Updated on February 24

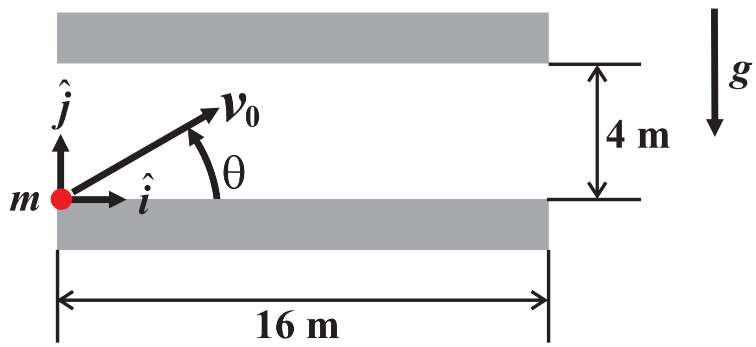
ENGRD/TAM 203: Dynamics (Spring 2006)

Solution of Homework 2 (assigned on Jan. 26, due on Feb. 2)

by Dennis Yang

1. Problem 2.2.31.

Statement. A mass particle m is launched from the left end of a 16-meter-long tunnel. What are the minimum launch speed and the associated launch angle that will allow it to travel through the tunnel without contacting the ceiling or floor after launch?



Solution. We let $\vec{v}_0$ be the launch velocity with v_0 being the launch speed and θ being the launch angle. (It is obvious that the launch angle must satisfy $0° < \theta < 90°$ in order to get the particle travel through the tunnel from the left end to the right exit without touching the floor.) Since the acceleration $\vec{g}$ is a **constant vector** through out the motion, $\vec{r}_{m/L}(t)$, the position vector of the mass particle m relative to the launch point L as a function of time t, can be given as

$$\vec{r}_{m/L}(t) = \vec{r}_{m/L}(0\,\mathrm{s}) + \vec{v}(0\,\mathrm{s})\,t + \frac{1}{2}\vec{g}\,t^2\,. \tag{1.1}$$

Please note that (1.1) is the vector form of formula (2.12) in the textbook. If we set the launch point L to be the origin of our coordinate system and let $t = 0\,\mathrm{s}$ correspond to the time instant when the particle is launched, then $\vec{r}_{m/L}(0\,\mathrm{s}) = \vec{0}$, $\vec{v}(0\,\mathrm{s}) = \vec{v}_0$, and (1.1) becomes

$$\vec{r}_{m/L}(t) = \vec{v}_0\,t + \frac{1}{2}\vec{g}\,t^2\,. \tag{1.2}$$

Furthermore, letting $\hat{\imath}$ point to the right and $\hat{\jmath}$ point upwards, we have $\vec{v}_0 = v_0\cos\theta\,\hat{\imath} + v_0\sin\theta\,\hat{\jmath}$ and $\vec{g} = -g\,\hat{\jmath}$, where g denotes the magnitude of $\vec{g}$ (e.g., $g = 9.81\,\mathrm{m/s^2}$.) Then (1.2) can be written as

$$\begin{aligned}\vec{r}_{m/L}(t) &= (v_0\cos\theta\,\hat{\imath} + v_0\sin\theta\,\hat{\jmath})\,t + \frac{1}{2}(-g\,\hat{\jmath})\,t^2\\ &= v_0\cos\theta\;t\,\hat{\imath} + (v_0\sin\theta\;t - \frac{1}{2}g\,t^2)\,\hat{\jmath}\,.\end{aligned} \tag{1.3}$$

Let $x(t)$ be the $\hat{\imath}$ component of $\vec{r}_{m/L}(t)$ and $y(t)$ be the $\hat{\jmath}$ component of $\vec{r}_{m/L}(t)$, i.e., $x(t) = \vec{r}_{m/L}(t) \bullet \hat{\imath}$ and $y(t) = \vec{r}_{m/L}(t) \bullet \hat{\jmath}$. By (1.3), we have

$$x(t) = \left(v_0 \cos\theta\, t\, \hat{\imath} + (v_0 \sin\theta\, t - \frac{1}{2} g\, t^2)\, \hat{\jmath}\right) \bullet \hat{\imath} = v_0 \cos\theta\, t\,; \tag{1.4}$$

$$y(t) = \left(v_0 \cos\theta\, t\, \hat{\imath} + (v_0 \sin\theta\, t - \frac{1}{2} g\, t^2)\, \hat{\jmath}\right) \bullet \hat{\jmath} = v_0 \sin\theta\, t - \frac{1}{2} g\, t^2\,. \tag{1.5}$$

