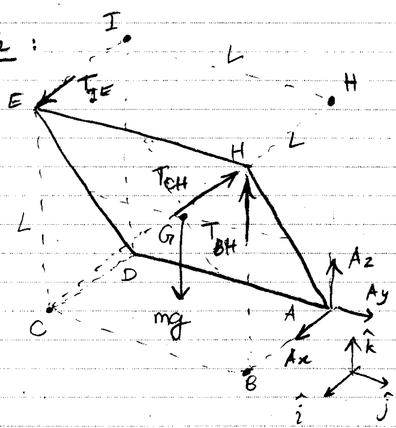
Basant Sharma Spring

① $\boxed{4.86}$

FBD of Plate :



Like problem 4.87 in HW2, part b) let us try finding as many unknown reactions as possible by using balance of moment about an arbitrary axis.

Taking moment about x-axis through H –

$\sum \underset{\sim}{M}_{/H} \cdot \hat{i} = 0 : \quad A_y . L + mg . \frac{L}{2} = 0$

So $\boxed{A_y = -\frac{mg}{2}}$

Taking moment about AH –

$\sum \underset{\sim}{M}_{/H} \cdot \underset{\sim}{AH} = 0 : \quad \frac{T_{IE}}{\sqrt{2}} . L + \frac{mg}{\sqrt{2}} \frac{L}{2} = 0$

So $\boxed{T_{IE} = -\frac{mg}{2}}$

Taking moments about x-axis Through A (ie. $\underset{\sim}{AB}$) –

$\sum \underset{\sim}{M}_{/A} \cdot \hat{i} = 0 : \quad -T_{CH} \frac{L}{\sqrt{2}} + mg . \frac{L}{2} = 0$

So $\boxed{T_{CH} = \frac{mg}{\sqrt{2}}}$

Now :

$\sum F_x = 0 : \quad T_{IE} + A_x = 0 \Rightarrow \boxed{A_x = -T_{IE} = \frac{mg}{2}}$

$\sum F_z = 0 : \quad -mg + T_{BH} + \frac{1}{\sqrt{2}} T_{CH} + A_z = 0$ —— ①

$\sum \underset{\sim}{M}_{/H} \cdot \underset{\sim}{EH} = 0 : \quad -mg \frac{L}{2} + A_z . L - A_x . L = 0$ —— ②

② $\Rightarrow \quad A_z = A_x + \frac{mg}{2}$ or $\boxed{A_z = mg}$

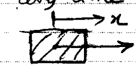
① $\Rightarrow \quad T_{BH} = mg - \frac{1}{\sqrt{2}} T_{CH} - A_z$ or $\boxed{T_{BH} = -\frac{mg}{2}}$

Note: L doesn't matter!

② $\boxed{5.9}$ Given a sinusoidal force acts on a 1 kg mass. Initially mass is at rest.

Need : Position of mass at any time t.

FBD of mass :



By Newton's second law (in one dimension)

$ma = F(t)$

$\Leftrightarrow \quad m\ddot{x}(t) = F(t)$

here $m = 1\,kg$.

Given that the mass is initially still ; ie

$x(0) = 0,$

$v(0) = \dot{x}(0) = 0.$

Also $F(t) = (5\,N)\sin t$

Thus the equation of motion of the mass is

$\ddot{x}(t) = 5\sin t$ (x in meters, t in seconds)

Integrating w.r.t t

$\Rightarrow \quad \dot{x}(t) = -5\cos t + C_1$ —— (1)

as $\dot{x}(0) = 0, \quad C_1 = 5$

Integrating (1) w.r.t t

$x(t) = -5\sin t + 5t + C_2$

as $x(0) = 0, \quad C_2 = 0$

So $\boxed{x(t) = 5(t - \sin t)}$

a) $v(t) = \dot{x}(t) = 5(1 - \cos t)$

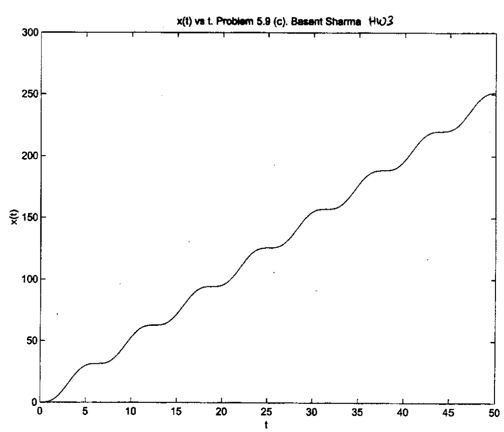
at $t = 2\pi$ seconds, $v(t) = 5(1 - \cos 2\pi)$

$\therefore \boxed{v = 0}$

b) at $t = 2\pi$ seconds, $x(t) = 5(2\pi - \sin 2\pi)$

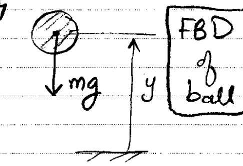
$\therefore \boxed{x = 10\pi}$

c)



③ $\boxed{5.14}$ Given, a ball of mass m which is dropped vertically from rest at a height h.

