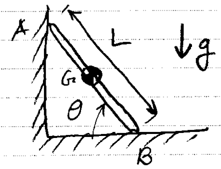
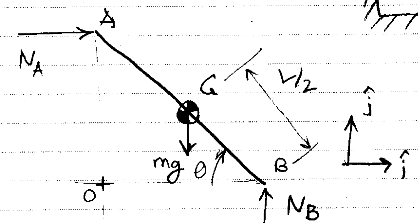
Basant Sharma Spring May 6^{th} '02

(1) $\boxed{10.30}$

Uniform thin rod of mass m rests against a frictionless wall and on a frictionless floor, in the presence of gravity



a) FBD of rod



b) at $t=0$, rod is at rest.

Using Energy balance (since no friction) as total Energy is conserved,

$$E(t) = \left\{\frac{1}{2} m |\underset{\sim}{V}_G|^2 + \frac{1}{2} I_G \omega^2\right\} + m g h_G = \text{Const} \quad (*)$$

Let (x_G, y_G) be the location of G (the center of gravity), then

$$\left.\begin{aligned} x_G &= \frac{L}{2}\cos\theta \\ y_G &= \frac{L}{2}\sin\theta \end{aligned}\right\} (1)$$

So by (1),
$$\left.\begin{aligned} \dot{x}_G &= -\frac{L}{2}\sin\theta\,\dot\theta \\ \dot{y}_G &= \frac{L}{2}\cos\theta\,\dot\theta \end{aligned}\right\} (2)$$

$$(2) \Rightarrow \underset{\sim}{V}_G = \dot{x}_G\hat{i} + \dot{y}_G\hat{j} = \frac{L}{2}\dot\theta(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

So $|\underset{\sim}{V}_G| = \left(\frac{L}{2}\dot\theta\right)$; $\omega = \dot\theta$; $h_G = y_G$

$$(*) \Rightarrow E(t) = \underbrace{\left(\frac{1}{2} m \frac{L^2}{4}\dot\theta^2 + \frac{1}{2} I_G \dot\theta^2\right)}_{\text{Kinetic Energy}} + \underbrace{\left(mg\frac{L}{2}\sin\theta\right)}_{\text{Potential energy}}$$

here $I_G = \frac{mL^2}{12}$

So $E(t) = \frac{mL^2}{6}\dot\theta^2 + mg\frac{L}{2}\sin\theta$

At $t=0$, $\theta(0) = \theta_0$, $\dot\theta(0) = 0$, so

$E(0) = mg\frac{L}{2}\sin\theta_0$

Using $(*)$: $\frac{mL^2}{6}\dot\theta^2 + \frac{mgL}{2}\sin\theta = \frac{mgL}{2}\sin\theta_0$

or $\dot\theta^2 + \frac{3g}{L}(\sin\theta - \sin\theta_0) = 0$ — (3)

Differentiating (3) w.r.t. time t we get the equation of motion (since $\dot\theta \neq 0$)

$$\boxed{\ddot\theta + \frac{3g}{2L}\cos\theta = 0} \quad (4)$$

Note: (4) is EXACTLY the equation of a simple pendulum for $\phi = \theta + \pi/2$

c) $\dot{\underset{\sim}{\omega}}_{AB} = -\ddot\theta\,\hat{k}$

So at $t=0$, $\dot{\underset{\sim}{\omega}}_{AB}\Big|_{t=0} = -\ddot\theta\Big|_{t=0}\hat{k}$
$= \frac{3g}{L}\cos\theta\Big|_{t=0}\hat{k}$ (by (4))

$$\therefore \boxed{\dot{\underset{\sim}{\omega}}_{AB}\Big|_{t=0} = \frac{3g}{L}\cos\theta_0}$$

Now $\underset{\sim}{a}_G = \ddot{x}_G\hat{i} + \ddot{y}_G\hat{j}$

$= \left(-\frac{L}{2}\sin\theta\,\ddot\theta - \frac{L}{2}\cos\theta\,\dot\theta^2\right)\hat{i}$ (Differentiating (2))