In this problem, there are two ways to look at the trajectory of the particle. One is to parameterize the x and y coordinates of the particle by time variable "t" using (1.5) and (1.4), i.e.,

$$(x(t),\, y(t)) = \left(v_0 \cos\theta\, t,\ v_0 \sin\theta\, t - \frac{1}{2} g\, t^2\right)\,.$$

The other one is to parameterize y by x. Specifically, we can write $t = \frac{x}{v_0 \cos\theta}$ by (1.4), then substitute this to (1.5) to obtain

$$\begin{aligned}(x,\, y(x)) &= \left(x,\ v_0 \sin\theta \left(\frac{x}{v_0 \cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta}\right)^2\right) \\ &= \left(x,\ \tan\theta\, x - \frac{g}{2 v_0^2 \cos^2\theta}\, x^2\right)\,. \end{aligned} \tag{1.6}$$

The convenience brought by (1.6) is that we can look at the trajectory as the graph of the function

$$\boxed{y(x) = \tan\theta\, x - \frac{g}{2 v_0^2 \cos^2\theta}\, x^2\,,\ \text{for } x \geq 0\,\text{m}\,,} \tag{1.7}$$

where the domain of x is determined by the fact that $x = 0\,\text{m}$ at the launch point and the particle travels to the right. We see that the appearance of the time variable t is eliminated in (1.7). Indeed, our only concern is that the graph of (1.7) does not touch the ceiling or the floor and there is no need of the involvement of t in this.

Let $x = x^*$ be the critical point of $y(x) = \tan\theta\, x - \frac{g}{2 v_0^2 \cos^2\theta}\, x^2$. That is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=x^*} = 0 \implies \tan\theta - \frac{g}{v_0^2 \cos^2\theta}\, x^* = 0 \\ \implies x^* = \frac{v_0^2 \sin\theta \cos\theta}{g} \\ \implies x^* = \frac{v_0^2 \sin 2\theta}{2g}\,. \end{aligned} \tag{1.8}$$

For $0^\circ < \theta < 90^\circ$, we have $x^* > 0\,\text{m}$. Thus, the graph of (1.7) is ascending (i.e., y increases monotonically) from $x = 0\,\text{m}$ up to $x = x^*$, where y reaches its highest value (call it y^*), and the graph becomes descending (i.e., y decreases monotonically) for $x > x^*$. In addition, $y^* = y(x^*)$ is given by

$$y^* = \tan\theta \left(\frac{v_0^2 \sin\theta \cos\theta}{g}\right) - \frac{g}{2 v_0^2 \cos^2\theta} \left(\frac{v_0^2 \sin\theta \cos\theta}{g}\right)^2$$

$$
\begin{aligned}
&= \frac{v_0^2 \sin^2\theta}{g} - \frac{v_0^2 \sin^2\theta}{2g} \\
&= \frac{v_0^2 \sin^2\theta}{2g} \\
&= \frac{v_0^2}{2g}\left(\frac{1-\cos 2\theta}{2}\right) \\
&= \frac{v_0^2(1-\cos 2\theta)}{4g}\,. \qquad (1.9)
\end{aligned}
$$

To avoid contacting the ceiling or floor, we must consider the following two situations.

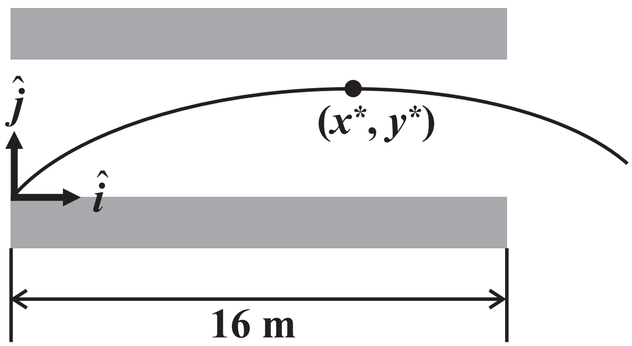


Figure 1

(1) Point (x^*, y^*) appears inside the tunnel. (Figure 1.) Mathematically, this means $x^* \leq 16\,\text{m}$. To avoiding contacting the ceiling, we must have $y^* \leq 4\,\text{m}$. In addition, since y decreases monotonically for $x^* \leq x \leq 16\,\text{m}$, we only need to ensure that $y(16\,\text{m}) \geq 0\,\text{m}$ at the exit to avoid touching the floor. Putting these conditions together and using (1.7)–(1.9), we can derive

