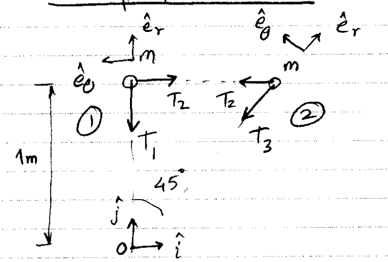
Solutions: Basant Sharma April 5^{th}/02

① $\boxed{7.65}$

Two point masses connected by three massless rods and the structure rotates at a constant rate, $\omega = 60\,\text{rpm} = 2\pi\ \text{rad/s}$.

FBD of point masses



from FBD ①:

LMB $\Rightarrow$ $m\underset{\sim}{a}_1 = -T_1\hat{j} + T_2\hat{i}$ ——(1)

here $\underset{\sim}{a}_1 = (\overset{0}{\ddot{r}} - r\dot{\theta}^2)\hat{e}_r + r\overset{0}{\ddot{\theta}}\hat{e}_\theta$
$= -r\omega^2\hat{j}$

So From (1)$\cdot\hat{i}$: $\boxed{T_2 = 0}$

(1)$\cdot\hat{j}$: $T_1 = m r_1\omega^2 = (0.5)(1)(2\pi)^2\,N$

$\boxed{T_1 \approx 19.74\,N}$

from FBD ②

LMB $\Rightarrow$ $m\underset{\sim}{a}_2 = -\cancel{T_2}\hat{i} - \frac{T_3}{\sqrt{2}}(\hat{i}+\hat{j})$

So $m(-r_2\omega^2)\hat{e}_r = \frac{-T_3}{\sqrt{2}}(\hat{i}+\hat{j})$,

but $\hat{e}_r = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$ So $T_3 = m r_2\omega^2 = (0.5)(\sqrt{2})(2\pi)^2\,N$

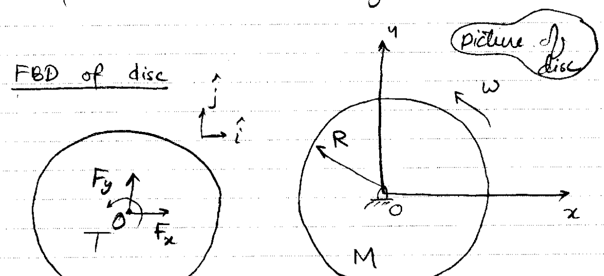
$\boxed{T_3 \approx 27.92\,N}$

② $\boxed{7.68}$

Uniform circular disc rotating at constant rate ω.

FBD of disc

picture of disc



Since O is the center of mass of the disc.

LMB: $M\underset{\sim}{a}_G = \underset{\sim}{F} = F_x\hat{i} + F_y\hat{j}$, $\underset{\sim}{a}_G = 0$ as G doesnot move

So $F_x = F_y = 0$.

AMB: $I_{zz}\overset{0}{\ddot{\phi}}\hat{k} = M\hat{k}$; $\dot{\theta} = \omega =$ constant

So $M = 0$.

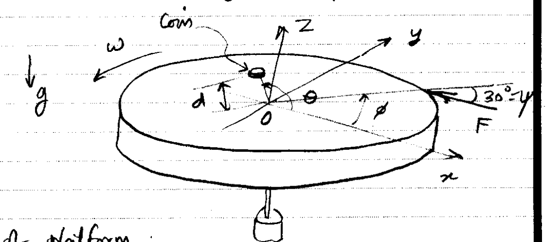
Total kinetic energy of the disk

$$KE = \frac{1}{2}M\cancel{v_G^2} + \frac{1}{2}I_{zz}\omega^2 = \frac{1}{2}\left(\frac{MR^2}{2}\right)\omega^2$$

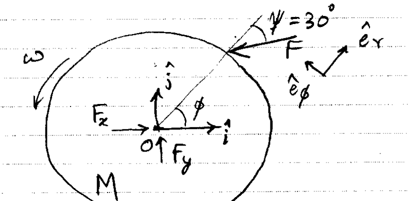
$$\boxed{\therefore KE = \frac{MR^2\omega^2}{4}}$$

③ $\boxed{7.70}$

Small coin on a rotating flat platform.



