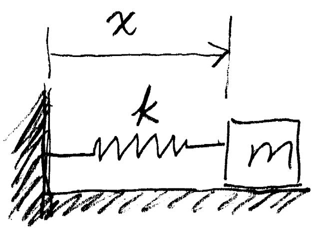
ENGRD/TAM 203 Spring 2006

HW13 (Assigned on Mar 7, due on Mar 14)

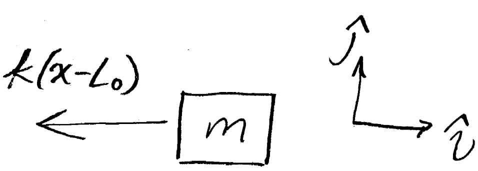
Solution by Dennis Yang

4.1.2 A 0.5 kg mass block is attached to a extensional spring with spring constant $k = 40$ N/m. The unstretched length of the spring is 0.3 m, and when initially released the mass is 2 m from the vertical wall. If released from rest on a frictionless surface, how fast will the mass be moving when it is 0.6 m from the wall?



Solution

F.B.D.



("L_0" is the original length of the spring)

$$\frac{1}{2} m V_2^2 = \frac{1}{2} m V_1^2 + W$$

$$= \frac{1}{2} m V_1^2 + \int_{x_1}^{x_2} \left(-k(x - L_0)\right) dx$$

$$= \frac{1}{2} m V_1^2 - k\left(\frac{x^2}{2} - L_0 x\right)\Big|_{x_1}^{x_2}$$

$$= \frac{1}{2} m V_1^2 + k\left(\frac{x_1^2}{2} - L_0 x_1\right) - k\left(\frac{x_2^2}{2} - L_0 x_2\right)$$

In our case, $V_1 = 0$ since it starts at rest.

Thus, $\frac{1}{2} m V_2^2 = k\left(\frac{x_1^2}{2} - L_0 x_1\right) - k\left(\frac{x_2^2}{2} - L_0 x_2\right)$

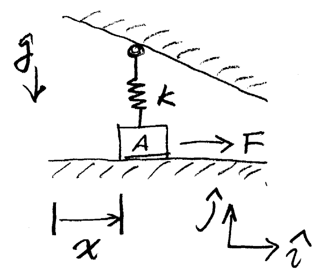
$$\Longrightarrow \quad V_2 = \sqrt{\frac{k}{m}\left(x_1^2 - x_2^2 - 2L_0(x_1 - x_2)\right)}$$

$$= \sqrt{\frac{40\ N/m}{0.5\ kg}\left((2m)^2 - (0.6m)^2 - 2 * 0.3m * (2m - 0.6m)\right)}$$

$$\boxed{V_2 \approx 14.97\ m/s}$$

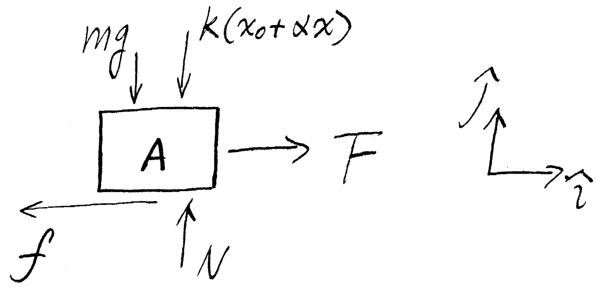
$\square$

4.1.14 Block A ($m_A = 2$ kg) is acted by $\vec{F} = 12\,N\,\hat{\imath}$ and it moves to the right. The force exerted by the string on A is given by $k(x_0 + \alpha x)$, with $k = 100$ N/m, $x_0 = 0.05$ m, and $\alpha = 0.27$. $\mu_s = \mu_d = 0.4$. Assume tha A starts from rest at $x = 0$, what is its speed at the point that the force resisting its motion is equal in magnitude to $\vec{F}$? Assume that the spring/wall interface is frictionless.



Solution

F.B.D.



$$\sum_i \vec{F}_i = m\vec{a} \Rightarrow F\hat{\imath} + N\hat{\jmath} + mg(-\hat{\jmath}) + k(x_0 + \alpha x)(-\hat{\jmath}) + f(-\hat{\imath})$$
$$= m\ddot{x}\,\hat{\imath} \qquad (*)$$

$$(*) \cdot \hat{\jmath} \Rightarrow N - mg - k(x_0 + \alpha x) = 0$$
$$\Rightarrow N = mg + k(x_0 + \alpha x)$$

Thus, $f = N\mu_d = \big(mg + k(x_0 + \alpha x)\big)\mu_d$

Note that $d\vec{r} = dx(\hat{\imath})$ when the mass block moves to the right.