$+ \left(\frac{L}{2}\cos\theta\,\ddot\theta - \frac{L}{2}\sin\theta\,\dot\theta^2\right)\hat{j}$

$= \frac{L}{2}\ddot\theta(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$
$-\frac{L}{2}\dot\theta^2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$ — (5)

at $t=0$, $\ddot\theta = -\frac{3g}{L}\cos\theta_0$, $\dot\theta = 0$, so

$$\underset{\sim}{a}_G\Big|_{t=0} = \frac{L}{2}\left(-\frac{3g}{L}\cos\theta_0\right)(-\sin\theta_0\,\hat{i} + \cos\theta_0\,\hat{j})$$

So $\boxed{\underset{\sim}{a}_G\Big|_{t=0} = \frac{3}{2}g\cos\theta_0(\sin\theta_0\,\hat{i} - \cos\theta_0\,\hat{j})}$

d) Using FBD of rod (part (a))

LMB: $\{m\underset{\sim}{a}_G = \Sigma\underset{\sim}{F}\}$

$\{\,\}\cdot\hat{i}$ and (5) $\Rightarrow N_A = m(\underset{\sim}{a}_G\cdot\hat{i}) = m\left(-\frac{L}{2}\sin\theta\,\ddot\theta - \frac{L}{2}\cos\theta\,\dot\theta^2\right)$

Using (3) and (4) we get

$$N_A = m\frac{L}{2}\left(-\sin\theta\left(-\frac{3g}{2L}\cos\theta\right) - \cos\theta\left(-\frac{3g}{L}(\sin\theta - \sin\theta_0)\right)\right)$$
$$= \frac{mg}{2}\left(\frac{3}{2}\sin\theta\cos\theta + 3\cos\theta(\sin\theta - \sin\theta_0)\right)$$

$$\boxed{\therefore N_A = 3\frac{mg}{2}\cos\theta\left(\frac{3}{2}\sin\theta - \sin\theta_0\right)}$$

So at $t=0$, $N_A\Big|_{t=0} = \frac{3mg}{4}\cos\theta_0\sin\theta_0$.

Now $\{\,\}\cdot\hat{j}$ and (5) $\Rightarrow N_B - mg = m(\underset{\sim}{a}_G\cdot\hat{j})$
$= m\frac{L}{2}(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2)$

Using (3) and (4), $N_B = mg + \frac{mL}{2}\left(-\frac{3g}{2L}\cos^2\theta + \frac{3g}{L}\sin\theta(\sin\theta - \sin\theta_0)\right)$

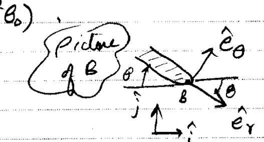
So $N_B = mg + \frac{mg}{2}\left(-\frac{3}{2}\cos^2\theta + 3\sin^2\theta - 3\sin\theta\sin\theta_0\right)$

$$\boxed{N_B = mg\left(1 + \frac{3}{2}\left(-\frac{1}{2} + \frac{3}{2}\sin^2\theta - 3\sin\theta\sin\theta_0\right)\right)}$$

at $t=0$, $N_B\Big|_{t=0} = mg\left(1 + \frac{3}{2}\left(-\frac{1}{2} - \frac{3}{2}\sin^2\theta_0\right)\right)$
$= mg\left(\frac{1}{4} - \frac{9}{4}\sin^2\theta_0\right)$

$\therefore N_B\Big|_{t=0} = \frac{mg}{4}(1 - 9\sin^2\theta_0)$.

e)



$\underset{\sim}{a}_B = \underset{\sim}{a}_G + \underset{\sim}{a}_{B/G}$

$= \underset{\sim}{a}_G + \left(-\frac{L}{2}(-\dot\theta)^2\hat{e}_r + \frac{L}{2}(-\ddot\theta)\hat{e}_\theta\right)$