$$
x^* \leq 16\,\text{m} \implies \frac{v_0^2 \sin 2\theta}{2g} \leq 16\,\text{m}\,. \qquad (1.10)
$$

$$
y^* \leq 4\,\text{m} \implies \frac{v_0^2(1-\cos 2\theta)}{4g} \leq 4\,\text{m}\,. \qquad (1.11)
$$

$$
\begin{aligned}
y(16\,\text{m}) \geq 0\,\text{m} &\implies \tan\theta\,(16\,\text{m}) - \frac{g}{2v_0^2\cos^2\theta}\,(16\,\text{m})^2 \geq 0\,\text{m} \\
&\implies \tan\theta - \frac{g}{2v_0^2\cos^2\theta} 16\,\text{m} \geq 0 \\
&\implies \tan\theta \geq \frac{g}{2v_0^2\cos^2\theta} 16\,\text{m} \\
&\implies \frac{2v_0^2 \sin\theta\cos\theta}{g} \geq 16\,\text{m} \\
&\implies \frac{v_0^2 \sin 2\theta}{g} \geq 16\,\text{m}\,. \qquad (1.12)
\end{aligned}
$$

Furthermore, since both $\sin 2\theta > 0$ and $1 - \cos 2\theta > 0$ for $0° < \theta < 90°$, we have

$$(1.10) + (1.12) \implies 16\,\text{m} \le \frac{v_0^2 \sin 2\theta}{g} \le 32\,\text{m}$$

$$\implies \boxed{\frac{16\,\text{m}}{\sin 2\theta} \le \frac{v_0^2}{g} \le \frac{32\,\text{m}}{\sin 2\theta}\,.} \tag{1.13}$$

$$(1.11) \implies \boxed{\frac{v_0^2}{g} \le \frac{16\,\text{m}}{1 - \cos 2\theta}\,.} \tag{1.14}$$

In the $\left(\frac{v_0^2}{g}, \theta\right)$ space, the region that satisfies both (1.13) and (1.14) is depicted in Figure 2.

