Updated on February 24

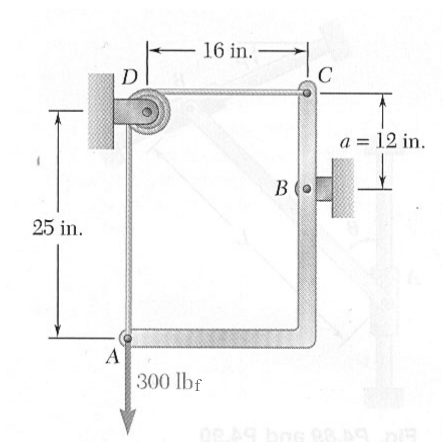
ENGRD/TAM 203: Dynamics (Spring 2006)

Solution of Homework 1 (assigned on Jan. 24, due on Jan. 31)

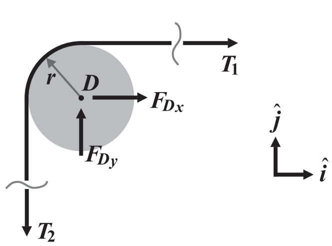
by Dennis Yang

1. Problem 4.81 from Beer, Johnston, and Eisenberg, page 189.

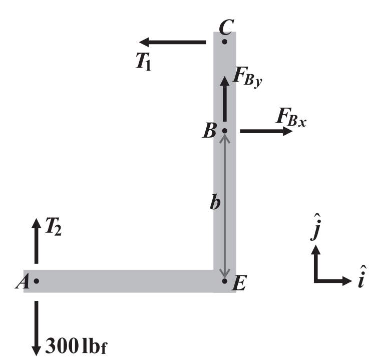
Statement. Member ABC is supported by a pin and bracket at B and by an inextensible cord attached at A and C and passing over a frictionless pulley at D (see the figure below.) The tension may be assumed to be the same in portion AD and CD of the cord. For the loading shown and neglecting the size of the pulley, determine the tension in the cord and the reaction at B.



Free Body Diagrams.



FDB 1.



FBD 2.

Solution. In FBD 1, the pulley is at static equilibrium. Thus the sum of moments about point D

is zero, i.e., $\sum_i \vec{M}_{i/D} = \vec{0}$, from which we have

$$\begin{aligned}\sum_i \vec{M}_{i/D} = \vec{0} &\implies r\hat{\jmath} \times T_1\hat{\imath} + r(-\hat{\imath}) \times T_2(-\hat{\jmath}) = \vec{0} \\ &\implies -rT_1\hat{k} + rT_2\hat{k} = \vec{0} \\ &\implies (-rT_1 + rT_2)\hat{k} = \vec{0}\,. \end{aligned} \tag{1.1}$$

Taking dot products on the both sides of (1.1) with $\hat{k}$ gives

$$\begin{aligned}(1.1) \bullet \hat{k} &\implies (-rT_1 + rT_2)\hat{k} \bullet \hat{k} = \vec{0} \bullet \hat{k} \\ &\implies -rT_1 + rT_2 = 0\,,\end{aligned}$$

which immediately yields that $T_1 = T_2$. This is indeed why we can assume the tension to be the same in portion AD and CD of the cord.

Now in FBD 2, the member is at static equilibrium. Thus the sum of the forces on the member is zero, i.e., $\sum_i \vec{F}_i = \vec{0}$, from which we have

$$\begin{aligned}\sum_i \vec{F}_i = \vec{0} &\implies (F_{Bx}\hat{\imath} + F_{By}\hat{\jmath}) + T_1(-\hat{\imath}) + T_2\hat{\jmath} + 300\,\mathrm{lb_f}(-\hat{\jmath}) = \vec{0} \\ &\implies (F_{Bx} - T_1)\hat{\imath} + (F_{By} + T_2 - 300\,\mathrm{lb_f})\hat{\jmath} = \vec{0}\,.\end{aligned} \tag{1.2}$$

We dot product the both sides of (1.2) with $\hat{\imath}$ to obtain

$$\begin{aligned}(1.2) \bullet \hat{\imath} &\implies (F_{Bx} - T_1)\hat{\imath} \bullet \hat{\imath} + (F_{By} + T_2 - 300\,\mathrm{lb_f})\hat{\jmath} \bullet \hat{\imath} = \vec{0} \bullet \hat{\imath} \\ &\implies F_{Bx} - T_1 = 0\,.\end{aligned} \tag{1.3}$$

