Solutions - Basant Sharma Mar 31st 02

① $\boxed{7.1}$

The trajectory of given particle is

$$\underset{\sim}{r}(t) = R\cos(\dot\theta t)\hat{i} + R\sin(\dot\theta t)\hat{j}$$

where $\dot\theta$ = Constant.

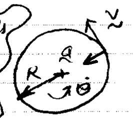
so $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t)$

$$= R\dot\theta\left(-\sin(\dot\theta t)\hat{i} + \cos(\dot\theta t)\hat{j}\right)$$

a) $\underset{\sim}{v} \neq \underset{\sim}{0}$ as $\underset{\sim}{v}$ depends on time. (take $t=0$, for example)

b) $\underset{\sim}{v} \neq$ Constant (see a))

This corresponds to a constant rate circular motion.



c) $|\underset{\sim}{v}| = R\dot\theta$ = constant. Since R and $\dot\theta$ are constant.

d) $\underset{\sim}{a} = \dot{\underset{\sim}{v}} = -R\dot\theta^2\left(\cos(\dot\theta t)\hat{i} + \sin(\dot\theta t)\hat{j}\right)$

$\neq 0$ (take $t=0$ for example)

e) $\underset{\sim}{a} \neq$ constant as it is clearly a function of time t.

f) $|\underset{\sim}{a}| = R\dot\theta^2$ = constant (see c))

g) $\underset{\sim}{v}\cdot\underset{\sim}{a} = -R^2\dot\theta^3\left(-\cancel{\sin(\dot\theta t)\cos(\dot\theta t)} + \cancel{\cos(\dot\theta t)\sin(\dot\theta t)}\right)$

$= 0$

$\therefore \underset{\sim}{v} \perp \underset{\sim}{a}$.

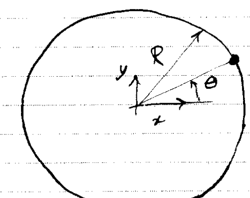
② $\boxed{7.8}$

Bead on a circular path

The rate of change of angular speed is proportional to the angular position as

$\alpha = c\theta^{3/2}$

So $\ddot\theta = c\,\theta^{3/2}$ ——(1)



Given the bead starts from rest at $\theta = 0$,

So $\theta(0) = 0$, $\dot\theta(0) = 0$. One could guess that $\theta = 0$ is the solution. Let's do it more rigorously. Multiplying both sides by $\dot\theta$ (say $\dot\theta \neq 0$),

$\dot\theta\ddot\theta = c\,\theta^{3/2}\dot\theta$

or

$$\frac{1}{2}\frac{d}{dt}\dot\theta^2 = \frac{c}{5/2}\frac{d\theta^{5/2}}{dt} \text{ — (2) (use "inverse" chain rule)}$$

Integrating (2) w.r.t. t we get

$$\frac{\dot\theta^2}{2} = \frac{2c}{5}\theta^{5/2} + C_1$$

at $t=0$, $\theta = \dot\theta = 0 \Rightarrow C_1 = 0$

So $\dot\theta^2 = \frac{4c}{5}\theta^{5/2}$

Consider only forward motion, $\dot\theta = \sqrt{\frac{4c}{5}}\,\theta^{5/4}$

Using separation of variables

$$\theta^{1/4} = \frac{-1/4}{\sqrt{\frac{4c}{5}}\,t + c_2}$$

Since $t=0$, $\theta = 0$ So $c_2 = \infty$!

So $\dot\theta \neq 0$ assumption was not good. So

Only solution is $\boxed{\theta = 0}$.

Solution after Correction :

Here $\ddot\theta = c\,\theta^{1/2}$ ——— (3)

Assume $\dot\theta \neq 0$ then $\frac{\dot\theta^2}{2} = \frac{c}{3/2}\theta^{3/2} + C_1$

(Same reasoning as above)

Since at $t=0$, $\theta = \dot\theta = 0$, $C_1 = 0$

So $\dot\theta = \sqrt{\frac{4c}{3}}\,\theta^{3/4}$

By separation of variables and using $\theta = 0$ at $t = 0$

$$\theta^{1/4} = \frac{1}{4}\sqrt{\frac{4c}{3}}\,t + \cancel{c_2}^{\,0}$$

So $\theta = \left(\frac{\sqrt{c}}{2\sqrt{3}}\right)^4 t^4 = \frac{c^2}{144}t^4$

Other solution is clearly $\dot\theta \equiv 0$ i.e. $\theta \equiv 0$

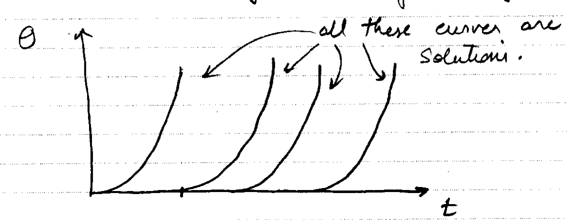
Since $\theta = 0$ at $t = 0$.

