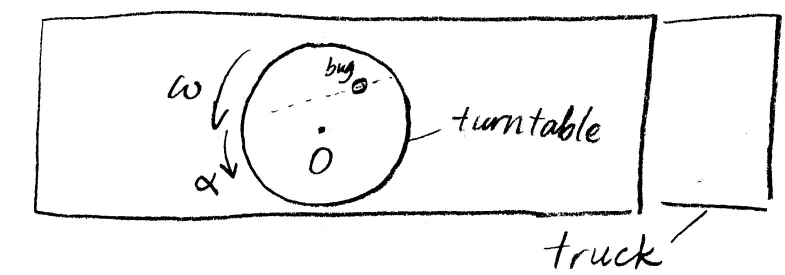
ENGRD/TAM 203

HW 20 (Assigned April 6, due on April 13)

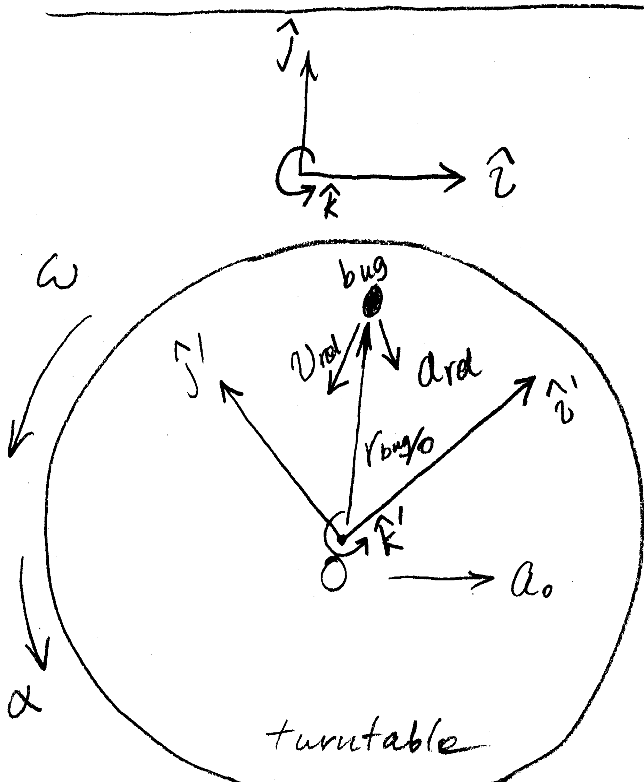
Solution by Dennis Yang

RP 9.30



Discuss the "five-term" acceleration formular.

a) Draw a picture of the situation, define all variables and reference frames.



$\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, are the unit vectors fixed on the ground. They do NOT move or turn.

$\hat{\imath}'$, $\hat{\jmath}'$ and $\hat{k}'$ are the unit vectors fixed on the turntable. $\hat{\imath}'$ and $\hat{\jmath}'$ turn with the turntable.

$\vec{a}_0$ ~ the acceleration (w.r.t. $\hat{\imath}$-$\hat{\jmath}$-$\hat{k}$ frame) of point O, the center of the turntable.

$\vec{a}_0 = a_{0x}\hat{\imath} + a_{0y}\hat{\jmath}$, which has NO component in $\hat{k}$ since we assume the truck only moves in the plane.

Assume the rotational axis of the turntable is perpendicular to the plane

$\vec{\alpha}$ ~ the angular acceleration of the turntable

$\vec{\alpha}$ is in the form of $\vec{\alpha} = \alpha\hat{k} = \alpha\hat{k}'$, since $\hat{k}$ and $\hat{k}'$ are always the same.

$\vec{\omega}$ ~ the angular velocity of the turntable

$\vec{\omega}$ is in the form of $\vec{\omega} = \omega\hat{k} = \omega\hat{k}'$, since $\hat{k}$ and $\hat{k}'$ are always the same

$\vec{r}_{bug/o}$ ~ the position of the bug relative to the turn table. Assume the bug is always on the table, then we can express $\vec{r}_{bug/o}$

by $\vec{r}_{bug/o} = x\hat{\imath} + y\hat{\jmath}$ ①

or $\vec{r}_{bug/o} = x'\hat{\imath}' + y'\hat{\jmath}'$ ②

in general, $x \neq x'$, $y \neq y'$

In either way, there is NO $\hat{k}$ or $\hat{k}'$ component.

$$\vec{v}_{rel} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' \neq \frac{d}{dt}(\vec{r}_{bug/o})$$

$$\vec{a}_{rel} = \ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}' \neq \frac{d^2}{dt^2}(\vec{r}_{bug/o})$$

Note using ①, $\frac{d}{dt}(\vec{r}_{bug/o}) = \frac{d}{dt}(x\hat{\imath}) + \frac{d}{dt}(y\hat{\jmath}) = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, since $\hat{\imath}$ and $\hat{\jmath}$ are CONSTANT VECTORS !