Next, we dot product the both sides of (1.2) with $\hat{\jmath}$ to obtain

$$\begin{aligned}(1.2) \bullet \hat{\jmath} &\implies (F_{Bx} - T_1)\hat{\imath} \bullet \hat{\jmath} + (F_{By} + T_2 - 300\,\mathrm{lb_f})\hat{\jmath} \bullet \hat{\jmath} = \vec{0} \bullet \hat{\jmath} \\ &\implies F_{By} + T_2 - 300\,\mathrm{lb_f} = 0\,.\end{aligned} \tag{1.4}$$

In addition, the sum of the moments on the member about point B is zero, i.e., $\sum_i \vec{M}_{i/B} = \vec{0}$, from which we have

$$\sum_i \vec{M}_{i/B} = \vec{0} \implies \vec{R}_{C/B} \times T_1(-\hat{\imath}) + \vec{R}_{A/B} \times T_2\hat{\jmath} + \vec{R}_{A/B} \times 300\,\mathrm{lb_f}(-\hat{\jmath}) = \vec{0}\,. \tag{1.5}$$

Since both $F_{Bx}\hat{\imath}$ and $F_{By}\hat{\jmath}$ pass through point B, they make no appearance in (1.5). Furthermore, by the given dimensions of the member, $\vec{R}_{C/B} = 12\,\mathrm{in}\,\hat{\jmath}$ and $\vec{R}_{A/B} = \vec{R}_{E/B} + \vec{R}_{A/E} = b(-\hat{\jmath}) + 16\,\mathrm{in}\,(-\hat{\imath})$, where $\vec{R}_{A/E} = 16\,\mathrm{in}\,(-\hat{\imath})$ and $\vec{R}_{E/B} = b(-\hat{\jmath})$ with b being the distance between point B and point E. Thus,

$$\begin{aligned}(1.5) &\implies 12\,\mathrm{in}\,\hat{\jmath} \times T_1(-\hat{\imath}) + \Big(b(-\hat{\jmath}) + 16\,\mathrm{in}\,(-\hat{\imath})\Big) \times T_2\hat{\jmath} + \Big(b(-\hat{\jmath}) + 16\,\mathrm{in}\,(-\hat{\imath})\Big) \times 300\,\mathrm{lb_f}(-\hat{\jmath}) = \vec{0} \\ &\implies 12\,\mathrm{in} \cdot T_1\hat{k} + 16\,\mathrm{in} \cdot T_2(-\hat{k}) + 16\,\mathrm{in} \cdot 300\,\mathrm{lb_f}\,\hat{k} = \vec{0} \\ &\implies \Big(12\,\mathrm{in} \cdot T_1 - 16\,\mathrm{in} \cdot T_2 + 16\,\mathrm{in} \cdot 300\,\mathrm{lb_f}\Big)\hat{k} = \vec{0}\,.\end{aligned} \tag{1.6}$$

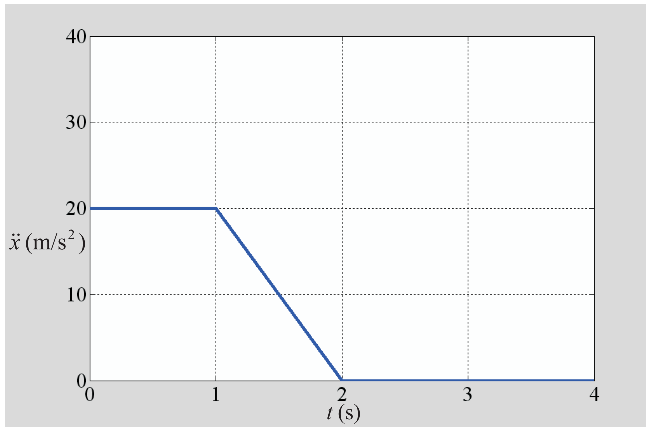
Again, we dot product the both sides of (1.6) with $\hat{k}$ to obtain

$$\begin{aligned}(1.6) \bullet \hat{k} &\implies \Big(12\,\text{in} \cdot T_1 - 16\,\text{in} \cdot T_2 + 16\,\text{in} \cdot 300\,\text{lb}_\text{f}\Big)\hat{k} \bullet \hat{k} = \vec{0} \bullet \hat{k} \\ &\implies 12\,\text{in} \cdot T_1 - 16\,\text{in} \cdot T_2 + 16\,\text{in} \cdot 300\,\text{lb}_\text{f} = 0\,. \qquad (1.7)\end{aligned}$$

