Outline

I. Restate book's solution

II. Show the error in the book's reasoning

III. Derive the correct solution.

I – The reasoning is that to minimize the total velocity, we need to minimize its vertical velocity, i.e. get the ball as high as the window – max height of the flight trajectory = 7 m. The ball will only go through the window if that max height is 2 m away from the origin.

Fig. 1

– In short: a trajectory whose maximum point is the position (2, 7). See Fig 1.

The solution:

$$y = y_0 + V_{oy} t + \frac{g}{2} t^2 \Rightarrow g = -9.81\ m/s^2,\ y_0 = 0$$

$$y = V_{oy} t + \frac{g}{2} t^2$$

$$y' = V_{oy} + gt = 0 \Rightarrow \boxed{t_{max} = \frac{-V_{oy}}{g}}$$

$$y_{max} = V_{oy}\left(\frac{-V_{oy}}{g}\right) + \frac{g}{2}\left(\frac{V_{oy}^2}{g^2}\right) = \frac{-V_{oy}^2}{2g} = y_{max} \Rightarrow V_{oy} = \sqrt{-2 g y_{max}}$$

$$V_{oy} = \sqrt{-2(-9.81)(7)} = 11.7192\ m/s$$

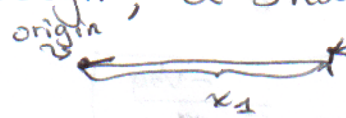
$$x = V_{ox} t;\ x_{max} = V_{ox} t_{max} = \frac{-V_{ox} V_{oy}}{g} \Rightarrow V_{ox} = \frac{-x_{max}\, g}{V_{oy}}$$

$$= \frac{-2 \cdot (-9.81)}{11.7192} = 1.67418\ m/s$$

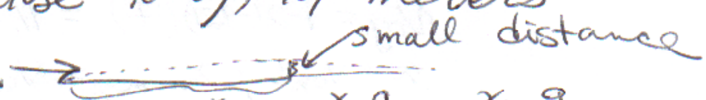
$$V = \sqrt{V_{oy}^2 + V_{ox}^2} = \sqrt{1.67418^2 + 11.7192^2} = 11.8382\ m/s$$

II- On the surface, this strategy seems reasonable, but we must rember that a minimum V_{oy} doesn't lead to a minimum V. If the decrease in V_{ox} is enough to compensate for the increases in V_{oy}, the overall velocity may be lower. Now the question is: can we find such a velocity vector. If we disprove that the book's solution works for the general case, we have good reason to believe there exists a better solution for this specific problem. We'll use the simplest specific case to show the book's strategy won't yield a minimum v for all cases.

Suppose there is a window x_1 meters away from the origin, a shade above the ground (almost 0). See Fig. 2.



The book's strategy would suggest we reach the maximum altitude (of close to 0), x_1 meters from the origin. See Fig. 3.



Using our formula for V_{ox}, we find $V_{ox} = \frac{x_1 g}{V_{oy}} = \frac{x_1 g}{\sqrt{2 g y_1}}$.

All the values in that expression are finite and do not approach 0, except for y_1 which is very close to 0. Thus the whole expression yields a large number that approaches ∞ as $y_1 \to 0$.

Surely we don't need an infinite initial velocity to cover a finite distance. Consider a trajectory where the ball goes through the window on its way down. See Fig. 4.

The maximum y is not important, what is important is that we can make it as small or as large as we want to, i.e. it doesn't have to approach 0.

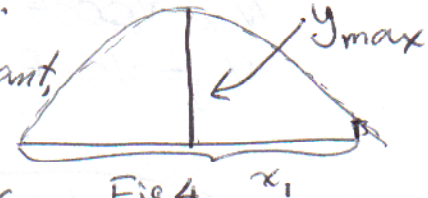


Fig.4

Now $v_{oy} = \sqrt{2gy_{max}}$ and

$$v_{ox} = \frac{x_1 g}{\sqrt{2gy_{max}}}$$

All the values in the two expressions are finite and don't approach 0. Thus, both are finite, so v is finite, which seems to be better than the infinite velocity yielded by the book's strategy. Thus, we have shown that the book's strategy won't work for all cases, and maybe it won't work for our specific example.