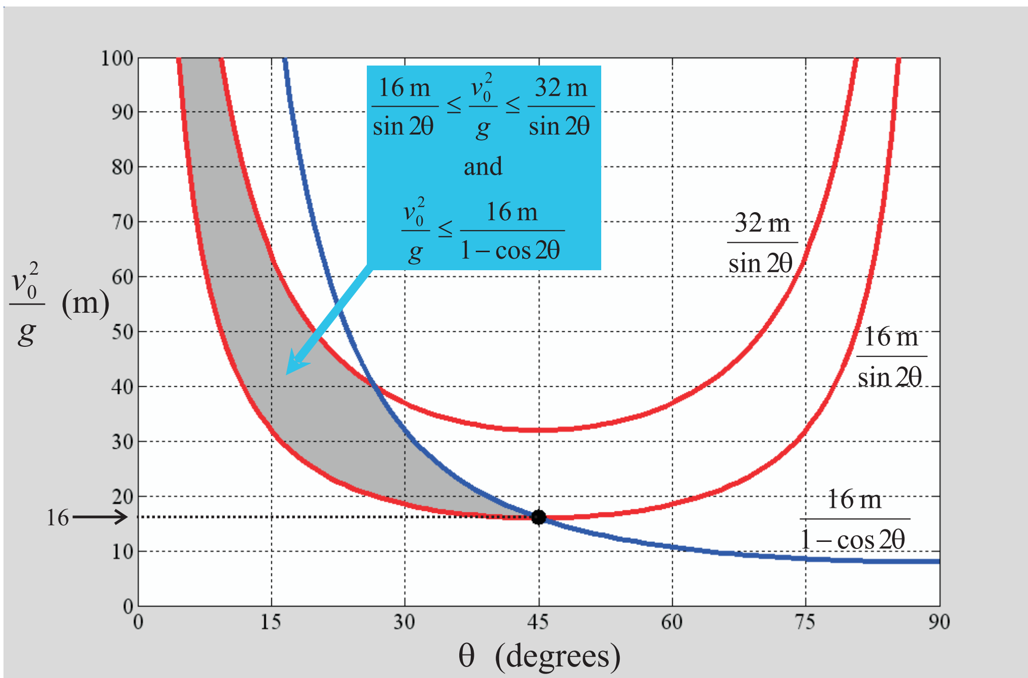


Figure 2

Our goal is to look for the minimum value of v_0, equivalently the minimum value of $\frac{v_0^2}{g}$, while satisfying both (1.13) and (1.14). A quick inspection on (1.13) reveals that the minimum value for the lower bound, i.e., $\frac{16\,\text{m}}{\sin 2\theta}$, occurs at $\theta = 45°$. Thus, for $0° < \theta < 90°$, the minimum value of $\frac{v_0^2}{g}$ allowed by (1.13) alone is that

$$\frac{v_0^2}{g} = \frac{16\,\text{m}}{\sin(2 \cdot 45°)} = 16\,\text{m}\,.$$

Next, we should check if (1.14) is satisfied by $\theta = 45°$ and $\frac{v_0^2}{g} = 16\,\text{m}$. Indeed,

$$\frac{16\,\text{m}}{1 - \cos(2 \cdot 45°)} = 16\,\text{m}\,.$$

Thus, (1.14) is just satisfied. Therefore, under the constraints of both (1.13) and (1.14), the minimum value of $\frac{v_0^2}{g}$ is $16\,\mathrm{m}$ and the associated value of θ is $45°$. **Please note that there is nothing magical about $\theta = 45°$. In this problem, the height of the tunnel is ARTIFICIALLY set to be $4\,\mathrm{m}$ to make this happen!**

(2) Point (x^*, y^*) appears outside the tunnel. (Figure 3.) Mathematically, this means $x^* \geq 16\,\mathrm{m}$. Since y increases monotonically for $0\,\mathrm{m} \leq x \leq x^*$ with $y(0\,\mathrm{m}) = 0\,\mathrm{m}$, we only need to ensure that $y(16\,\mathrm{m}) \leq 4\,\mathrm{m}$ at the exit to avoid touching the ceiling. Putting these conditions together and using (1.7)–(1.9), we can derive

$$x^* \geq 16\,\mathrm{m} \implies \frac{v_0^2 \sin 2\theta}{2g} \geq 16\,\mathrm{m}\,. \tag{1.15}$$

$$y(16\,\mathrm{m}) \leq 4\,\mathrm{m} \implies \tan\theta\,(16\,\mathrm{m}) - \frac{g}{2v_0^2\cos^2\theta}\,(16\,\mathrm{m})^2 \leq 4\,\mathrm{m}\,. \tag{1.16}$$

However, before we throw ourselves into the madness of solving (1.15) and (1.16) together, we should pay some attention to the similarity between (1.12) and (1.15). Again, since $\sin 2\theta > 0$ for $0° < \theta < 90°$, we have

$$(1.15) \implies \frac{v_0^2}{g} \geq \frac{32\,\mathrm{m}}{\sin 2\theta}\,. \tag{1.17}$$

Thus, by (1.17) alone, the smallest possible value of $\frac{v_0^2}{g}$ is $32\,\mathrm{m}$, which is something already larger than what we have found in case (1). Therefore, our analysis up to this point has been sufficient to make the conclusion.

Conclusion. In order to allow the particle to travel through the tunnel without contacting the ceiling or floor, the minimum launch speed is

$$v_{0\min} = \sqrt{16\,\mathrm{m}\cdot g} = \sqrt{16\,\mathrm{m}\cdot 9.81\,\mathrm{m/s^2}} \approx \underline{12.53\,\mathrm{m/s}}$$