With the fact that $T_1 = T_2$, solving (1.7) gives $T_1 = T_2 = 1200\,\text{lb}_\text{f}$. Substituting this result to (1.3) and (1.4), we can easily obtain $F_{Bx} = 1200\,\text{lb}_\text{f}$ and $F_{By} = -900\,\text{lb}_\text{f}$. Therefore, the tension in the cord is $1200\,\text{lb}_\text{f}$ and the reaction at B is $F_{Bx}\hat{\imath} + F_{By}\hat{\jmath} = 1200\,\text{lb}_\text{f}\,\hat{\imath} - 900\,\text{lb}_\text{f}\,\hat{\jmath}$. □

2. Problem 2.2.6.

Statement. A plot of acceleration versus time for a particle is shown below. What's the difference between its position at $t = 4\,\mathrm{s}$ and $t = 0\,\mathrm{s}$ if $\dot{x}(0\,\mathrm{s}) = -4\,\mathrm{m/s}$?



Solution. For $0\,\mathrm{s} \le t \le 1\,\mathrm{s}$, by inspection we have that

$$\ddot{x}(t) = 20\,\mathrm{m/s^2}\,.$$

For $1\,\mathrm{s} < t < 2\,\mathrm{s}$, the slope of the line segment is given by

$$\frac{\ddot{x}(t) - \ddot{x}(1\,\mathrm{s})}{t - 1\,\mathrm{s}} = \frac{\ddot{x}(2\,\mathrm{s}) - \ddot{x}(1\,\mathrm{s})}{2\,\mathrm{s} - 1\,\mathrm{s}} = \frac{0\,\mathrm{m/s^2} - 20\,\mathrm{m/s^2}}{2\,\mathrm{s} - 1\,\mathrm{s}} = -20\,\mathrm{m/s^3}\,,$$

which yields that

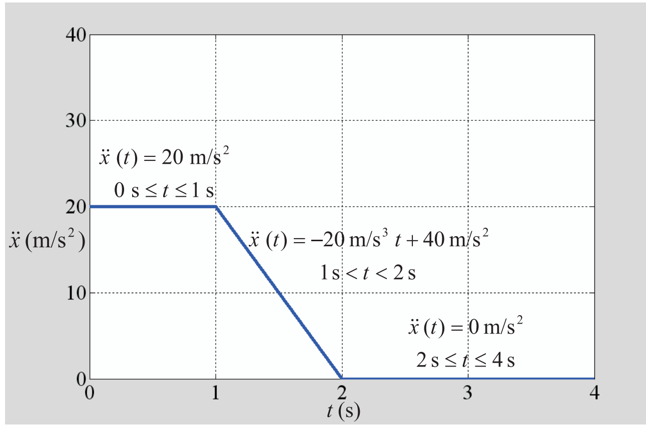
$$\begin{aligned} \ddot{x}(t) &= -20\,\mathrm{m/s^3} \cdot (t - 1\,\mathrm{s}) + \ddot{x}(1\,\mathrm{s}) \\ &= -20\,\mathrm{m/s^3} \cdot (t - 1\,\mathrm{s}) + 20\,\mathrm{m/s^2} \\ &= -20\,\mathrm{m/s^3} \cdot t + 40\,\mathrm{m/s^2}\,. \end{aligned}$$

For $2\,\mathrm{s} \le t \le 4\,\mathrm{s}$, by inspection we have that

$$\ddot{x}(t) = 0\,\mathrm{m/s^2}\,.$$

Thus, the expression of $\ddot{x}(t)$ for $0\,\mathrm{s} \le t \le 4\,\mathrm{s}$ is (also see the following figure)

$$\boxed{\ddot{x}(t) = \begin{cases} 20\,\mathrm{m/s^2} & \text{if } 0\,\mathrm{s} \le t \le 1\,\mathrm{s}\,, \\ -20\,\mathrm{m/s^3} \cdot t + 40\,\mathrm{m/s^2} & \text{if } 1\,\mathrm{s} < t < 2\,\mathrm{s}\,, \\ 0\,\mathrm{m/s^2} & \text{if } 2\,\mathrm{s} \le t \le 4\,\mathrm{s}\,. \end{cases}} \tag{2.1}$$