FBD of platform:
(Ignore the coin as it is very small)



By AMB about point O: $\Sigma\underset{\sim}{M}_{/O} = \dot{\underset{\sim}{H}}_{/O}$ ——(1)

Now $\Sigma\underset{\sim}{M}_{/O} = R\hat{e}_r \times (-F\cos\psi\,\hat{e}_r + F\sin\psi\,\hat{e}_\phi)$
$= RF\sin\psi\,\hat{k}$,

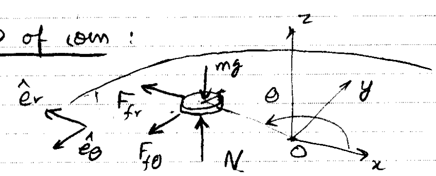
And $\dot{\underset{\sim}{H}}_{/O} = \int \underset{\sim}{r}\times\underset{\sim}{a}\,dm = \left(\frac{M}{\pi R^2}\right)\int_0^R\int_0^{2\pi} \underset{\sim}{r}\times(-r\omega^2\hat{e}_r + r\dot{\omega}\hat{e}_\theta)\,r\,dr\,d\theta$

$= \frac{M}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\dot{\omega}\hat{k}\,dr\,d\theta = \frac{MR^2}{2}\dot{\omega}\hat{k}$.

(F_x and F_y are x, y components of F)

Now (1)$\cdot\hat{k}$ $\Rightarrow$ $\boxed{\dot{\omega} = \frac{2F\sin\psi}{MR}}$ ——(1)

FBD of coin:



LMB: $m((\ddot{r} - r\dot{\theta}^2)\hat{e}_r + r\ddot{\theta}\hat{e}_\theta) = F_{fr}\hat{e}_r + F_{f\theta}\hat{e}_\theta + (N - mg)\hat{k}$

Since $r = d =$ constant

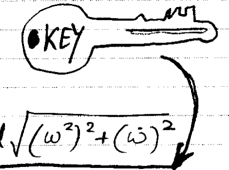
So (LMB)$\cdot\hat{e}_r$: $F_{fr} = -md\dot{\theta}^2$ ——(2)

(LMB)$\cdot\hat{e}_\theta$: $F_{f\theta} = md\ddot{\theta}$ ——(3)

(LMB)$\cdot\hat{k}$: $N = mg$ ——(4)

As the coin is moving with the platform

$\dot{\theta} = \omega$
$\ddot{\theta} = \dot{\omega}$



So from ②, ③

$$\sqrt{(F_{fr})^2 + (F_{f\theta})^2} = md\sqrt{(\omega^2)^2 + (\dot{\omega})^2}$$

For No slipping : $\boxed{\sqrt{(F_{fr})^2 + (F_{f\theta})^2} \le \mu N} = \mu mg$

So we require

$$\cancel{m}d\sqrt{\omega^4 + \dot{\omega}^2} \le \mu\cancel{m}g$$

or $d \le \frac{\mu g}{\sqrt{\omega^4 + \dot{\omega}^2}}$

from ①

$$d \le \frac{\mu g}{\sqrt{\omega^4 + \left(\frac{2FR\sin\psi}{MR^2}\right)^2}}$$

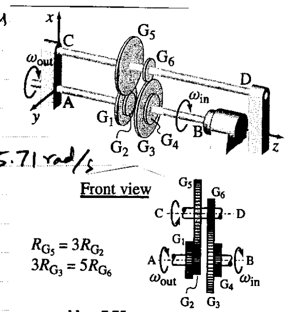
given $\psi = 30°$, So

$$\boxed{d \le \frac{\mu g}{\sqrt{\omega^4 + \frac{F^2}{M^2R^2}}}}$$

Note: the result is independent of θ, ϕ and m. But the dependence on m is implicit in the assumption that it is very small (negligible) compared to M. If m is large than the frictional force will appear in the FBD of the platform; in other words the computation of I_{zz} depends on the position of the coin!