So we have clearly two distinct solutions to (3) with given initial conditions.

Infact there are infinite number of solutions and each such solution is given by

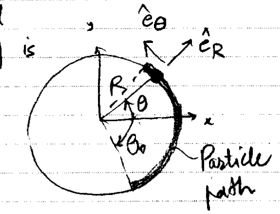
$$\theta = \begin{cases} 0 & 0 \le t \le t_1, \\ \frac{c^2}{144}t^4 & t_1 \le t. \end{cases}$$

where t_1 is any number of choice !



Note: That an ODE has more than one solution with given ICs (like this) is exceptional (you won't see it too often).

③ $\boxed{7.14}$ We have to find the conditions for which $|\underset{\sim}{a}|$ is max at $\theta = 0$.



Given $\theta = \theta_0\cos\lambda t$

So $\dot\theta = -\theta_0\lambda\sin\lambda t$

$\ddot\theta = -\theta_0\lambda^2\cos\lambda t$.

And $\underset{\sim}{a} = -R\dot\theta^2\hat{e}_R + R\ddot\theta\hat{e}_\theta$

where $\dot\theta^2 = \theta_0^2\lambda^2\sin^2\lambda t$

$= \theta_0^2\lambda^2(1 - \cos^2\lambda t)$

$= \lambda^2(\theta_0^2 - \theta^2)$; $\ddot\theta = -\lambda^2\theta$.

So $|\underset{\sim}{a}|^2 = R^2\dot\theta^4 + R^2\ddot\theta^2$

$= R^2\left(\lambda^4(\theta_0^2 - \theta^2)^2 + \lambda^4\theta^2\right)$

Note: explicit time dependence is eliminated!

$$= R^2\lambda^4\underbrace{\left((\theta_0^2 - \theta^2)^2 + \theta^2\right)}_{\text{Call } f(\theta)}$$

Maximum acceleration occurs at the maximum value of $f(\theta) = \theta_0^4 + \theta^4 + \theta^2(1 - 2\theta_0^2)$ as λ, R are given constants.

at $\theta = 0$, $f(\theta) = \theta_0^4$

So we need $f(\theta) \le \theta_0^4$ for any θ

ie. $\theta^4 + \theta^2(1 - 2\theta_0^2) \le 0$

$\Rightarrow \theta^2 + 1 - 2\theta_0^2 \le 0$ (as $\theta^2 \ge 0$)

We know that $\theta = \theta_0\cos\lambda t$

So $|\theta| \le |\theta_0|$ or $\theta^2 \le \theta_0^2$

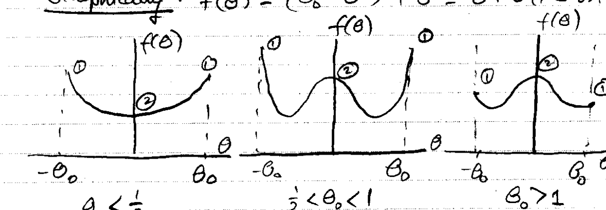
Therefore, for maximum acceleration we need

$1 - \theta_0^2 \le 0$ or $\boxed{\theta_0 \ge 1}$

Then the maximum acceleration becomes

$$\boxed{|\underset{\sim}{a}| = R\lambda^2\theta_0^2}$$

Graphically : $f(\theta) = (\theta_0^2 - \theta^2)^2 + \theta^2 = \theta^4 + \theta^2(1 - 2\theta_0^2) + \theta_0^4$

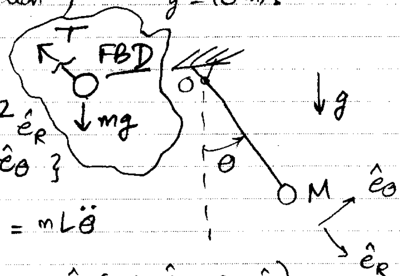


① $f(\theta)|_{\theta = \theta_0} = \theta_0^2$; ② $f(\theta = 0) = \theta_0^4$

$\therefore$ Max value of $|\underset{\sim}{a}|$ will be at $\theta = 0$ only when $\theta_0 \ge 1 \Rightarrow |\underset{\sim}{a}|_{max} = |\underset{\sim}{a}|_{\theta = 0} = R\lambda^2\theta_0^2$.