with the associated launch angle $\underline{\theta = 45°}$. □

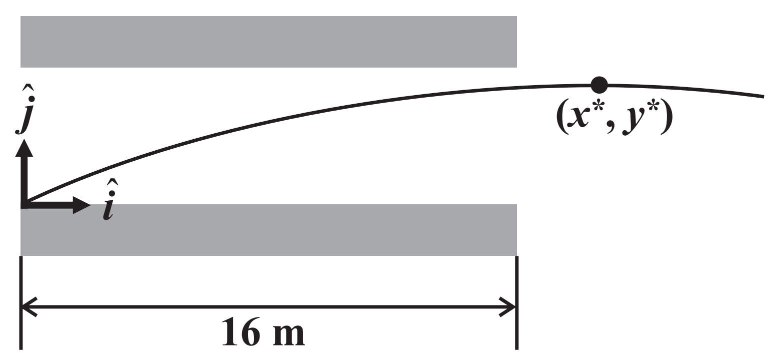
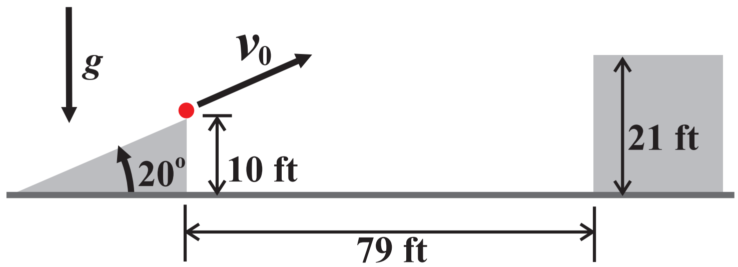


Figure 3

2. Problem 2.2.35.

Statement. What is the minimum speed v_0 that will allow the car (treated as a mass particle) to take off from the top edge of the incline and land on the platform as shown below?



Solution. We let $\vec{v}_0$ be the take-off velocity with v_0 being the take-off speed. The launch angle is 20° because the velocity of the car is parallel to the incline as the car takes off. Since the acceleration $\vec{g}$ is a **constant vector** through out the motion, $\vec{r}_{c/L}(t)$, the position vector of the car c relative to the take-off point L (i.e., the top edge of the incline) as a function of time t, can be given as

$$\vec{r}_{c/L}(t) = \vec{r}_{c/L}(0\,\mathrm{s}) + \vec{v}(0\,\mathrm{s})\,t + \frac{1}{2}\vec{g}\,t^2\,. \tag{2.1}$$

If we set the take-off point L to be the origin of our coordinate system and let $t = 0\,\mathrm{s}$ correspond to the time instant when the car takes off, then $\vec{r}_{c/L}(0\,\mathrm{s}) = \vec{0}$, $\vec{v}(0\,\mathrm{s}) = \vec{v}_0$, and (2.1) becomes

$$\vec{r}_{c/L}(t) = \vec{v}_0\,t + \frac{1}{2}\vec{g}\,t^2\,. \tag{2.2}$$

Furthermore, letting $\hat{\imath}$ point to the right and $\hat{\jmath}$ point upwards, we have $\vec{v}_0 = v_0\cos 20^\circ\,\hat{\imath} + v_0\sin 20^\circ\,\hat{\jmath}$ and $\vec{g} = -g\,\hat{\jmath}$, where g denotes the magnitude of $\vec{g}$ (e.g., $g = 32.2\,\mathrm{ft/s^2}$.) Then (2.2) can be written as

$$\begin{aligned}\vec{r}_{c/L}(t) &= (v_0\cos 20^\circ\,\hat{\imath} + v_0\sin 20^\circ\,\hat{\jmath})\,t + \frac{1}{2}(-g\,\hat{\jmath})\,t^2\\ &= v_0\cos 20^\circ\;t\,\hat{\imath} + (v_0\sin 20^\circ\;t - \frac{1}{2}g\,t^2)\,\hat{\jmath}\,.\end{aligned} \tag{2.3}$$

Let $x(t)$ be the $\hat{\imath}$ component of $\vec{r}_{c/L}(t)$ and $y(t)$ be the $\hat{\jmath}$ component of $\vec{r}_{c/L}(t)$, i.e., $x(t) = \vec{r}_{c/L}(t)\bullet\hat{\imath}$ and $y(t) = \vec{r}_{c/L}(t)\bullet\hat{\jmath}$. By (2.3), we have

$$x(t) = \Big(v_0\cos 20^\circ\;t\,\hat{\imath} + (v_0\sin 20^\circ\;t - \frac{1}{2}g\,t^2)\,\hat{\jmath}\Big)\bullet\hat{\imath} = v_0\cos 20^\circ\;t\,; \tag{2.4}$$

$$y(t) = \Big(v_0\cos 20^\circ\;t\,\hat{\imath} + (v_0\sin 20^\circ\;t - \frac{1}{2}g\,t^2)\,\hat{\jmath}\Big)\bullet\hat{\jmath} = v_0\sin 20^\circ\;t - \frac{1}{2}g\,t^2\,. \tag{2.5}$$