However, using ②, $\frac{d}{dt}(\vec{r}_{bug/o}) = \frac{d}{dt}(x'\hat{\imath}') + \frac{d}{dt}(y'\hat{\jmath}')$ since $\hat{\imath}'$ & $\hat{\jmath}'$ are NOT constant vectors!

$$= \dot{x}'\hat{\imath}' + x'\dot{\hat{\imath}}' + \dot{y}'\hat{\jmath}' + y'\dot{\hat{\jmath}}'$$

Finally, $\vec{a}_{bug} = \vec{a}_o + \vec{a}_{bug/o}$, where

$\vec{a}_{bug}$ is the acceleration of the bug w.r.t. $\hat{i}$-$\hat{j}$-$\hat{k}$ frame

and $\vec{a}_{bug/o} = \frac{d^2}{dt^2}(\vec{r}_{bug/o})$

$$= \underbrace{\vec{\alpha} \times \vec{r}_{bug/o} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{bug/o}) + \vec{a}_{rel} + 2\vec{\omega} \times \vec{v}_{rel}}_{\substack{\text{the last 4 terms in the} \\ \text{"5-term" acceleration formula:}}}$$

$$\vec{a}_{bug} = \vec{a}_o + \vec{\alpha} \times \vec{r}_{bug/o} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{bug/o}) + \vec{a}_{rel} + 2\vec{\omega} \times \vec{v}_{rel}$$

b) Find situations all but one term is zero in the above formula.

c) Find situations all but 2 terms are zero in the above formula.

For b) and c), we just discuss when each of the 5 terms is and is not zero.

1 — $\vec{a}_o$: it's the acceleration of the center of the turntable, which is fixed on the truck. Thus $\vec{a}_o$ is/isn't zero if the truck hasn't/has any acceleration.

2 —— $\vec{\alpha} \times \vec{r}_{bug/o}$:

$$\vec{\alpha} \times \vec{r}_{bug/o} = \alpha \hat{k}' \times (x'\hat{\imath}' + y'\hat{\jmath}')$$
$$= \alpha x' \hat{\jmath}' + \alpha y'(-\hat{\imath}')$$
$$= \alpha(-y'\hat{\imath}' + x'\hat{\jmath}')$$

which is zero if $\alpha = 0$ (then $\vec{\alpha} = \vec{0}$)

or $x' = y' = 0$ (then $\vec{r}_{bug/o} = \vec{0}$)

Thus, $\vec{\alpha} \times \vec{r}_{bug/o} = \vec{0}$ if one/both of the following is/are true

i) turntable has no angular acceleration
ii) the bug is at the center of the turntable

3 —— $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{bug/o})$:

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{bug/o}) = \omega \hat{k}' \times (\omega \hat{k}' \times (x'\hat{\imath}' + y'\hat{\jmath}'))$$
$$= \omega^2 x'(-\hat{\imath}') + \omega^2 y'(-\hat{\jmath}')$$
$$= -\omega^2 (x'\hat{\imath}' + y'\hat{\jmath}')$$

which is zero if $\omega = 0$ or $x' = y' = 0$

Thus, $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{bug/o}) = \vec{0}$ if one/both of the following is true :

i) turntable is NOT turning

ii) the bug is at the center of the turn table.

4 — $\vec{a}_{rel}$:

$\vec{a}_{rel} = \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' = \vec{0}$ if and only if both $\ddot{x}'$ and $\ddot{y}'$ is zero, i.e., Viewed by Some one sits on the turntable, the bug moves along a straight line at a constant rate or the bug is NOT moving at all.

5 — $2\vec{\omega} \times \vec{v}_{rel}$:

$$
\begin{aligned}
2\vec{\omega} \times \vec{v}_{rel} &= 2\omega\hat{k}' \times (\dot{x}'\hat{i}' + \dot{y}'\hat{j}') \\
&= 2\omega\dot{x}'\hat{j}' + 2\omega\dot{y}'(-\hat{i}') \\
&= 2\omega(-\dot{y}'\hat{i}' + \dot{x}'\hat{j}')
\end{aligned}
$$

$2\vec{\omega} \times \vec{v}_{rel} = \vec{0}$ if $\omega = 0$ or $\dot{y}' = \dot{x}' = 0$