(4) 7.75

Constant speed gear train with friction loss bearings and contacts.



problem 7.75:

$M_{input} = 500\,Nm.$

$\omega_{input} = 150\,rev/min = 5\pi\,rad/s. \approx 15.71\,rad/s$

a) $P_{input} = M_{input} \times \omega_{input} = 2.5\pi\,kNm/s = 2.5\pi\,kW$ $\boxed{\approx 7.85\,kW}$

b) $\boxed{P_{output} = P_{input}}$, as there is no dissipation of energy. (due to absense of friction)

c) $\omega_{input} = \omega_4$ (from picture),

Now $\omega_4 = \omega_3 = \frac{R_{G_6}}{R_{G_3}}\omega_6$

$\omega_6 = \omega_5 = \frac{R_{G_2}}{R_{G_5}}\omega_2$

$\omega_2 = \omega_{output}$

So $\omega_{output} = \frac{R_{G_5}}{R_{G_2}} \cdot \frac{R_{G_6}}{R_{G_3}} \cdot \omega_{input}$

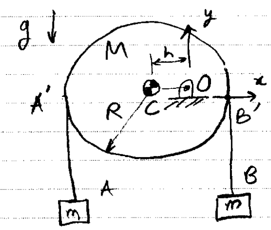
$= 3 \cdot \frac{5}{3} \cdot 5\pi\,rad/s$

$= 25\pi\,rad/s$ $\boxed{\approx 78.54\,rad/s}$.

d) $M_{output} = \frac{P_{output}}{\omega_{output}} = \frac{2.5\pi\,kW}{25\pi\,rad/s} = \boxed{100\,Nm.}$

(5) 7.84

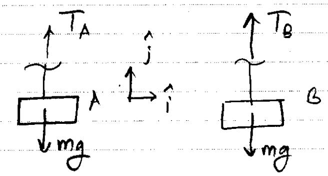
Find the acceleration of the two disks and the angular acceleration of the pulley.



a) If the strings are inextensible (assume) then

$$\left.\begin{aligned} \underset{\sim}{a}_A &= \underset{\sim}{a}_{A'} = (R+h)\alpha(-\hat{j}) \\ \text{and } \underset{\sim}{a}_B &= \underset{\sim}{a}_{B'} = (R-h)\alpha(+\hat{j}) \end{aligned}\right\} \quad \text{—— (1)}$$

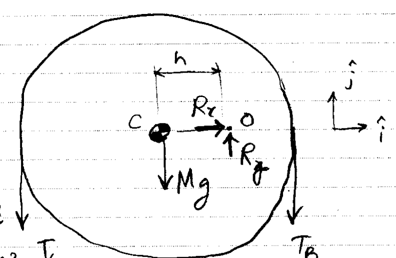
b). FBD of the masses A, B:



LMB: $m\underset{\sim}{a}_A = (T_A - mg)\hat{j}$; $m\underset{\sim}{a}_B = (T_B - mg)\hat{j}$

(LMB)$\cdot\hat{j}$: $T_A = m(g-(R+h)\alpha)$; $T_B = m(g-(R-h)\alpha)$ —— (2)
(Using (1))

FBD of the pulley:



AMB: $\dot{\underset{\sim}{H}}_{/O} = \underset{\sim}{M}_{/O}$

Now $\dot{\underset{\sim}{H}}_{/O} = \int \underset{\sim}{r} \times \underset{\sim}{a}\, dm$

$= \int r^2 \alpha\, dm$

Say $\underset{\sim}{r} = \underset{\sim}{u} + h(-\hat{i})$

So $r^2 = u^2 + h^2 - 2\underset{\sim}{h}\cdot\underset{\sim}{u}$

(Let $\underset{\sim}{h} = h\hat{i}$)

Now $\int u^2 dm = \frac{MR^2}{2}$

$\underset{\sim}{h}\cdot\int \underset{\sim}{u}\, dm = 0$ (as C is the center of mass)

So $\dot{\underset{\sim}{H}}_{/O} = \frac{MR^2}{2}\alpha + Mh^2\alpha$

Now $\underset{\sim}{M}_{/O} = Mg\cdot h\hat{k} + [T_A(R+h) - T_B(R-h)]\hat{k}$.