Using (2.4) and (2.5), we parameterize y by x. Specifically, we write $t = \frac{x}{v_0\cos 20^\circ}$ by (2.4), then substitute this to (2.5) to obtain

$$\begin{aligned}(x,\,y(x)) &= \left(x,\;v_0\sin 20^\circ\left(\frac{x}{v_0\cos 20^\circ}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos 20^\circ}\right)^2\right)\\ &= \left(x,\;\tan 20^\circ\,x - \frac{g}{2v_0^2\cos^2 20^\circ}\,x^2\right)\,.\end{aligned} \tag{2.6}$$

With (2.6), we can look at the trajectory as the graph of the function

$$\boxed{y(x) = \tan 20^\circ\, x - \frac{g}{2v_0^2 \cos^2 20^\circ}\, x^2\,, \text{ for } x \geq 0\,\text{ft}\,,} \tag{2.7}$$

where the domain of x is determined by the fact that $x = 0\,\text{ft}$ at the take-off point and the car travels to the right.

In order to ensure that the car will land on the platform, we need $y(79\,\text{ft}) \geq 21\,\text{ft} - 10\,\text{ft} = 11\,\text{ft}$, which is the difference between the height of the platform and that of the take-off point (see Figure 4). Expressed by (2.7), this condition yields that

$$\begin{aligned}
y(79\,\text{ft}) \geq 11\,\text{ft} &\implies \tan 20^\circ\, 79\,\text{ft} - \frac{g}{2v_0^2\cos^2 20^\circ}\,(79\,\text{ft})^2 \geq 11\,\text{ft} \\
&\implies \tan 20^\circ\, 79\,\text{ft} \cdot v_0^2 - \frac{(79\,\text{ft})^2 \cdot g}{2\cos^2 20^\circ} \geq 11\,\text{ft} \cdot v_0^2 \\
&\implies (\tan 20^\circ\, 79\,\text{ft} - 11\,\text{ft})\, v_0^2 \geq \frac{(79\,\text{ft})^2 \cdot g}{2\cos^2 20^\circ} \\
&\implies v_0^2 \geq \frac{(79\,\text{ft})^2 \cdot g}{2\cos^2 20^\circ\,(\tan 20^\circ\, 79\,\text{ft} - 11\,\text{ft})} \\
&\implies \boxed{v_0 \geq \sqrt{\frac{(79\,\text{ft})^2 \cdot g}{2\cos^2 20^\circ\,(\tan 20^\circ\, 79\,\text{ft} - 11\,\text{ft})}}\,.}
\end{aligned} \tag{2.8}$$

It is obvious from (2.8) that the minimum speed needed at the take-off point is

$$\begin{aligned}
v_{0\text{min}} &= \sqrt{\frac{(79\,\text{ft})^2 \cdot g}{2\cos^2 20^\circ\,(\tan 20^\circ\, 79\,\text{ft} - 11\,\text{ft})}} \\
&= \sqrt{\frac{(79\,\text{ft})^2 \cdot 32.2\,\text{ft/s}^2}{2\cos^2 20^\circ\,(\tan 20^\circ\, 79\,\text{ft} - 11\,\text{ft})}} \\
&\approx \underline{80.06\,\text{ft/s}\,.}
\end{aligned}$$

□

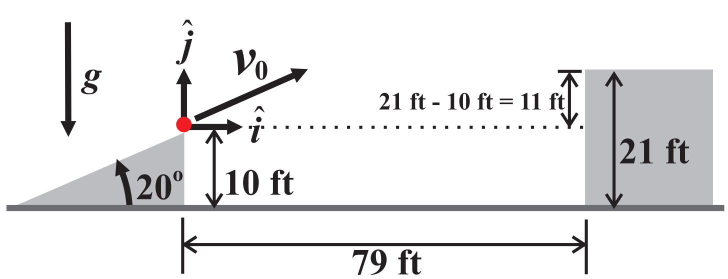


Figure 4