$$(*) \cdot d\vec{r} \Longrightarrow F\hat{\imath}\cdot dx\hat{\imath} + N\hat{\jmath}\cdot dx\hat{\imath} + mg(-\hat{\jmath})\cdot dx\hat{\imath} + k(x_0+\alpha x)(-\hat{\jmath})\cdot dx\hat{\imath} + f(-\hat{\imath})\cdot dx\hat{\imath} = m\ddot{x}\,\hat{\imath}\cdot dx\hat{\imath}$$

$$\Longrightarrow F\,dx - f\,dx = m\ddot{x}\,dx$$

$$\Longrightarrow F\,dx - f\,dx = m\dot{x}\,d\dot{x}$$

$$\Longrightarrow (F-f)\,dx = \tfrac{1}{2}m\,d(\dot{x}^2)$$

$$\Longrightarrow \left[F - (mg + k(x_0+\alpha x))\mu_d\right]dx = \tfrac{1}{2}m\,d(\dot{x}^2) \qquad (**)$$

Now integrate both sides of $(**)$:

$$\int_{x_1=0}^{x_2}\left[F-(mg+k(x_0+\alpha x))\mu_d\right]dx = \int_{\dot{x}_1^2=0}^{\dot{x}_2^2}\tfrac{1}{2}m\,d(\dot{x}^2)$$

(it starts from rest)

$$\Longrightarrow \boxed{F x_2 - mg\mu_d x_2 - k x_0 \mu_d x_2 - \tfrac{1}{2}k\alpha\mu_d x_2^2 = \tfrac{1}{2}m\dot{x}_2^2}$$

$(***)$

$$(***) \Longrightarrow \left(F - mg\mu_d - kx_0\mu_d - \tfrac{1}{2}k\alpha\mu_d x_2\right)x_2$$
$$= \tfrac{1}{2}m\dot{x}_2^2 \qquad (****)$$

In our case, we want to evaluate above at x_2 such that $F = f$ (i.e., the resisting force equal to the magnitude of F). That is,

$$F = f = (mg + k(x_0 + \alpha x_2))\mu_d$$

$$\Longrightarrow F = mg\mu_d + kx_0\mu_d + k\alpha\mu_d x_2 \quad ①$$

Substitute this to $(****)$, we have

$$\tfrac{1}{2}k\alpha\mu_d x_2 \times x_2 = \tfrac{1}{2}m\dot{x}_2^2$$

$$\Longrightarrow \dot{x}_2 = \sqrt{\frac{k}{m}\alpha\mu_d x_2^2} = \sqrt{\frac{k}{m}\alpha\mu_d}\cdot x_2,$$

where x_2 satisfies ①, i.e.,

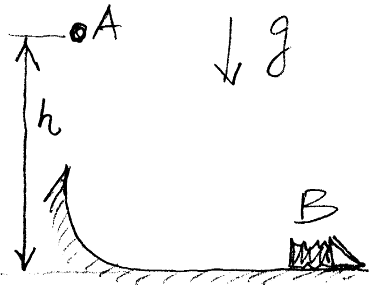
$$x_2 = \frac{F - mg\mu_d - kx_0\mu_d}{k\alpha\mu_d}$$

Therefore, $\boxed{\dot{x}_2 = \sqrt{\frac{1}{mk\alpha\mu_d}}\cdot(F - mg\mu_d - kx_0\mu_d)}$

$$\dot{x}_2 = \sqrt{\frac{1}{2\,kg \cdot 100\,N/m \cdot 0.27 \cdot 0.4}} \cdot \left(12\,N - 2\,kg \cdot 9.81\,m/s^2 \cdot 0.4 - 100\,N/m \cdot 0.05\,m \cdot 0.4\right)$$

$$\boxed{\dot{x}_2 \approx 0.463 \ m/s}$$

4.2.4 A 0.2 kg mass particle A is dropped from $h = 1.5$ m. It smoothly contacts the curve (no friction) track ($r = 0.5$m), and slides along the ground, and finally impacts an elastic cushion B. The spring constant of the cushion is 40 N/m. What's the cushion's maximum compression?



Solution

The track is frictionless

$\Longrightarrow$ the total mechanical energy is conserved!

$$\Longrightarrow mgh + \cancelto{0}{\tfrac{1}{2}mv_{initial}^2} = \tfrac{1}{2}kx^2 + \cancelto{0}{\tfrac{1}{2}mv_{final}^2}$$

$$\Longrightarrow mgh = \tfrac{1}{2}kx^2$$

$$\Longrightarrow x = \sqrt{\frac{2mgh}{k}}$$

$$\Longrightarrow x = \sqrt{\frac{2 \cdot 0.2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 1.5\,\text{m}}{40\,\text{N/m}}}$$

$$\boxed{x \approx 0.384\ \text{m}}$$

□