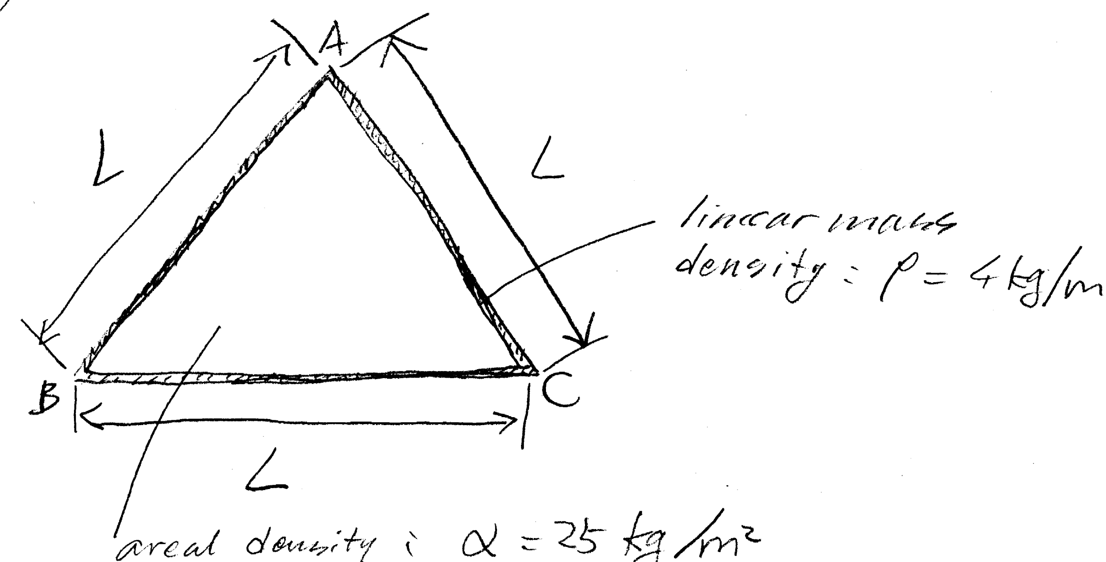
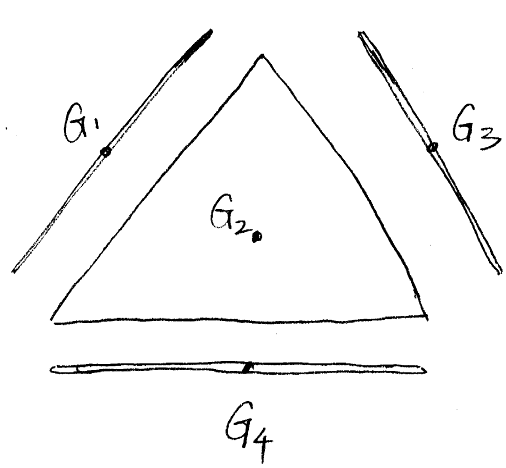
ENGRD/TAM203
HW23 (Assigned on April 18, due on April 25)
Solution by Dennis Yang

<u>7.2.11</u> Find the position of the center of mass and determine I_A (in-plane rotation) for the illustrated body.

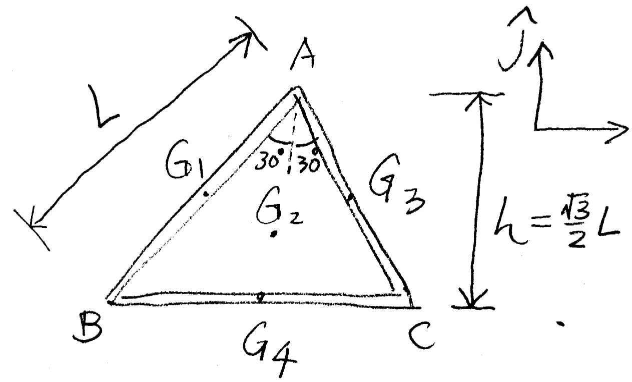


<u>Solution</u>

Take the illustrated body into 4 parts :



When these four parts are assembled into the illustrated body, we have:



where $\vec{r}_{G1/A} = -\frac{L}{2}\sin 30°\,\hat{i} - \frac{L}{2}\cos 30°\,\hat{j}$