Need : Position and velocity as a function of time.



By Newton's Second law

$m\ddot{y} = -mg$.

So $\ddot{y}(t) = -g$ —— ①

Given $y(0) = h, \ \dot{y}(0) = 0$ (namely ball drops from 'rest', from 'height' h).

Integrating ① twice w.r.t time t

$y(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$

Using initial conditions, $C_1 = 0, \ C_2 = h$.

So $\boxed{y(t) = h - \frac{1}{2}gt^2}$

and $\boxed{v(t) = \dot{y}(t) = -gt}$

When the ball hits the ground, $y = 0$, let this happen at $t = t_0$, then

$y(t_0) = h - \frac{1}{2}g t_0^2 = 0$

$\therefore \ t_0 = \sqrt{\frac{2h}{g}}$ (Since $t_0 > 0$).

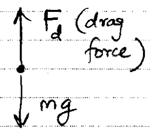
at $t = t_0$, $v(t) = -g t_0 = -g\sqrt{\frac{2h}{g}}$

So $\boxed{V_0 = -\sqrt{2gh}}$

And Speed $= |V_0| = \sqrt{2gh}$

(4) $\boxed{5.18}$ Given: a particle falls under gravity while a drag force $\propto V^2$ acts on it, where v is velocity.

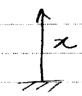
Need: Equation of motion and its numerical soln.



(a) FBD of particle

$$m\ddot{x} = -mg + F_d \quad \text{——(1)}$$

(By Newton's second law)



But given $F_d \propto v^2$

ie. $F_d = k(\dot{x})^2$, $k>0$.

So $(1) \Leftrightarrow \ddot{x} - \frac{k}{m}(\dot{x})^2 + g = 0$ —— (2)

This is the equation of motion of the particle with $x(t)$ the height of particle at time t and k, m, g given constants.

(b) for constant speed $\dot{x}(t) \equiv V = \text{constant}$
$\Rightarrow \ddot{x}(t) \equiv 0$

$$\therefore (2) \Rightarrow 0 - \frac{k}{m}V^2 + g = 0 \Rightarrow \boxed{V = \sqrt{\frac{mg}{k}}}$$

(Since we assumed that the particle is falling down due to gravity, the negative root is neglected, so that $F_d = kV^2$ is a drag force and not an accelerating force.)

(c) Let $m=1$, $k=2$, $g=10$.

$$\therefore (2) \Rightarrow \ddot{x} - 2(\dot{x})^2 + 10 = 0 \quad \text{——(3)}$$

We further assume $x(0) = h$, $\dot{x}(0) = 0$ (namely the particle starts from a state of rest at height h).

(i) We set up the differential equation (3) for use in MATLAB by rewriting it as a system of first-order differential equations.

Let $z_1(t) = x(t)$, $z_2(t) = \dot{x}(t)$

Then

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = 2(z_2(t))^2 - 10$$

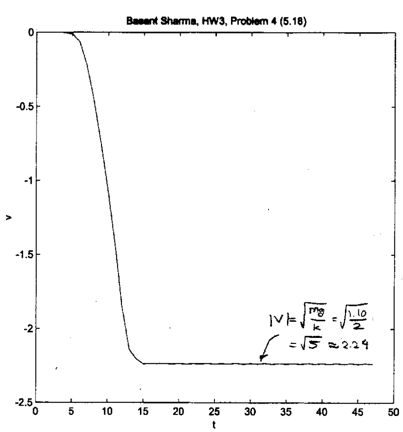
with $z_1(0) = h$
$z_2(0) = 0$.

