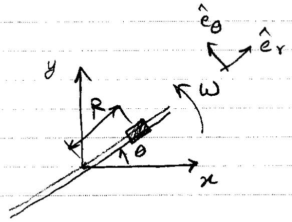
④ $\boxed{10.8}$

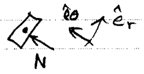


Assume $\omega = \omega_0$ is constant, the bead (mass) is free to slide; at $t=0$ the bead is at 1 ft from the origin and $\frac{dR}{dt} = 0$.

$$\text{a)}\quad \underset{\sim}{v}_{bead} = \dot{R}\hat{e}_r + R\dot{\theta}\hat{e}_\theta$$

$$\underset{\sim}{a}_{bead} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_r + (2\dot{R}\dot{\theta} + R\cancelto{0}{\ddot{\theta}})\hat{e}_\theta$$

FBD of bead



by <u>LMB</u>: $\{m\underset{\sim}{a}_{bead} = N\hat{e}_\theta\}$

$$\text{So}\ \{\}\cdot\hat{e}_r \Rightarrow m\underset{\sim}{a}_{bead}\cdot\hat{e}_r = 0$$

$$\Leftrightarrow \ddot{R} - R\dot{\theta}^2 = 0$$

$$\Leftrightarrow \boxed{\ddot{R} = R\omega_0^2} \quad \text{———} (1)$$

$$\text{b)}\quad \text{Since}\ \dot{\theta} = \omega_0 = \text{const.}$$

$$\theta = \omega_0 t \quad (\text{as at } t=0,\ \theta=0)$$

$$\text{So}\quad \frac{d^2R}{dt^2} = \frac{d^2R}{d\theta^2}\cdot\left(\frac{d\theta}{dt}\right)^2 = \omega_0^2\frac{d^2R}{d\theta^2}$$

$$\text{So}\ (1) \Leftrightarrow \boxed{\frac{d^2R(\theta)}{d\theta^2} = R(\theta)} \quad \text{———} (2)$$

$$\text{Solving (2)}\quad R(\theta) = \frac{1}{2}(e^{\theta} + e^{-\theta})\ \text{ft}$$

$$(\text{Use } R(0) = 1\text{ft},\ \dot{R}(0) = 0)$$

c) after 1 revolution

$$R_1 = R(2\pi) = \frac{1}{2}(e^{2\pi} + e^{-2\pi})\ \text{ft}\ \boxed{\approx 267.7\ \text{ft}}$$

after 2 revolutions

$$R_2 = R(4\pi) = (e^{4\pi} + e^{-4\pi})\ \text{ft}\ \boxed{\approx 1.43\times10^5\ \text{ft}}$$

$$\text{d)}\quad \dot{R}(t) = \frac{dR(\theta)}{d\theta}\cdot\frac{d\theta}{dt} = \omega_0\frac{dR(\theta)}{d\theta} = \frac{\omega_0}{2}(e^{\theta} - e^{-\theta}),\ \text{using}$$

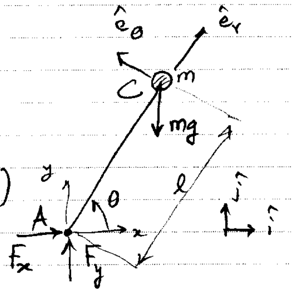
$\omega_0 = 2\pi\,\text{rad/s}$,

$$\therefore v_1 = \dot{R}(1) = \frac{2\pi}{2}(e^{2\pi} - e^{-2\pi})\frac{ft}{s}\ \boxed{\approx 1682.3\ \text{ft/s}}$$

$$\text{e)}\quad KE_{bead} = \frac{1}{2}mv_1^2$$

$$\boxed{\approx (283\times10^6)\times\text{mass}\ (\text{ft/s})^2} \quad [\text{Use } v_1 = \dot{R}(1)]$$

⑤ $\boxed{10.24}$ (Balancing a broom)



Kinematics :

$$\underset{\sim}{a}_C = \underset{\sim}{a}_A + \underset{\sim}{a}_{C/A}$$

$$= a\hat{i} + (-\ell\dot{\theta}^2\hat{e}_r + \ell\ddot{\theta}\hat{e}_\theta)$$

by AMB/A : $\dot{\underset{\sim}{H}}_{/A} = \Sigma \underset{\sim}{M}_{/A}$

$$\Leftrightarrow \ell\hat{e}_r \times m\underset{\sim}{a}_c = -mg\ell\cos\theta\,\hat{k}$$

$$\Leftrightarrow \{\ell m(a\,\hat{e}_r\times\hat{i} + \ell\ddot{\theta}\,\hat{e}_r\times\hat{e}_\theta) = -mg\ell\cos\theta\,\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow m\ell(-a\sin\theta + \ell\ddot{\theta}) = -mg\ell\cos\theta$$