$\vec{r}_{G2/A} = -\frac{2}{3}h\hat{j} = -\frac{\sqrt{3}}{3}L\hat{j}$

$\vec{r}_{G3/A} = \frac{L}{2}\sin 30°\,\hat{i} - \frac{L}{2}\cos 30°\,\hat{j}$

$\vec{r}_{G4/A} = -h\hat{j} = -\frac{\sqrt{3}}{2}L\hat{j}$

Note: "ΔABC" stands for the interior trangular part with areal density $\alpha = 25\ kg/m^2$

Thus, $$\vec{r}_{G/A} = \frac{m_{\overline{AB}}\vec{r}_{G1/A} + m_{\overline{AC}}\vec{r}_{G3/A} + m_{\overline{BC}}\vec{r}_{G4/A} + m_{\Delta ABC}\vec{r}_{G2/A}}{m_{\overline{AB}} + m_{\overline{AC}} + m_{\overline{BC}} + m_{\Delta ABC}}$$

$$= \frac{L\rho\,\vec{r}_{G1/A} + L\rho\,\vec{r}_{G3/A} + L\rho\,\vec{r}_{G4/A} + \frac{1}{2}L\cdot h\alpha\,\vec{r}_{G2/A}}{3L\rho + \frac{1}{2}Lh\alpha}$$

$$= \frac{L\rho(-\sqrt{3}L\hat{j}) + \frac{1}{2}Lh\alpha(-\frac{\sqrt{3}}{3}L\hat{j})}{3L\rho + \frac{1}{2}Lh\alpha}$$

$$= \frac{3L\rho\left(-\frac{\sqrt{3}}{3}L\hat{j}\right) + \frac{1}{2}Lh\alpha\left(-\frac{\sqrt{3}}{3}L\hat{j}\right)}{3L\rho + \frac{1}{2}Lh\alpha}$$

$$= \frac{\cancel{\left(3L\rho + \frac{1}{2}Lh\alpha\right)}\left(-\frac{\sqrt{3}}{3}L\hat{j}\right)}{\cancel{3L\rho + \frac{1}{2}Lh\alpha}}$$

$$\boxed{\vec{r}_{G/A} = -\frac{\sqrt{3}}{3}L\hat{j}} \approx \underline{-0.462\ m\,\hat{j}}$$

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for $\overline{AC}$: $I_{G_3} = \frac{\overbrace{(L\rho)}^{m_{\overline{AC}}}L^2}{12} = \frac{\rho L^3}{12}$

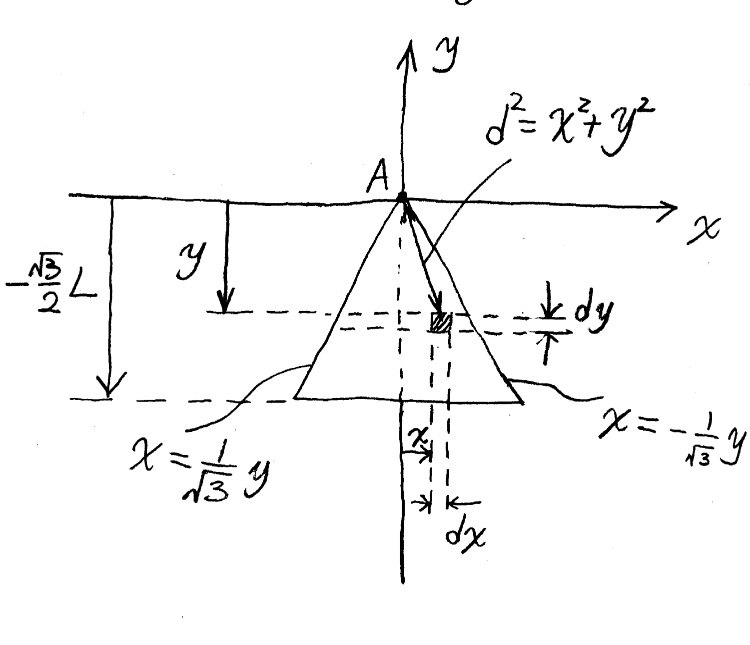
for $\overline{AB}$: $I_{G_1} = \frac{(L\rho)L^2}{12} = \frac{\rho L^3}{12}$

