

TAM 203 Spring '06

Bonus problem I

assigned Jan 26, done Feb 12

This problem concerns example 2.3 from Tongue and Sheppard page 25

In physics I you were taught the formula  $v_2^2 = v_1^2 + 2\bar{a}(x_2 - x_1)$ . The formula basically states that given the distance traveled and the starting and ending velocities the value of acceleration could be obtained. You were probably taught that  $\bar{a}$  is constant acceleration when in fact  $\bar{a}$  is really the average acceleration regardless of whether acceleration is really constant. We can prove that by starting with the definition of acceleration (This duplicates the derivation in the textbook on page 24).

$$\frac{dv}{dt} = a$$

We can manipulate the equation by multiplying both sides by  $v$ .

$$v \frac{dv}{dt} = av$$

$$v dv = a v dt$$

$v dt$  is  $dx$  given the change in time multiplied by the velocity is the change in the distance.

$$v dv = a dx$$

We can then integrate both sides:

$$\int_{v_1}^{v_2} v dv = \int_{x_1}^{x_2} a dx$$

We will then look at the definition of the average value of a function. It is the integral divided by its interval:

$$\bar{a} = \frac{\int_{x_1}^{x_2} a dx}{(x_2 - x_1)}$$

$$\bar{a}(x_2 - x_1) = \int_{x_1}^{x_2} a dx$$

Now we substitute and get the formula:

$$\frac{v_2^2}{2} - \frac{v_1^2}{2} = \bar{a}(x_2 - x_1)^*$$

So indeed the  $\bar{a}$  is the value of the average acceleration rather than constant acceleration. Note, the textbook vaguely states that  $\bar{a}$  is a constant without noting that the formula is exact for an arbitrary acceleration function. Let's look at example 2.3 on page 25. By using the formula the value of the average acceleration turns out to be  $3.86 \text{ m/s}^2$ . Let's see if we get this answer using  $a(t) \neq \text{constant}$ . For example, acceleration is linear:

$$a(t) = ct + c_2 \quad c_2 = 0 \text{ since initial acceleration is 0.}$$

$$\Rightarrow a(t) = ct$$

We integrate to get velocity:

$$v(t) = \frac{1}{2} ct^2 + c_3 \quad c_3 = 0 \text{ since initial velocity is 0.}$$

$$\Rightarrow v(t) = \frac{1}{2} ct^2$$

We integrate to get position:

$$x(t) = \frac{1}{6} ct^3 + c_4 \quad c_4 = 0 \text{ since initial position is 0.}$$

$$\Rightarrow x(t) = \frac{1}{6} ct^3$$

We are given the final velocity is  $100 \text{ km/hr}$ .

$$(100 \text{ km/hr}) \left( \frac{1000 \text{ m/km}}{3600 \text{ s/hr}} \right) = \frac{1}{2} ct^2$$

$$27.7 \text{ m/s} = \frac{1}{2} ct^2$$

We know the final position is  $100 \text{ m}$ :

$$100 \text{ m} = \frac{1}{6} ct^3$$

We solve the system of equations to obtain the value of the finishing time:

$$\begin{cases} 55.55 \text{ m/s} = ct^2 & \textcircled{1} \\ 600 \text{ m} = ct^3 & \textcircled{2} \end{cases} \Rightarrow \frac{600 \text{ m}}{t^3} = c \quad \textcircled{3}$$

Substitute  $\textcircled{3}$  into  $\textcircled{1}$ :

$$55.55 \text{ m/s} = \frac{600 \text{ m}}{t^3} t^2 \Rightarrow t = 10.8 \text{ s} \Rightarrow \frac{600 \text{ m}}{(10.8 \text{ s})^3} = 0.476 = c$$

$$a(t) = 0.476 t$$

We can now find the average acceleration.

$$\bar{a} = \frac{1}{10.8} \int_0^{10.8} 0.476 t dt$$

$$= \frac{1}{10.8} (0.476 \cdot \frac{1}{2} t^2) \Big|_0^{10.8} = 2.57 \text{ m/s}^2$$

$$\bar{a} = 2.57 \text{ m/s}^2 \neq 3.86 \text{ m/s}^2 \text{ (The answer using eqn. *) ?!!}$$

This seems like a paradox since we are solving for the average acceleration in both cases. Why does eqn \* not give the same  $\bar{a}$  for this linearly varying acceleration?

Let's look again at our  $a(t) = 0.476t$ . We have 2 different definitions of  $\bar{a}$ . The formula uses

$$\bar{a} = \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} a \, dx$$

$$\text{We use } \bar{a} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a \, dt.$$

$$\text{We can calculate: } \bar{a} = \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} a \, dx$$

We start by writing  $a(t)$  in terms of  $x$ :

$$x(t) = \frac{1}{6} \cdot 0.476 t^3$$

$$\frac{6x}{0.476} = t^3 \Rightarrow \left( \frac{6x}{0.476} \right)^{\frac{1}{3}} = t$$

We substitute  $t$  into the acceleration function:

$$a(x) = 0.476 \left( \frac{6x}{0.476} \right)^{\frac{1}{3}}$$

We substitute  $a(x)$  into the  $\bar{a}$  function.

$$\bar{a} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} 0.476 \left( \frac{6x}{0.476} \right)^{\frac{1}{3}} dx$$

$$= \frac{1}{100\text{m} - 0\text{m}} \int_{0\text{m}}^{100\text{m}} 0.476 \left( \frac{6x}{0.476} \right)^{\frac{1}{3}} dx$$

$$= \frac{1}{100\text{m}} \left( 1.108 \cdot \frac{3}{4} x^{\frac{4}{3}} \right) \Big|_0^{100}$$

$$= \frac{1}{100\text{m}} 1.108 \cdot \frac{3}{4} (100)^{\frac{4}{3}}$$

$$= 3.86 \text{ m/s}^2 \text{ (the answer using the formula.)}$$

Where is the paradox ?!!

We saw before that using  $\bar{a} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a dt$  we get  $2.57 \text{ m/s}^2$ .

So here's the resolution: The textbook solution to sample 2.3 is correct for  $\bar{a}$  associated with the position of the car.

So as long as  $\bar{a} \equiv \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} a dx$  it works but not for the

more usual definition of average  $\bar{a} \equiv \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a dt$ .