From (AMB)$\cdot\hat{k}$,

$\left(\frac{MR^2}{2} + Mh^2\right)\alpha = Mgh + m(g-(R+h)\alpha)(R+h) - m(g-(R-h)\alpha)(R-h)$

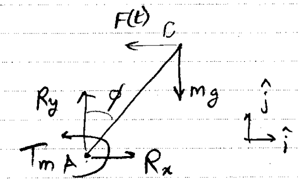
$= Mgh + 2mgh - m\alpha((R+h)^2 - (R-h)^2)$

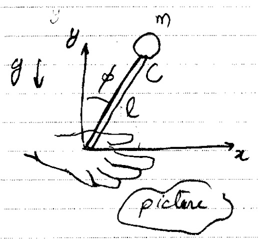
(By Parallel Axis Theorem)

So $\boxed{\alpha = \frac{\left(\frac{M}{m}+2\right)2gh}{\left(\frac{M}{m}\left(1+\frac{2h^2}{R^2}\right) + \frac{8h}{R}\right)MR^2}}$

(6) 7.99

(a) FBD





(b) by AMB $\sum \underset{\sim}{M}_{/A} = \dot{\underset{\sim}{H}}_{/A}$

Now $\sum \underset{\sim}{M}_{/A} = \underset{\sim}{r}_{C/A} \times (-mg\hat{j} - F(t)\hat{i}) + T_m\hat{k}$

$= (-mgl\sin\phi + F(t)l\cos\phi + T_m)\hat{k}$

$\dot{\underset{\sim}{H}}_{/A} = \sum \underset{\sim}{r}_{/A} \times \underset{\sim}{a}_c m_c$

$= -ml^2\ddot{\phi}\hat{k}$

$\therefore \{\}\cdot\hat{k}$: $ml^2\ddot{\phi} = mgl\sin\phi - Fl\cos\phi - T_m$

& $\boxed{\ddot{\phi} - \frac{g}{l}\sin\phi + \frac{F}{ml}\cos\phi + \frac{T_m}{ml^2} = 0}$ —— (1)

(c) T_m can be any function of ϕ and $\dot{\phi}$. One guess is that T_m should be in such a form that makes equation (1) look like a damped harmonic oscillator:
$\ddot{\phi} + c\dot{\phi} + k\phi = 0$ with $c \geq 0, k \geq 0$.

If $\frac{T_m}{ml^2} = \frac{g}{l}\sin\phi + c\dot{\phi}$ then for $F(t) \equiv 0$
(1) becomes
$\ddot{\phi} + c\dot{\phi} = 0$

(d) Say $T_m = ml^2(c\dot{\phi} + k\phi)$; $c>0, k>0$ constant.

$\therefore \ddot{\phi} - \frac{g}{l}\sin\phi + c\dot{\phi} + k\phi = 0$ by (1)

Linearizing the above equation about $\phi = 0$, we approximate $\sin\phi$ by ϕ so

$\ddot{\phi} + c\dot{\phi} + \left(k - \frac{g}{l}\right)\phi = 0$;

Here if $c^2 > 4\left(k-\frac{g}{l}\right)$, $\phi = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$; $\lambda_1 < \lambda_2 < 0$