III - Now we are going to try to find a solution for the problem where the trajectory's max point isn't at the window. See Fig. 5a, b.

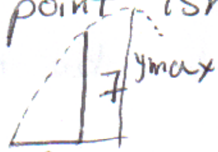


Fig.5a

Fig. 5b

We can't conclusively say yet wether the ball will go through the window before or or after it reaches the peak of its flight, but we'll be able to answer that after we solve the main problem. ~~Find the minimum velocity th~~

$V_{ox} = V\cos\theta$

$V_{oy} = V\sin\theta$

$y = \cancel{y_0} + v\sin\theta t + \frac{g}{2}t^2 = 7$ $(y_0 = 0, g = -9.81\ m/s)$

$= v\sin\theta t + \frac{g}{2}t^2 = 7$

$x = \boxed{v\cos\theta t = 2}$

$\hookrightarrow t = \frac{2}{v\cos\theta}$

The problem: find the initial velocity of a trajectory that passes through (2,7) that minimizes that said initial velocity.

Now plugging t into y.

$$y = v\sin\left(\frac{2}{\cos\theta v}\right) + \frac{g}{2}\left(\frac{2}{\cos\theta v}\right)$$

$$\boxed{= 2\tan\theta + \frac{2g}{\cos^2\theta v^2} = 7}$$

$$\frac{2g}{\cos^2\theta v^2} = 7 - 2\tan\theta$$

$$2g = v^2(7\cos^2\theta - 2\sin\theta\cos\theta)$$

$$\boxed{v(\theta) = \sqrt{\frac{2g}{7\cos^2\theta - 2\sin\theta\cos\theta}}}$$

Now we have an explicit equation that relates all angle-velocity pairs that pass through (2,7). We need to minimize v, so $|7\cos^2\theta - 2\sin\theta\cos\theta|$ need to be maximized. Because $2g < 0$, that means $7\cos^2\theta - 2\sin\theta\cos\theta < 0$, $\therefore 7\cos^2\theta - 2\sin\theta\cos\theta$ needs to be minimized

$f(\theta) = 7\cos^2\theta - 2\sin\theta\cos\theta$, Double angle formulas yield: $f(\theta) = 7\left(\frac{1+\cos 2\theta}{2}\right) - \sin 2\theta$

$$= \frac{7}{2} + \frac{7}{2}\cos 2\theta - \sin 2\theta \quad \text{, let } 2\theta = \phi$$

$$f(\phi) = 7/2 + 7/2\cos\phi - \sin\phi$$

$$f'(\phi) = -7/2\sin\phi - \cos\phi = 0$$

$$\tan\phi = -2/7$$

$$\phi = \tan^{-1}(-2/7) = 164.055^\circ$$ ← only result in the right range

$$\theta = \frac{\phi}{2} = \frac{164.055}{2} = \boxed{82.0273^\circ}$$

$$\boxed{\begin{array}{l}\theta : [0, 90] \\ \therefore \phi : [0, 180]\end{array}}$$

By the second derivative: $f''(\phi) = -7/2\cos\phi + \sin\phi$

$$f''(164.055) = -7/5\cos(164.055) + \sin(164.055)$$
$$= 3.64005 > 0$$

$\therefore \phi = 164.055^\circ$ is a minimum of $f(\phi)$. ✓✓

Now by substituting $\theta = 82.0273^\circ$

$$v(82.0273) = \sqrt{\frac{2g}{7\cos^2(82.0273) - 2\sin(82.0273)\cos(82.0273)}} = \sqrt{\frac{-19.62}{-.14005}}$$

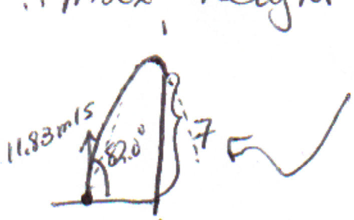
$$= \sqrt{140.088} = \boxed{11.8359\ m/s}$$

$11.8359 < 11.8382$ (the book's solution).

Now, one final thing to sort out, does the ball go through the window before or after it reaches y_{max}.

$$x_{max} = \frac{-V_{0x}V_{0y}}{g} = \frac{-v^2\cos\theta\sin\theta}{g} = \boxed{1.962\ m}$$

$\therefore$ Max height before the window



correct trajectory