Now we integrate (2.1) to obtain the expression of $\dot{x}(t)$. For $0\,\mathrm{s} \le t \le 1\,\mathrm{s}$, we have

$$\begin{aligned}\dot{x}(t) &= \dot{x}(0\,\mathrm{s}) + \int_{0\,\mathrm{s}}^{t} \ddot{x}(\tau)\,d\tau \\ &= -4\,\mathrm{m/s} + \int_{0\,\mathrm{s}}^{t} 20\,\mathrm{m/s^2}\,d\tau \\ &= -4\,\mathrm{m/s} + 20\,\mathrm{m/s^2}\cdot t\,.\end{aligned}$$

It follows that $\dot{x}(1\,\mathrm{s}) = -4\,\mathrm{m/s} + 20\,\mathrm{m/s^2}\cdot 1\,\mathrm{s} = 16\,\mathrm{m/s}$. For $1\,\mathrm{s} < t < 2\,\mathrm{s}$, we have

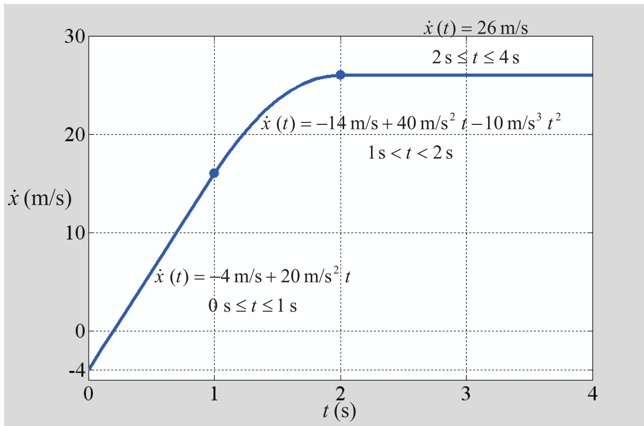
$$\begin{aligned}\dot{x}(t) &= \dot{x}(1\,\mathrm{s}) + \int_{1\,\mathrm{s}}^{t} \ddot{x}(\tau)\,d\tau \\ &= 16\,\mathrm{m/s} + \int_{1\,\mathrm{s}}^{t} \left(-20\,\mathrm{m/s^3}\cdot\tau + 40\,\mathrm{m/s^2}\right)\,d\tau \\ &= 16\,\mathrm{m/s} + \left(-10\,\mathrm{m/s^3}\cdot\tau^2 + 40\,\mathrm{m/s^2}\cdot\tau\right)\Big|_{1\,\mathrm{s}}^{t} \\ &= -14\,\mathrm{m/s} + 40\,\mathrm{m/s^2}\cdot t - 10\,\mathrm{m/s^3}\cdot t^2\,.\end{aligned}$$

It follows that $\dot{x}(2\,\mathrm{s}) = -14\,\mathrm{m/s} + 40\,\mathrm{m/s^2}\cdot 2\,\mathrm{s} - 10\,\mathrm{m/s^3}\cdot(2\,\mathrm{s})^2 = 26\,\mathrm{m/s}$. Then, for $2\,\mathrm{s} \le t \le 4\,\mathrm{s}$,

$$\begin{aligned}\dot{x}(t) &= \dot{x}(2\,\mathrm{s}) + \int_{2\,\mathrm{s}}^{t} \ddot{x}(\tau)\,d\tau \\ &= 26\,\mathrm{m/s} + \int_{2\,\mathrm{s}}^{t} 0\,\mathrm{m/s^2}\,d\tau \\ &= 26\,\mathrm{m/s}\,.\end{aligned}$$