④ $\boxed{7.41}$ Simple Pendulum, $M = 1\,kg$, $L = 1\,m$
(no air friction) $g = 10\,m/s^2$

a)



LMB: $\sum \underset{\sim}{F} = m\underset{\sim}{a}$

$\{-T\hat{e}_R + mg\,\hat{i} = -mL\dot{\theta}^2\hat{e}_R + mL\ddot{\theta}\,\hat{e}_\theta\}$

$\{\}\cdot\hat{e}_\theta \Rightarrow mg(\hat{i}\cdot\hat{e}_\theta) = mL\ddot{\theta}$

where $\hat{i}\cdot\hat{e}_\theta = \hat{i}\cdot(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = -\sin\theta$

$\therefore\ -mg\sin\theta = mL\ddot{\theta}$

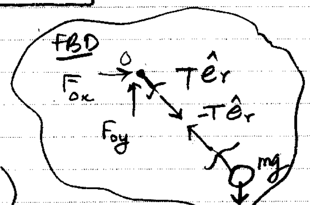
$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0} \text{ ——— (1)}$$

b) LMB. $\hat{e}_R \Rightarrow -T + mg(\hat{i}\cdot\hat{e}_R) = -mL\dot{\theta}^2$

where $\hat{i}\cdot\hat{e}_R = \hat{i}\cdot(+\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = \cos\theta$

$$\Rightarrow \boxed{T = mg\cos\theta + mL\dot{\theta}^2} \text{ ——— (2)}$$

c) The force at the hinge support is Simple $\underset{\sim}{F_o} = -T\hat{e}_R$ since the string is massless.



$\underset{\sim}{F_o} = -T\hat{e}_R = -T(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

So $F_{ox}\,\hat{i} + F_{oy}\,\hat{j} = -(mg\cos\theta + mL\dot{\theta}^2)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

or $\boxed{F_{ox} = -(mg\cos\theta + mL\dot{\theta}^2)\cos\theta;\ F_{oy} = -(mg\cos\theta + mL\dot{\theta}^2)\sin\theta}$

d) Let $\omega = \dot{\theta} \Rightarrow \dot{\omega} = \ddot{\theta} = -\frac{g}{L}\sin\theta$ by (1)

$$\Rightarrow \boxed{\begin{array}{l}\dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{L}\sin\theta\end{array}}$$

e) see next page.

f) Max. tension $= 29.999\,N$

g) Time period $= 2.3452\,s$ ← by fine tuning tspan, interpolation, or event detection.

h) Note:

① Maximum Tension is independent of string length.
Since Energy (total) is conserved,

$$\frac{1}{2}m(L\dot{\theta})^2 + mg(L - L\cos\theta) = \underset{\text{initial}}{mgL}$$

i.e. $mL\dot{\theta}^2 + 2mg(1 - \cos\theta) = 2mg$

So $mL\dot{\theta}^2 = 2mg\cos\theta$

So by (2) $T = 3mg\cos\theta$

$$\Rightarrow \boxed{T_{max} = 3mg}$$

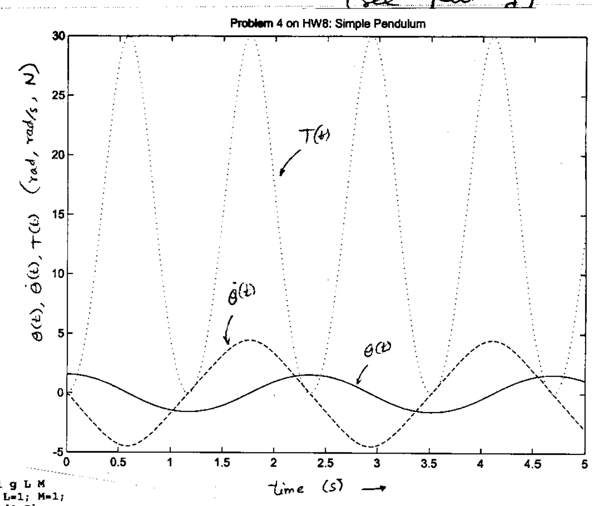
If you compute numerically T_{max} then you will always obtain value close to 30

L	T_{max} (N)
1	29.9993
5	29.9935
10	29.9977

② Maximum tension can be made closer to 30N by decreasing the 'RelTol' and 'AbsTol'. For example:

AbsTol	RelTol	T_{max}
1e-6	1e-6	29.9993
1e-4	1e-4	29.9820
1e-4	1e-8	29.9947
1e-8	1e-8	30.0000