if $c^2 = 4\left(k-\frac{g}{l}\right)$, $\phi = (A_1 + A_2t)e^{-\frac{c}{2}t}$

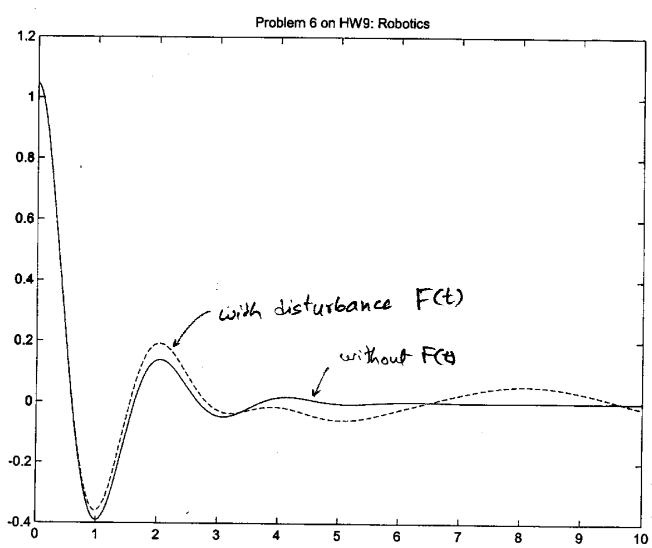
if $c^2 < 4\left(k-\frac{g}{l}\right)$, $\phi = Ae^{-\frac{c}{2}t}\sin\left(\sqrt{4\left(k-\frac{g}{l}\right) - c^2}\,t + \psi\right)$

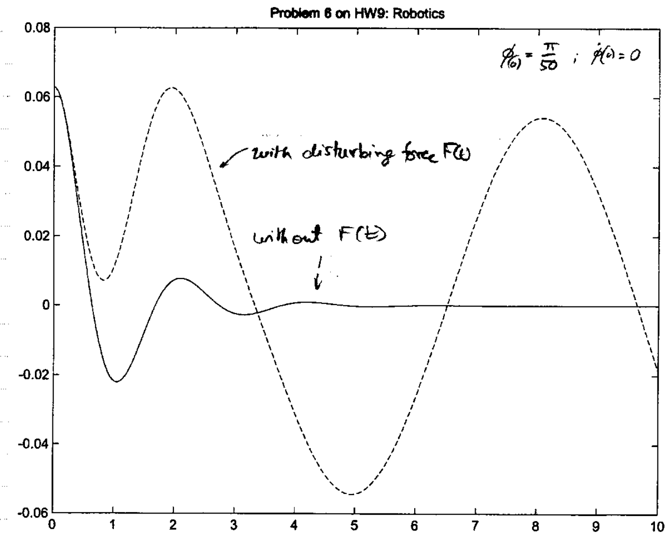
(in 1st case $\lambda_1 = -\frac{c}{2} + \sqrt{c^2 - 4\left(k-\frac{g}{l}\right)}$, $\lambda_2 = -\frac{c}{2} - \sqrt{c^2 - 4\left(k-\frac{g}{l}\right)}$)

Thus in all the three cases ϕ decays finally to 0.

e) For the nonlinear equation
$$\ddot{\phi} - \frac{g}{\ell}\sin\phi + c\dot{\phi} + k\phi = 0,$$
we implement a MATLAB program as given below.

```
global g L m c k
g=10; L=1; m=1; c=2; k=20;
tspan=[0 10];
phi0=pi/3; phidot0=0; % initial conditions
z0=[phi0 phidot0];
options=odeset('AbsTol',1e-8,'RelTol',1e-8);

[t,z]=ode45('invertedpendulum',tspan,z0,options);
phi=z(:,1);phidot=z(:,2);

plot(t,phi);
hold on;

[t,z]=ode45('perturbedpendulum',tspan,z0,options);
phi=z(:,1);phidot=z(:,2);

plot(t,phi,'--');

title('Problem 6 on HW9: Robotics');


function zdot=invertedpendulum(t,z)
global g L m c k

phi=z(1);
omega=z(2);

phidot=omega;
omegadot=(g/L)*sin(phi)-c*omega-k*phi;
zdot=[phidot omegadot]';


function zdot=perturbedpendulum(t,z)
global g L m c k

phi=z(1);
omega=z(2);

phidot=omega;
omegadot=(g/L)*sin(phi)-c*omega-k*phi+0.5*sin(t)*cos(phi);;
zdot=[phidot omegadot]';
```





f) In this (above) numerical experiment we have compared the solution without (—) and with (- -) a perturbating force $F(t)$. The disturbing force $F(t)$ becomes dominant after a small amount of time!
From the above plots one can conclude that assuming "small" fluctuations near the vertical alignment of the pendulum are allowed the present control scheme devised by us works fine.