$$\Rightarrow \ddot{\theta} = \frac{1}{\ell}(-g\cos\theta + a\sin\theta)$$

by LMB : $\{m\underset{\sim}{a}_C = F_x\hat{i} + F_y\hat{j} - mg\hat{j}\}$

$$\{\}\cdot\hat{i} \Rightarrow F_x = m\underset{\sim}{a}_c\cdot\hat{i}$$

$$= m(a - \ell\dot{\theta}^2\hat{e}_r\cdot\hat{i} + \ell\ddot{\theta}\hat{e}_\theta\cdot\hat{i})$$

$$= m(a - \ell\dot{\theta}^2\cos\theta + \ell\ddot{\theta}(-\sin\theta))$$

$$= m(a - \ell\dot{\theta}^2\cos\theta + g\sin\theta\cos\theta - a\sin^2\theta)$$

$$\{\}\cdot\hat{j} \Rightarrow F_y = mg + m(-\ell\dot{\theta}^2\hat{e}_r\cdot\hat{j} + \ell\ddot{\theta}\hat{e}_\theta\cdot\hat{j})$$

$$= m(g - \ell\dot{\theta}^2\sin\theta + (-g\cos\theta + a\sin\theta)\cos\theta)$$

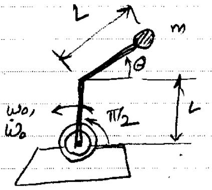
$$= m(g - \ell\dot{\theta}^2\sin\theta - g\cos^2\theta + a\sin\theta\cos\theta)$$

So the force of the hand is

$$\boxed{\begin{aligned}\underset{\sim}{F} &= m(a - \ell\dot{\theta}^2\cos\theta + g\sin\theta\cos\theta - a\sin^2\theta)\,\hat{i}\\ &\quad + m(g - \ell\dot{\theta}^2\sin\theta - g\cos^2\theta + a\sin\theta\cos\theta)\,\hat{j}\end{aligned}}$$

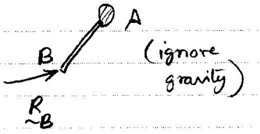
⑥ $\boxed{10.26}$

A motor at O turns at $\omega_0, \dot{\omega}_0$.
At the end of a stick connected to this motor is a frictionless hinge attached to another massless stick.
Both sticks have length L.
At the end of the second stick is a mass m. what is $\ddot{\theta}$?



AMB/B : $\Sigma\underset{\sim}{M}_{/B} = \dot{\underset{\sim}{H}}_{/B}$

FBD of 2nd stick with mass



$$\Sigma\underset{\sim}{M}_{/B} = \underset{\sim}{0}$$

$$\dot{\underset{\sim}{H}}_{/B} = \underset{\sim}{r}_{A/B} \times m\underset{\sim}{a}_A$$

Now using kinematics

$$\underset{\sim}{a}_A = \underset{\sim}{a}_B + \underset{\sim}{a}_{A/B}$$

$$= \underset{\sim}{\omega}_0 \times (\underset{\sim}{\omega}_0 \times \underset{\sim}{r}_{B/O}) + \dot{\underset{\sim}{\omega}}_0 \times \underset{\sim}{r}_{B/O}$$

$$+ \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times \underset{\sim}{r}_{A/B}) + \ddot{\theta}\hat{k} \times \underset{\sim}{r}_{A/B}$$

$$= (-\dot{\omega}_0 L - \dot{\theta}^2 L\cos\theta - \ddot{\theta}L\sin\theta)\,\hat{i}$$
$$+ (-\omega_0^2 L - \dot{\theta}^2 L\sin\theta + \ddot{\theta}L\cos\theta)\,\hat{j}$$

From AMB/B above, $\dot{\underset{\sim}{H}}_{/B} = \underset{\sim}{0}$

$$\Rightarrow \underset{\sim}{r}_{A/B} \times m\underset{\sim}{a}_A = \underset{\sim}{0}$$

$$\Rightarrow mL(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \times [(-\dot{\omega}_0 L - \dot{\theta}^2 L\cos\theta - \ddot{\theta}L\sin\theta)\,\hat{i} + (-\omega_0^2 L - \dot{\theta}^2 L\sin\theta + \ddot{\theta}\cos\theta L)\,\hat{j}] = \underset{\sim}{0}$$

$$\Rightarrow \{mL^2(-\omega_0^2\cos\theta - \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\cos^2\theta + \dot{\omega}_0\sin\theta + \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\sin^2\theta)\,\hat{k} = \underset{\sim}{0}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \omega_0^2\cos\theta - \dot{\omega}_0\sin\theta}$$