Thus, the expression of $\dot{x}(t)$ for $0\,\mathrm{s} \le t \le 4\,\mathrm{s}$ is (also see the following figure)

$$\boxed{\dot{x}(t) = \begin{cases} -4\,\mathrm{m/s} + 20\,\mathrm{m/s^2}\cdot t & \text{if } 0\,\mathrm{s} \le t \le 1\,\mathrm{s}\,, \\ -14\,\mathrm{m/s} + 40\,\mathrm{m/s^2}\cdot t - 10\,\mathrm{m/s^3}\cdot t^2 & \text{if } 1\,\mathrm{s} < t < 2\,\mathrm{s}\,, \\ 26\,\mathrm{m/s} & \text{if } 2\,\mathrm{s} \le t \le 4\,\mathrm{s}\,. \end{cases}} \tag{2.2}$$



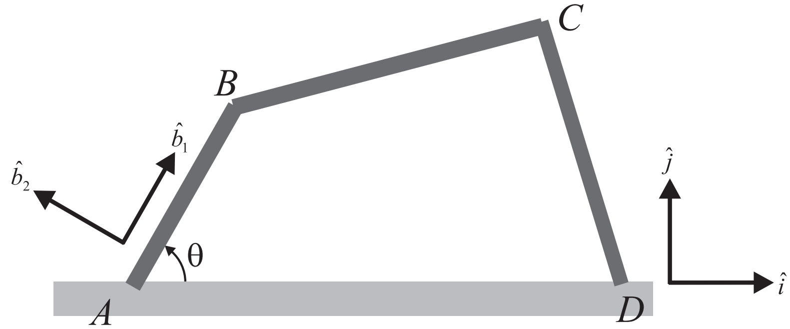
Finally, the difference between the position at $t = 4\,\mathrm{s}$ and $t = 0\,\mathrm{s}$, i.e., $x(4\,\mathrm{s}) - x(0\,\mathrm{s})$, is obtained by integrating (2.2) from $t = 0\,\mathrm{s}$ to $t = 4\,\mathrm{s}$,

$$
\begin{aligned}
x(4\,\mathrm{s}) - x(0\,\mathrm{s}) &= \int_{0\,\mathrm{s}}^{4\,\mathrm{s}} \dot{x}(\tau)\,d\tau \\
&= \int_{0\,\mathrm{s}}^{1\,\mathrm{s}} \dot{x}(\tau)\,d\tau + \int_{1\,\mathrm{s}}^{2\,\mathrm{s}} \dot{x}(\tau)\,d\tau + \int_{2\,\mathrm{s}}^{4\,\mathrm{s}} \dot{x}(\tau)\,d\tau \\
&= \int_{0\,\mathrm{s}}^{1\,\mathrm{s}} \left(-4\,\mathrm{m/s} + 20\,\mathrm{m/s}^2 \cdot \tau\right)\,d\tau + \int_{1\,\mathrm{s}}^{2\,\mathrm{s}} \left(-14\,\mathrm{m/s} + 40\,\mathrm{m/s}^2 \cdot \tau - 10\,\mathrm{m/s}^3 \cdot \tau^2\right)\,d\tau \\
&\quad + \int_{2\,\mathrm{s}}^{4\,\mathrm{s}} 26\,\mathrm{m/s}\,d\tau \\
&= \left(-4\,\mathrm{m/s} \cdot \tau + 10\,\mathrm{m/s}^2 \cdot \tau^2\right)\Big|_{0\,\mathrm{s}}^{1\,\mathrm{s}} + \left(-14\,\mathrm{m/s} \cdot \tau + 20\,\mathrm{m/s}^2 \cdot \tau^2 - \frac{10}{3}\,\mathrm{m/s}^3 \cdot \tau^3\right)\Big|_{1\,\mathrm{s}}^{2\,\mathrm{s}} \\
&\quad + 26\,\mathrm{m/s} \cdot \tau\Big|_{2\,\mathrm{s}}^{4\,\mathrm{s}} \\
&= \underline{80\frac{2}{3}\,\mathrm{m}}\,.
\end{aligned}
$$

□

3. Problem 2.2.20.

Statement. Construct a coordinate transformation array from $\hat{\imath}$, $\hat{\jmath}$ to $\hat{b}_1$, $\hat{b}_2$ and express the vector $\vec{p} = 4\hat{\imath} - 8\hat{\jmath}$ in terms of $\hat{b}_1$, $\hat{b}_2$ for $\theta = 130°$. The $\hat{b}_1$, $\hat{b}_2$ unit vectors are attached to the illustrated link $\overline{AB}$.