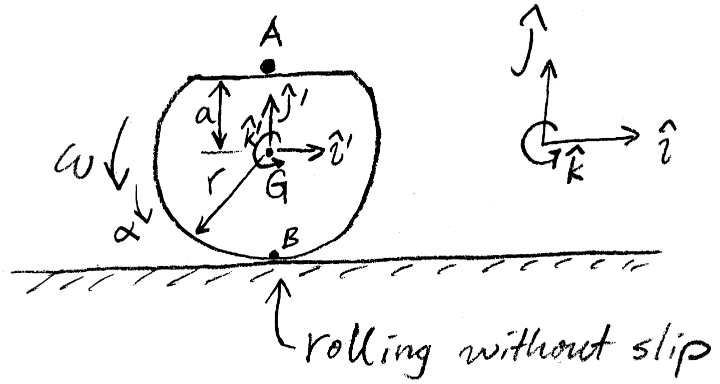
Thus, $2\vec{\omega} \times \vec{v}_{rel} = \vec{0}$ if one or both of the following is true :

i) the turntable is NOT turning

ii) Viewed by some one sits on the turntable, the bug is NOT moving.

$\Box$

6.4.21



$\vec{\omega} = \omega\hat{k} = 10\,\text{rad/s}\,\hat{k}$

$\vec{\alpha} = \alpha\hat{k} = 5\,\text{rad/s}^2\,\hat{k}$

$r = 1.5\,m$

$a = 1.1\,m$

particle A is moving to the right along the top flat surface with speed 2 m/s and acceleration 3 m/s²

Find $\vec{a}_A$.

Solution

We take $\hat{i}', \hat{j}', \hat{k}'$ that are fixed on the body G by letting $\hat{i}'$ be parallel to the flat surface, $\hat{j}'$ point to the flat surface, and $\hat{k}' = \hat{i}' \times \hat{j}'$ point out of the paper. In addition, at the instant shown, $\hat{i}' = \hat{i}$, $\hat{j}' = \hat{j}$, $\hat{k}' = \hat{k}$, although $\hat{i}' \neq \hat{i}$, $\hat{j}' \neq \hat{j}$ before and after this instant.

Let point B be a point on the body G and at the bottom of G, i.e., <u>B is at contact with the ground</u>

"rolling without slip" $\Longrightarrow \vec{a}_B \cdot \hat{i} = 0 \quad (*)$

Consider body G :

$$\vec{a}_B = \vec{a}_G + \vec{\alpha} \times \vec{r}_{B/G} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/G})$$

(B is fixed on the body, So we use this 3-term formula !)

$$= a_G \hat{\imath} + \alpha \hat{k} \times r(-\hat{\jmath}) + \omega \hat{k} \times (\omega \hat{k} \times r(-\hat{\jmath}))$$

$$= a_G \hat{\imath} + \alpha r \hat{\imath} + \omega^2 r \hat{\jmath}$$

$$= (a_G + \alpha r)\hat{\imath} + \omega^2 r \hat{\jmath}$$

by (*), $\left((a_G + \alpha r)\hat{\imath} + \omega^2 r \hat{\jmath}\right) \cdot \hat{\imath} = 0$

$$\Rightarrow \quad a_G + \alpha r = 0$$

$$\Rightarrow \quad a_G = -\alpha r$$

$$\Rightarrow \quad \boxed{\vec{a}_G = -\alpha r \hat{\imath}}$$

"Particle A is moving to the right along the top flat surface with speed 2 m/s and acceleration 3 m/s²"

$$\Longrightarrow \quad \vec{v}_{rel} = 2 \text{ m/s } \hat{\imath}' = 2 \text{ m/s } \hat{\imath}$$

only holds at this instant !

and $\vec{a}_{rel} = 3 \text{ m/s}^2 \hat{\imath}' = 3 \text{ m/s}^2 \hat{\imath}$

Now, we are in the position to apply the "5-term" formular to the moving point A on G

$$\vec{a}_A = \vec{a}_G + \vec{\alpha} \times \vec{r}_{A/G} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/G}) + \vec{a}_{rel} + 2\vec{\omega} \times \vec{v}_{rel}$$

$$= -\alpha r \hat{\imath} + \alpha \hat{k} \times a\hat{\jmath} + \omega \hat{k} \times (\omega \hat{k} \times a\hat{\jmath}) + 3\,m/s^2\,\hat{\imath} + 2\omega \hat{k} \times 2\,m/s\,\hat{\imath}$$

$$= -\alpha r \hat{\imath} + \alpha a(-\hat{\imath}) + \omega^2 a(-\hat{\jmath}) + 3\,m/s^2\,\hat{\imath} + \omega \cdot 4\,m/s\,\hat{\jmath}$$