(ii) for $h = 10$, the following MATLAB code solves the problem/equations stated in (i)

```
% Particle Falling in air: Homework Problem 4 (5.18)
% Numerical Solution of the ODE involved
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
options=odeset('RelTol',1e-6,'AbsTol',1e-8);
[t,z]=ODE23('eqnsofmotion', [0 10], [0 0]);
% plot(z(:,1),  z(:,2))
plot(z(:,2));
axis('square');
xlabel('t');
ylabel('v');
title('Basant Sharma, HW3, Problem 4 (5.18)');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Equations of Motion eqnsofmotion.m
function zdot=eqnsofmotion(t,z)
% These seven lines are in the file iknowthis.m
x = z(1);
v = z(2);
% assuming we are on earth, take g to be 10
g = 10;
% c is the measure of drag
c = 2;
xdot=  v;
vdot= c*v^2-g;
zdot=[xdot, vdot]';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

(iii) The following plot shows the velocity of particle as a function of time:



Aside:

(Analytical solution for $v(t)$)

The differential equation satisfied by the velocity $v(t)$ is

$$\dot{v}(t) - \frac{k}{m}v(t)^2 + g = 0 \qquad \left(\text{Let } V_\infty = \sqrt{\frac{mg}{k}}\right)$$

or

$$\frac{dv(t)}{dt} = -g + \frac{k}{m}v(t)^2 = \frac{k}{m}\left(-V_\infty^2 + v(t)^2\right)$$

This is a separable equation, so

$$\int \frac{dv}{v^2 - V_\infty^2} = \frac{k}{m}\int dt, \quad \text{say } v=0 \text{ at } t=0,$$

$$\Rightarrow \frac{1}{2V_\infty}\ln\left|\frac{v - V_\infty}{v + V_\infty}\right| = \frac{k}{m}t$$

or

$$\left|\frac{v(t) - V_\infty}{v(t) + V_\infty}\right| = e^{2\frac{k}{m}tV_\infty} \qquad \left(= \frac{V_\infty - v}{V_\infty + v} \because v < V_\infty \text{ in our case}\right)$$

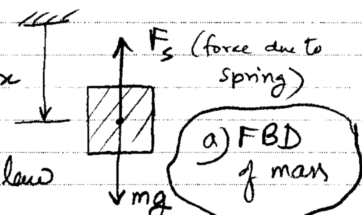
or

$$\boxed{v(t) = V_\infty \frac{e^{\frac{2k}{m}tV_\infty} - 1}{e^{2k/m\, tV_\infty} + 1}}$$

(5) $\boxed{5.40}$

Given: mass 'm', spring with constant 'k' and l_0 initial length.

need: position of mass at time t.

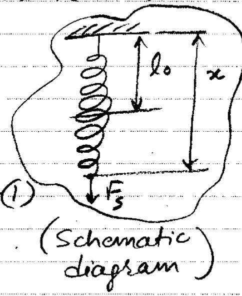


b) Using Newton's second law

$$m\ddot{x} = mg - F_s$$

c) Given the spring with constant k and length l_0 when relaxed,

$$F_s = K(x - l_0)$$



So $m\ddot{x} = mg - k(x - l_0)$

or $\boxed{\ddot{x} + \frac{k}{m}x - \left(g + \frac{kl_0}{m}\right) = 0}$ — (1)

d) Let $x(t) = l_0 + \frac{mg}{k}$

Then L.H.S. of (1) $= \ddot{x}(t) + \frac{k}{m}x(t) - \left(g + \frac{kl_0}{m}\right)$

$= 0 + \frac{k}{m}\left(\cancel{l_0} + \frac{mg}{k}\right) - \left(\cancel{g} + \cancel{\frac{kl_0}{m}}\right)$

$= 0 =$ R.H.S. of (1)

Hence (1) is satisfied.

So $x(t) = l_0 + \frac{mg}{k}$ is a solution of (1).

e) the solution in d) represents the static solution of the system (namely no oscillations). If one extends the spring by $\frac{mg}{k}$ and attaches a mass without any additional input of energy (no pushing) then the mass stays at a fixed position ever.

f) Let $\hat{x} = x - \left(l_0 + \frac{mg}{k}\right)$.

Then $\dot{\hat{x}} = \dot{x}$, $\ddot{\hat{x}} = \ddot{x}$

So (1) $\Leftrightarrow$ $\boxed{\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0}$ — (2)

(note: $\hat{x}$ represents the displacement of the mass about its position of static equilibrium)

g) Given $x(0) = D$, $\dot{x}(0) = 0$.

So $\hat{x}(0) = D - \left(l_0 + \frac{mg}{k}\right)$, $\dot{\hat{x}}(0) = 0$.

S. (2) has the solution $\hat{x}(t) = \hat{x}(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

In terms of $x(t)$, the solution is

$$x(t) = \hat{x}(t) + \left(l_0 + \frac{mg}{k}\right)$$

$\therefore$ $\boxed{x(t) = \left[D - \left(l_0 + \frac{mg}{k}\right)\right]\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \left(l_0 + \frac{mg}{k}\right)}$

This gives the motion of the mass.

(observe: if $D = l_0 + \frac{mg}{k}$, ie. $\hat{x}(0) = 0$, then $x(t) \equiv l_0 + \frac{mg}{k}$ as described as a solution in part d))

h) Let the period be T, then

$$x(t+T) = x(t) \Leftrightarrow \cos\sqrt{\frac{k}{m}}\,t = \cos\sqrt{\frac{k}{m}}(t+T)$$

$\Rightarrow \sqrt{\frac{k}{m}}\,T = 2\pi$ or $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$

i) In order that the mass never hits the 'roof'

$x(t) \geq 0$ at any time t

In particular $x(t)$ is minimum when t is an odd multiple of $\frac{T}{2} = \pi\sqrt{\frac{m}{k}}$

(you can derive this by putting $\frac{dx(t)}{dt} = 0$ and solving for t)

So we need (to make physical sense for a spring hanging from a surface)

$$\left[D - \left(l_0 + \frac{mg}{k}\right)\right](-1) + \left(l_0 + \frac{mg}{k}\right) \geq 0$$

$$\Rightarrow \quad D \leq 2\left(l_0 + \frac{mg}{k}\right)$$

Therefore, If $D > 2\left(l_0 + \frac{mg}{k}\right)$ Then tension becomes negative and as you might have seen most springs 'buckle' when compressed.