So $\underset{\sim}{a}_B = \underset{\sim}{a}_G - \frac{L}{2}\dot\theta^2(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) - \frac{L}{2}\ddot\theta(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

Using (5),

$\underset{\sim}{a}_B = \frac{L}{2}\ddot\theta(-\sin\theta\,\hat{i} + \cancel{\cos\theta\,\hat{j}}) - \frac{L}{2}\dot\theta^2(\cos\theta\,\hat{i} + \cancel{\sin\theta\,\hat{j}})$
$- \frac{L}{2}\ddot\theta(\sin\theta\,\hat{i} + \cancel{\cos\theta\,\hat{j}}) - \frac{L}{2}\dot\theta^2(\cos\theta\,\hat{i} - \cancel{\sin\theta\,\hat{j}})$
$= (-L\ddot\theta\sin\theta - L\dot\theta^2\cos\theta)\,\hat{i}$

[Alternatively: $x_B = L\cos\theta$ so $\dot{x}_B = -L\sin\theta\,\dot\theta$

So $\underset{\sim}{a}_B = \ddot{x}_B\,\hat{i} = (-L\cos\theta\,\dot\theta^2 - L\sin\theta\,\ddot\theta)\,\hat{i}$]

Using (3) and (4)

$$\underset{\sim}{a}_B = -L\left(-\frac{3g}{2L}\cos\theta\sin\theta - \frac{3g}{L}(\sin\theta - \sin\theta_0)\cos\theta\right)\hat{i}$$
$$= 3g\left(\frac{1}{2}\cos\theta\sin\theta + \sin\theta\cos\theta - \sin\theta_0\cos\theta\right)\hat{i}$$

$$\boxed{\underset{\sim}{a}_B = 3g\cos\theta\left(\frac{3}{2}\sin\theta - \sin\theta_0\right)\hat{i}}$$

(f) at $\theta = \frac{\theta_0}{2}$, by (3) $\dot\theta^2 = -\frac{3g}{L}\left(\sin\left(\frac{\theta_0}{2}\right) - \sin\theta_0\right)$
$= \frac{3g}{L}\sin\left(\frac{\theta_0}{2}\right)\left(2\cos\left(\frac{\theta_0}{2}\right) - 1\right)$

Since $0 \le \theta_0 \le \frac{\pi}{2}$, $\left(2\cos\left(\frac{\theta_0}{2}\right)-1\right) \ge 0$

So $\dot\theta = \pm\sqrt{\frac{3g}{L}\sin\left(\frac{\theta_0}{2}\right)\left(2\cos\left(\frac{\theta_0}{2}\right)-1\right)}$ $(\because \dot\theta < 0)$

So $$\boxed{\underset{\sim}{\omega}_{AB}\Big|_{\theta=\frac{\theta_0}{2}} = -\sqrt{\frac{3g}{L}\sin\left(\frac{\theta_0}{2}\right)\left(2\cos\left(\frac{\theta_0}{2}\right)-1\right)}\,\hat k}$$

($\because \omega_{AB}$ should be clockwise along $\hat k$)

As for part (e)

$$a_A = (L\sin\theta)^{\cdot\cdot}\,\hat j$$
$$= \left(-L\sin\theta\,\dot\theta^2 + L\cos\theta\,\ddot\theta\right)\hat j$$

Using (3) (4)

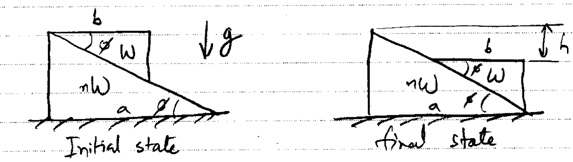
So $$a_A = \left(-L\sin\theta\left(-\frac{3g}{L}\right)(\sin\theta-\sin\theta_0) + L\cos\theta\left(-\frac{3g}{2L}\cos\theta\right)\right)\hat j$$
$$= \left(3g\sin\theta(\sin\theta-\sin\theta_0) - \frac{3}{2}g\cos^2\theta\right)\hat j$$
$$= 3g\left(\sin^2\theta - \sin\theta\sin\theta_0 - \frac12 + \frac12\sin^2\theta\right)\hat j$$

$$\therefore a_A = 3g\left(-\frac12 + \frac32\sin^2\theta - \sin\theta\sin\theta_0\right)\hat j$$

$$\boxed{a_A = 3g\left(-\frac12 + \sin\theta\left(\frac32\sin\theta - \sin\theta_0\right)\right)\hat j}$$