③ Use $\sin\theta \approx \theta$ and get Time Period $= 1.9869\,sec$ (see part g)




```
global g L M
g=10; L=1; M=1;
tspan=[0 5];
theta0=pi/2; thetadot0=0;
z0=[theta0 thetadot0];
options=odeset('AbsTol',1e-6,'RelTol',1e-6);

[t,z]=ode45('simplependulum',tspan,z0,options);

theta=z(:,1);
thetadot=z(:,2);
tension=M*g*cos(theta)+M*L*thetadot.^2;
max(tension)

plot(t,theta)
hold on
plot(t,thetadot,'--')
plot(t,tension,':')

title('Problem 4 on HW8: Simple Pendulum');
```

```
function zdot=simplependulum(t,z)
global g L M

theta=z(1);
omega=z(2);

thetadot=omega;
omegadot=(-g/L)*sin(theta);
zdot=[thetadot omegadot]';
```

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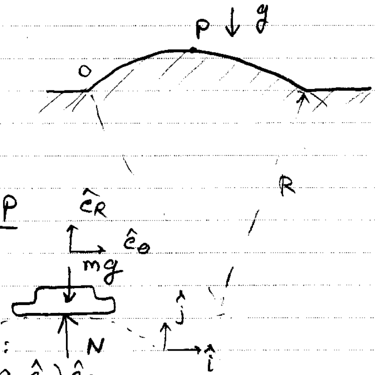
⑤ $\boxed{7.47}$

Car moves with speed v and climbs the hill at O



FBD of car at P

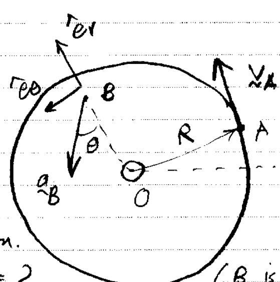
By the Second law in the radial direction at P:

$$(N - mg)\,\hat{e}_R = m(\underset{\sim}{a}\cdot\hat{e}_R)\,\hat{e}_R$$
$$= m\left(-\frac{v^2}{R}\right)\hat{e}_R \qquad (\because \dot{v} = 0)$$

If the car just barely leaves the ground at P then $N = 0$

$$\Rightarrow \quad mg = \frac{mv^2}{R} \quad \text{or} \quad \boxed{v = \sqrt{gR}}$$

⑥ $\boxed{7.59}$



Circular disc $R = 100$ mm rotating with center O.

$V_A = |\underset{\sim}{V}_A| = 0.8$ m/s at given instant

a_B makes angle θ with radial line as shown.

Angular acceleration $\alpha = ?$

Given $\tan\theta = 0.6$.

(B is any point)

By kinematics

$$\underset{\sim}{v} = \underset{\sim}{\omega}\times\underset{\sim}{r}$$
$$\underset{\sim}{a} = \underset{\sim}{\alpha}\times\underset{\sim}{r} - \omega^2\underset{\sim}{r}$$

where $\underset{\sim}{r} = r\hat{e}_r$, $\underset{\sim}{\omega} = \omega\hat{k}$, $\underset{\sim}{\alpha} = \alpha\hat{k}$ with ω and α instantaneous angular velocity and acceleration respectively. Using $\hat{k}\times\hat{e}_r = \hat{e}_\theta$ we get,

for point A : $V_A = \omega R \Rightarrow \omega = \dfrac{V_A}{R}$ —— (1)

for point B : $\underset{\sim}{a}_B = \alpha r\,\hat{e}_\theta - r\omega^2\,\hat{e}_r$ —— (2)

$$(2) \Rightarrow \tan\theta = \frac{a_B\cdot\hat{e}_\theta}{(-a_B\cdot\hat{e}_r)} = \frac{\alpha r}{\omega^2 r} = \frac{\alpha}{\omega^2}$$

$$\text{by (1)}, \quad \tan\theta = \frac{\alpha}{(V_A)^2/R^2}$$

0.6

Using the data given $\boxed{\alpha = \dfrac{V_A^2}{R^2}\tan\theta = 38.4\ \dfrac{\text{rad}}{s^2}}$

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⑦

Matlab Code – rotatepic.m

```
before=[1 2 1.5 1 1 2 2 1; 1 1 2 1 2 3 3 1];
line(before(1,:),before(2,:)); % input picture
axis equal;
hold on;

theta=2*pi/3;
Q=[cos(theta) -sin(theta); sin(theta) cos(theta)];
after=Q*before;
line(after(1,:),after(2,:)); % rotated picture

title('Rotation of a drawing by 120 deg');
```