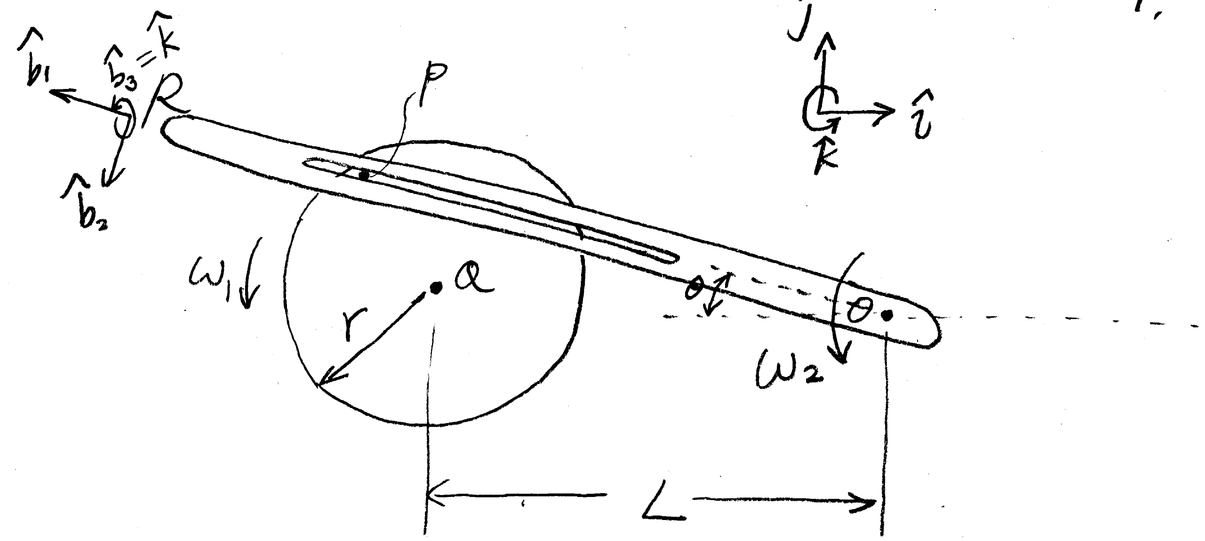
$$= (-\alpha(r+a) + 3\,m/s^2)\,\hat{\imath} + (-\omega^2 a + \omega \cdot 4\,m/s)\,\hat{\jmath}$$

$$= (-5\,rad/s^2\,(1{,}5\,m + 1{,}1\,m) + 3\,m/s^2)\,\hat{\imath} + (-(10\,rad/s)^2\,1{,}1\,m + 10\,rad/s \cdot 4\,m/s)\,\hat{\jmath}$$

$$\boxed{\vec{a}_A = -10\,m/s^2\,\hat{\imath} - 70\,m/s^2\,\hat{\jmath}}$$

$\square$

6.4.26



$\|\vec{r}_{P/Q}\| = r$, $\|\vec{r}_{Q/O}\| = L$, Q and O are at the same level.

$\vec{\omega}_1 = 12 \text{ rad/s } \hat{k}$, $r = 0.1 \text{ m}$, $L = 0.3 \text{ m}$

Determine the velocity of the pin w.r.t. the slot in arm $\overline{OR}$ when $\vec{r}_{P/Q} = r\hat{j}$.

Solution

In the frame $\hat{b}_1 - \hat{b}_2 - \hat{b}_3$ which is fixed on arm $\overline{OR}$, $\vec{v}_{P\,rel} = v_{P\,rel}\,\hat{b}_1$

Thus, $\vec{v}_P = \cancelto{\vec{0}}{\vec{v}_O} + \vec{\omega}_2 \times \vec{r}_{P/O} + \vec{v}_{P\,rel}$, where $\vec{\omega}_2$ is given by $\vec{\omega}_2 = \omega_2\hat{k} = \omega_2\hat{b}_3$

$$= \omega_2\hat{b}_3 \times \|\vec{r}_{P/O}\|\,\hat{b}_1 + v_{P\,rel}\,\hat{b}_1$$

$$= \omega_2\|\vec{r}_{P/O}\|\,\hat{b}_2 + v_{P\,rel}\,\hat{b}_1 \quad ①$$

On the other hand,

$$\vec{v}_P = \cancelto{\vec{0}}{\vec{v}_Q} + \vec{\omega}_1 \times \vec{r}_{P/Q}$$