Solution. Since $\hat{\imath}$, $\hat{\jmath}$ are a set of orthogonal unit vectors (i.e., they are of unit length and perpendicular to each other,) we can express $\hat{b}_1$, $\hat{b}_2$ in terms of $\hat{\imath}$, $\hat{\jmath}$ as

$$\begin{aligned} \hat{b}_1 &= (\hat{b}_1 \bullet \hat{\imath})\,\hat{\imath} + (\hat{b}_1 \bullet \hat{\jmath})\,\hat{\jmath}\,, \\ \hat{b}_2 &= (\hat{b}_2 \bullet \hat{\imath})\,\hat{\imath} + (\hat{b}_2 \bullet \hat{\jmath})\,\hat{\jmath}\,. \end{aligned}$$

On the other hand, $\hat{b}_1$, $\hat{b}_2$ are also a set of orthogonal unit vectors. Thus we can express $\hat{\imath}$, $\hat{\jmath}$ in terms of $\hat{b}_1$, $\hat{b}_2$ as

$$\begin{aligned} \hat{\imath} &= (\hat{b}_1 \bullet \hat{\imath})\,\hat{b}_1 + (\hat{b}_2 \bullet \hat{\imath})\,\hat{b}_2\,, \\ \hat{\jmath} &= (\hat{b}_1 \bullet \hat{\jmath})\,\hat{b}_1 + (\hat{b}_2 \bullet \hat{\jmath})\,\hat{b}_2\,. \end{aligned}$$

Therefore, to construct the coordinate transformation array

	$\hat{\imath}$	$\hat{\jmath}$
$\hat{b}_1$	$\hat{b}_1 \bullet \hat{\imath}$	$\hat{b}_1 \bullet \hat{\jmath}$
$\hat{b}_2$	$\hat{b}_2 \bullet \hat{\imath}$	$\hat{b}_2 \bullet \hat{\jmath}$

(3.1)

we need to compute four dot products $\hat{b}_1 \bullet \hat{\imath}$, $\hat{b}_1 \bullet \hat{\jmath}$, $\hat{b}_2 \bullet \hat{\imath}$, and $\hat{b}_2 \bullet \hat{\jmath}$. (**Note: (3.1) is the more general form of the coordiante transformation array. However, it is NOT in the textbook.**) Specifically, as shown in the figure on the next page, the angle between $\hat{\imath}$ and $\hat{b}_1$ is θ. Thus,

$$\hat{b}_1 \bullet \hat{\imath} = \cos\theta\,. \tag{3.2}$$

The angle between $\hat{\jmath}$ and $\hat{b}_1$ is $90° - \theta$. Thus,

$$\hat{b}_1 \bullet \hat{\jmath} = \cos(90° - \theta) = \sin\theta\,. \tag{3.3}$$

The angle between $\hat{\imath}$ and $\hat{b}_2$ is $\theta + 90°$. Thus,

$$\hat{b}_2 \bullet \hat{\imath} = \cos(\theta + 90°) = -\sin\theta\,. \tag{3.4}$$

The angle between $\hat{\jmath}$ and $\hat{b}_2$ is θ. Thus,

$$\hat{b}_2 \bullet \hat{\jmath} = \cos\theta\,. \tag{3.5}$$

Substituting (3.2)–(3.5) into (3.1), we obtain

	$\hat{\imath}$	$\hat{\jmath}$
$\hat{b}_1$	$\cos\theta$	$\sin\theta$
$\hat{b}_2$	$-\sin\theta$	$\cos\theta$

Now, the vector $\vec{p}$ in terms of $\hat{b}_1$, $\hat{b}_2$ is

$$\begin{aligned}\vec{p} &= 4\hat{\imath} - 8\hat{\jmath}\\ &= 4(\cos\theta\,\hat{b}_1 - \sin\theta\,\hat{b}_2) - 8(\sin\theta\,\hat{b}_1 + \cos\theta\,\hat{b}_2)\\ &= (4\cos\theta - 8\sin\theta)\,\hat{b}_1 + (-4\sin\theta - 8\cos\theta)\,\hat{b}_2\,.\end{aligned} \tag{3.6}$$

Evaluating (3.6) at $\theta = 130°$ yields that

$$\boxed{\vec{p} \approx -8.70\hat{b}_1 + 2.08\hat{b}_2\,.}$$

□