② 10.53

Two frictionless prisms problem! see below.



to find: final velocities!

Since all contacts are frictionless and the gravitational force is conservative, the total energy is conserved.
Since initial state is of rest, equivalently the final kinetic energy is the same as the change is potential energy of the system.

Let $\underset{\sim}{v} = v_x\hat i + v_y\hat j$ be the final velocity of upper block

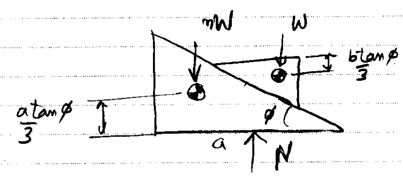
And $\underset{\sim}{u} = u\,\hat i$ be the final velocity of lower block. (Clearly it has no y-component)

So $\frac12\frac{W}{g}|\underset{\sim}{v}|^2 + \frac12\frac{nW}{g}|\underset{\sim}{u}|^2 = Wh$

here $h = (a-b)\tan\phi$

So $v_x^2+v_y^2+n\,u^2 = 2g(a-b)\tan\phi$ —— ①

FBD of blocks



So by LMB: $\left\{\left(\frac{W+nW}{g}\right)\underset{\sim}{a}_{cm} = \Sigma\underset{\sim}{F}\right\}$

$\{\}\cdot\hat i \Rightarrow a_{cm}\cdot\hat i = 0 \Rightarrow v_{cm}\cdot\hat i = $ Constant

at $t=0$, $v_{cm}=0$ So $v_{cm}\cdot\hat i \equiv 0$ —— ②

Using the definition of center of mass and linear momentum

$$\left\{\frac{W+nW}{g}(v_{cm})_{final} = \frac{W}{g}(v_x\hat i + v_y\hat j) + \frac{nW}{g}u\,\hat i\right\}$$

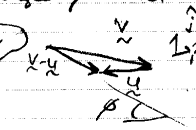
Since the COM fall vertically under gravity by (2),

$\{\}\cdot\hat i \Rightarrow v_x + un = 0$ or $\boxed{u = -\frac{v_x}{n}}$ —— (3)

If we now consider the relative velocity between the blocks then we know that it should be "parallel to angle ϕ" ie,

$\frac{v_y}{v_x - u} = \tan\phi$

Picture



Using (3) $\frac{v_y}{v_x(1+\frac1n)} = \tan\phi$

So $\boxed{v_y = v_x\left(1+\frac1n\right)\tan\phi}$ —— (4)

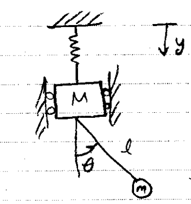
Solving (1), (3), (4) one can obtain the explicit solution. summarizing

$$\boxed{\begin{aligned} u &= -\frac{v_x}{n} \\ v_y &= v_x\left(1+\frac1n\right)\tan\phi \\ v_x &= \sqrt{\frac{2g(a-b)\tan\phi}{\left(1+\frac1n+\left(1+\frac1n\right)^2\tan^2\phi\right)}} \end{aligned}}$$

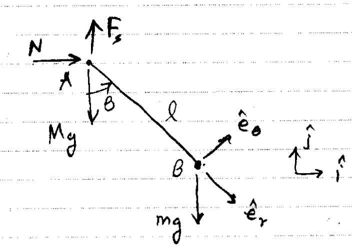
③ 10.60

find the differential equation governing the angle θ and y of the system shown.