$$= \omega_1\hat{k} \times r\hat{j} = -\omega_1 r\,\hat{i} \quad ②$$

$$① = ② \Longrightarrow \omega_2 \| \vec{r}_{P/O} \| \hat{b}_2 + v_{P_{rel}} \hat{b}_1 = -\omega_1 r \hat{\imath} \quad (*)$$

$$(*) \cdot \hat{b}_1 \Longrightarrow \omega_2 \| \vec{r}_{P/O} \| \hat{b}_2 \cdot \hat{b}_1 + v_{P_{rel}} \hat{b}_1 \cdot \hat{b}_1 = -\omega_1 r \hat{\imath} \cdot \hat{b}_1$$

$$\Longrightarrow v_{P_{rel}} = -\omega_1 r \cos(180° - \theta)$$

$$= \omega_1 r \cos\theta$$

$$= \omega_1 r \cdot \frac{L}{\sqrt{L^2 + r^2}}$$

Thus, $$v_{P_{rel}} = \frac{\omega_1 r L}{\sqrt{L^2 + r^2}}$$

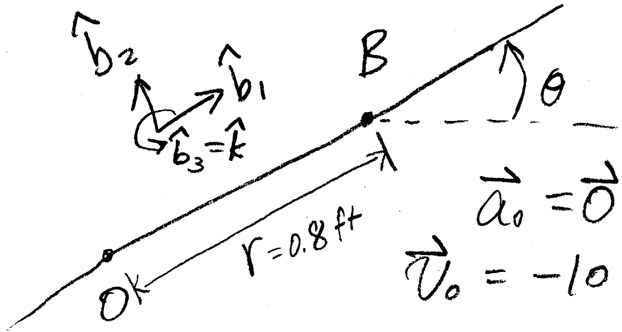
$$= \frac{12 \text{ rad/s} \cdot 0.1 \text{ m} \cdot 0.3 \text{ m}}{\sqrt{(0.3 \text{ m})^2 + (0.1 \text{ m})^2}}$$

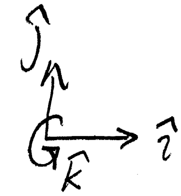
$$v_{P_{rel}} \approx 1.14 \text{ m/s}$$

$$\boxed{\vec{v}_{P_{rel}} \approx 1.14 \text{ m/s } \hat{b}_1}$$

□

6.4.29





$\vec{a}_0 = \vec{0}$

$\vec{v}_0 = -10 \text{ ft/s } \hat{\imath}, \quad \vec{r}_{B/0} = 0.8 \text{ ft } \hat{b}_1$

$\theta = 20°, \quad \vec{\omega} = \dot{\theta}\hat{k} = 18 \text{ rad/s } \hat{k}$

$\vec{\alpha} = \ddot{\theta}\hat{k} = \vec{0}$

w.r.t. O, $\vec{v}_{B\,rel} = 18 \text{ ft/s } (-\hat{b}_1)$

$\vec{a}_{B\,rel} = -5 \text{ ft/s}^2\, \hat{b}_1$

Find $\vec{a}_B$.

Solution

$$\vec{a}_B = \underbrace{\cancel{\vec{a}_0}}_{\vec{0}} + \underbrace{\cancel{\vec{\alpha} \times \vec{r}_{B/0}}}_{\vec{0}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/0}) + \vec{a}_{B\,rel} + 2\vec{\omega} \times \vec{v}_{B\,rel}$$

$$= \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times r\hat{b}_1) + (-5 \text{ ft/s}^2\, \hat{b}_1) + 2\dot{\theta}\hat{k} \times 18 \text{ ft/s} (-\hat{b}_1)$$

$$= \dot{\theta}^2 r(-\hat{b}_1) - 5 \text{ ft/s}^2\, \hat{b}_1 + 2\dot{\theta} \cdot 18 \text{ ft/s} (-\hat{b}_2)$$

$$= -\left(\dot{\theta}^2 r + 5 \text{ ft/s}^2\right)\hat{b}_1 - 2\dot{\theta} \cdot 18 \text{ ft/s}\, \hat{b}_2$$

$$= -\left((18 \text{ rad/s}^2)^2 \cdot 0.8 \text{ ft} + 5 \text{ ft/s}^2\right)\hat{b}_1 - 2 \cdot 18 \text{ rad/s} \cdot 18 \text{ ft/s}\, \hat{b}_2$$

$$\boxed{\vec{a}_B = -264.2 \text{ ft/s}^2\, \hat{b}_1 - 648 \text{ ft/s}^2\, \hat{b}_2}$$

□