Let the unstretched length of the spring be y_0.



FBD of the pendulum and mass



AMB/A : $\dot{\underset{\sim}{H}}_{/A} = \Sigma \underset{\sim}{M}_{/A}$

$\Rightarrow \{ l\hat{e}_r \times m\underset{\sim}{a}_B = -mgl\sin\theta\,\hat{k} \}$

Now $\underset{\sim}{a}_B = (-l\dot\theta^2\hat{e}_r + l\ddot\theta\hat{e}_\theta) + \underset{\sim}{a}_A$
$\quad = -l\dot\theta^2\hat{e}_r + l\ddot\theta\hat{e}_\theta - \ddot{y}\hat{j}$

So $\{\}\cdot\hat{k} \Rightarrow ml^2\ddot\theta - ml\ddot{y}(\hat{e}_r\times\hat{j})\cdot\hat{k} = -mgl\sin\theta$

$\Leftrightarrow ml^2\ddot\theta - ml\ddot{y}\sin\theta = -mgl\sin\theta$

So $\boxed{\ddot\theta + \frac{1}{l}(g-\ddot{y})\sin\theta = 0}$ —— (1)

LMB: $\{ N\hat{i} + (F_s - Mg - mg)\hat{j} = -(M+m)\ddot{y}\hat{j} + m(-l\dot\theta^2\hat{e}_r + l\ddot\theta\hat{e}_\theta) \}$

$\{\}\cdot\hat{i} \Rightarrow N = -ml\dot\theta^2(\hat{e}_r\cdot\hat{i}) + ml\ddot\theta(\hat{e}_\theta\cdot\hat{i})$
$\quad = -ml\dot\theta^2\sin\theta + ml\ddot\theta\cos\theta$

$\{\}\cdot\hat{j} \Rightarrow F_s - Mg - mg = -(M+m)\ddot{y} - ml\dot\theta^2\cos\theta + ml\ddot\theta\sin\theta$

But $F_s = k(y-y_0)$

Let y_s be such that $k(y_s - y_0) = (M+m)g$.

define $\tilde{y} = y - y_s$, so $\ddot{y} = \ddot{\tilde{y}}$ (for example in (1))

So $k(\tilde{y} + y_s - y_0) - (M+m)g = -(M+m)\ddot{\tilde{y}} - ml\dot\theta^2\cos\theta + ml\ddot\theta\sin\theta$

$\Leftrightarrow k\tilde{y} = -(M+m)\ddot{\tilde{y}} - ml\dot\theta^2\cos\theta + ml\ddot\theta\sin\theta$

Ignore ~ above y so that

$$\boxed{\ddot{y} + \frac{k}{(M+m)}y = \frac{ml}{(M+m)}(-\dot\theta^2\cos\theta + \ddot\theta\sin\theta)} \text{ —— (2)}$$

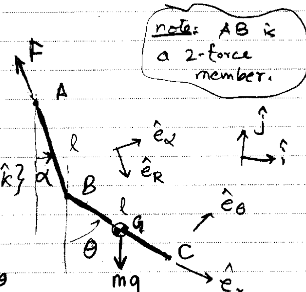
(1) and (2) are the required ODEs for y and θ and can be simplified further.

(4) 10.63

A double pendulum made of two uniform rigid rods of length l each. first rod massless.

find equations of motion of second rod!

FBD of pendulum



AMB/B: $\dot{\underset{\sim}{H}}_{/B} = \Sigma \underset{\sim}{M}_{/B}$

So $\{ \frac{l}{2}\hat{e}_r \times m\underset{\sim}{a}_G = -mg\frac{l}{2}\sin\theta\,\hat{k} \}$

here $\underset{\sim}{a}_G = \underset{\sim}{a}_B + \underset{\sim}{a}_{G/B}$
$\quad = -l\dot\alpha^2\hat{e}_R + l\ddot\alpha\hat{e}_\alpha$
$\quad - l\dot\theta^2\hat{e}_r + l\ddot\theta\hat{e}_\theta$

So $\{\}\cdot\hat{k} \Rightarrow \frac{ml^2}{2}(-\dot\alpha^2(\hat{e}_r\times\hat{e}_R\cdot\hat{k}) + \ddot\alpha(\hat{e}_r\times\hat{e}_\alpha\cdot\hat{k}) + \ddot\theta) = -\frac{mgl}{2}\sin\theta$

$\Rightarrow \boxed{\dot\alpha^2\sin(\theta-\alpha) + \ddot\alpha\cos(\theta-\alpha) + \ddot\theta + \frac{g}{l}\sin\theta = 0}$

LMB: $\{ m\underset{\sim}{a}_G = \Sigma \underset{\sim}{F} \}$

$\{\}\cdot\hat{e}_\alpha \Rightarrow m(l\ddot\alpha - l\dot\theta^2\sin(\theta-\alpha) + l\ddot\theta\cos(\theta-\alpha)) = -mg\hat{j}\cdot\hat{e}_\alpha$

$\Leftrightarrow \boxed{\ddot\alpha - \dot\theta^2\sin(\theta-\alpha) + \ddot\theta\cos(\theta-\alpha) + \frac{g}{l}\sin\alpha = 0}$

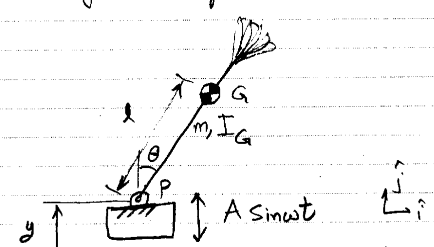
(The equations can be simplified further!)

(5) 10.67

Balancing the broom again! vertical shaking...

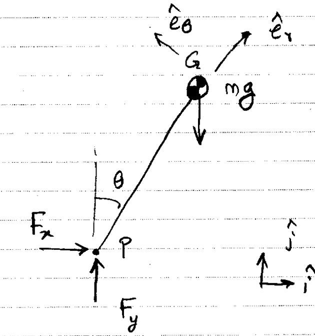
a) Picture and model

Assume the broom has center of mass at a distance l from the support (pin) and the moment of inertia is I_G (if the broom is modeled as uniform rod then $I_G = \frac{ml^2}{3}$). Assume the support at the base is a frictionless pin oscillating sinusoidally vertically.



Kinematics: $\underset{\sim}{a}_P = \ddot{y}\hat{j} = -A\omega^2\sin\omega t\,\hat{j}$

b) FBD



c) AMB/P: $\dot{\underset{\sim}{H}}_{/P} = \Sigma \underset{\sim}{M}_{/P}$

$\Leftrightarrow I_G(-\ddot\theta)\hat{k} + l\hat{e}_r \times m\underset{\sim}{a}_G = -mgl\sin\theta\,\hat{k}$

$\Leftrightarrow -I_G\ddot\theta\hat{k} + ml\,\hat{e}_r\times(\underset{\sim}{a}_P + \underset{\sim}{a}_{G/P}) = -mgl\sin\theta\,\hat{k}$

$\Leftrightarrow \{ -I_G\ddot\theta\hat{k} + ml\,\hat{e}_r\times(-A\omega^2\sin\omega t\,\hat{j} + (-l(-\dot\theta)^2\hat{e}_r + l(-\ddot\theta)\hat{e}_\theta)) = -mgl\sin\theta\,\hat{k} \}$

$\{\}\cdot\hat{k} \Rightarrow -I_G\ddot\theta - ml[A\omega^2\sin\omega t\sin\theta + l\ddot\theta] = -mgl\sin\theta$

$\Leftrightarrow (I_G + ml^2)\ddot\theta + mlA\omega^2\sin\omega t\sin\theta = mgl\sin\theta$ —— (1)

d), e) see part c).

f) writing (1) as system of first order equations define $\Omega = \dot\theta$ so

$$\dot\theta = \Omega$$
$$\dot\Omega = (mgl - mlA\omega^2\sin\omega t)\frac{\sin\theta}{(I_G + ml^2)}$$

Let $\alpha = \frac{I_G}{ml^2} = \frac{k^2}{l^2}$, take l = unit length.

where k is the radius of gyration of broom.

So
$$\boxed{\begin{aligned}\dot\theta &= \Omega \\ \dot\Omega &= (g - A\omega^2\sin\omega t)\frac{\sin\theta}{(\alpha+1)}\end{aligned}}$$

See plots on the next page for various simulations

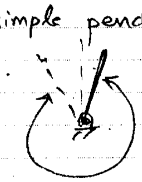
g) see plots. when ω is large, the gravitational forces don't get "enough time" to make broom "fall a lot" so inertial effects due to oscillation of base dominates!

```
global g a A f
g=10; a=1; A=0; f=2*pi*20;
tspan=[0 10];
theta0=(pi/2)/10; thetadot0=0;
z0=[theta0 thetadot0];
options=odeset('AbsTol',1e-4,'RelTol',1e-6);
[t,z]=ode45('balancebroom',tspan,z0,options);
theta=z(:,1);
thetadot=z(:,2);
plot(t,theta/pi)
xlabel('t');
ylabel('theta (in multiples of pi)');
title('Balancing broom problem on HW14: no forcing');
```

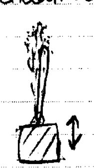
```
function zdot=balancebroom(t,z)
global g a A f
theta=z(1);
omega=z(2);
thetadot=omega;
omegadot=(g-A*f^2*sin(f*t))*sin(theta)/(1+a);
zdot=[thetadot omegadot]';
```

(Plots on next page:

1st: no vertical shaking
⇒ unstable motion
as the rod swings through the "simple pendulum" orientation.

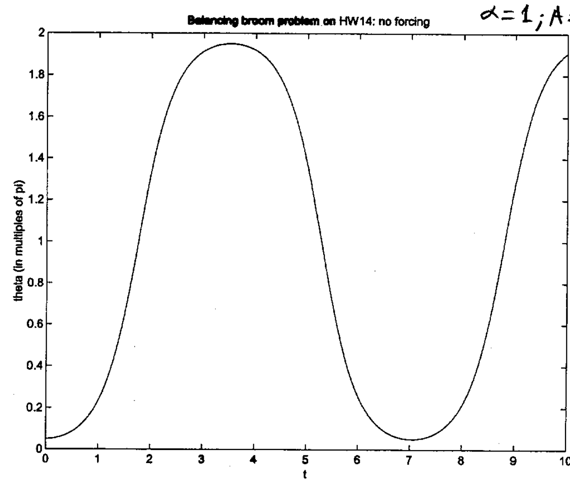


2nd and 3rd: stabilization of the "inverted pendulum" or slightly tilted broom achieved through the oscillation of the base.

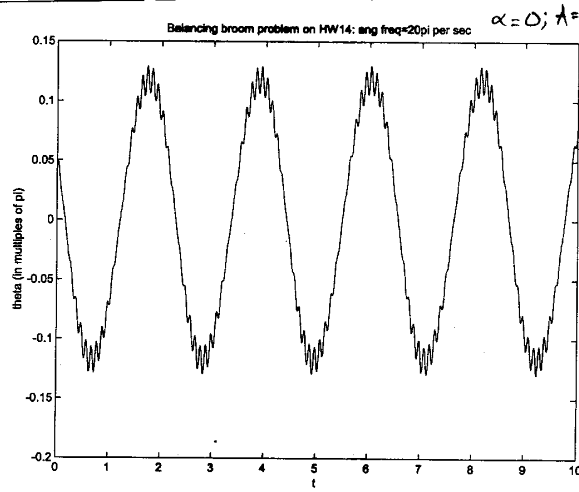


)

$\alpha = 1; A = 0$



$\alpha = 0; A = 0.1$



$\alpha = 1; A = 0.1$