for $\overline{BC}$: $I_{G_4} = \frac{(L\rho)L^2}{12} = \frac{\rho L^3}{12}$

Now, for ΔABC (the interior part only)

$$I_{\Delta ABC/A} = \int_{y=-\frac{\sqrt{3}}{2}L}^{y=0}\int_{x=\frac{1}{\sqrt{3}}y}^{x=-\frac{1}{\sqrt{3}}y} d^2 \underbrace{\alpha\, \overbrace{dx\,dy}^{dA}}_{dm}$$

$$= \int_{y=-\frac{\sqrt{3}}{2}L}^{y=0}\int_{x=\frac{1}{\sqrt{3}}y}^{x=-\frac{1}{\sqrt{3}}y} (y^2+x^2)\,\alpha\, dx\,dy$$



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$$= \int_{y=-\frac{\sqrt{3}}{2}L}^{y=0} \left[\alpha\left(y^2 x + \tfrac{1}{3}x^3\right) \Big|_{x=\frac{1}{\sqrt{3}}y}^{x=-\frac{1}{\sqrt{3}}y} \right] dy$$

$$= \int_{y=-\frac{\sqrt{3}}{2}L}^{y=0} \left(-\frac{2\alpha}{\sqrt{3}} y^3 - \frac{2\alpha}{9\sqrt{3}} y^3 \right) dy$$

$$= -\frac{20\alpha}{9\sqrt{3}} \int_{y=-\frac{\sqrt{3}}{2}L}^{y=0} y^3 \, dy$$

$$= -\frac{5\alpha}{9\sqrt{3}} y^4 \Big|_{y=-\frac{\sqrt{3}}{2}L}^{y=0}$$

$$I_{\Delta ABC/A} = \frac{5\alpha L^4}{16\sqrt{3}}$$

Finally, the whole thing has the total moment of inertia about point A :

$$I_A = \left[I_{G_1} + \|\vec{r}_{G_1/A}\|^2 \cdot m_{\overline{AB}} \right] + \left[I_{G_3} + \|\vec{r}_{G_3/A}\|^2 \cdot m_{\overline{AC}} \right]$$

$$+ \left[I_{G_4} + \|\vec{r}_{G_4/A}\|^2 \cdot m_{\overline{BC}} \right] + I_{\Delta ABC/A}$$

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$$= \left[\frac{\rho L^3}{12} + \left(\frac{L}{2}\right)^2 (\rho L)\right] + \left[\frac{\rho L^3}{12} + \left(\frac{L}{2}\right)^2 (\rho L)\right]$$

$$+ \left[\frac{\rho L^3}{12} + \left(\frac{\sqrt{3}}{2} L\right)^2 (\rho L)\right] + \frac{5\alpha L^4}{16\sqrt{3}}$$

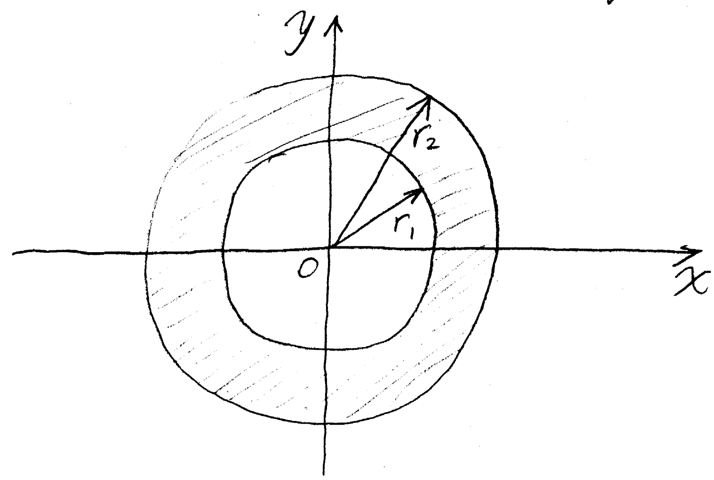
$$= \frac{\rho L^3}{3} + \frac{\rho L^3}{3} + \frac{5}{6}\rho L^3 + \frac{5\alpha L^4}{16\sqrt{3}}$$

$$\boxed{I_A = \frac{3}{2}\rho L^3 + \frac{5\alpha L^4}{16\sqrt{3}}} \approx 4.92\ \mathrm{kg \cdot m^2}$$

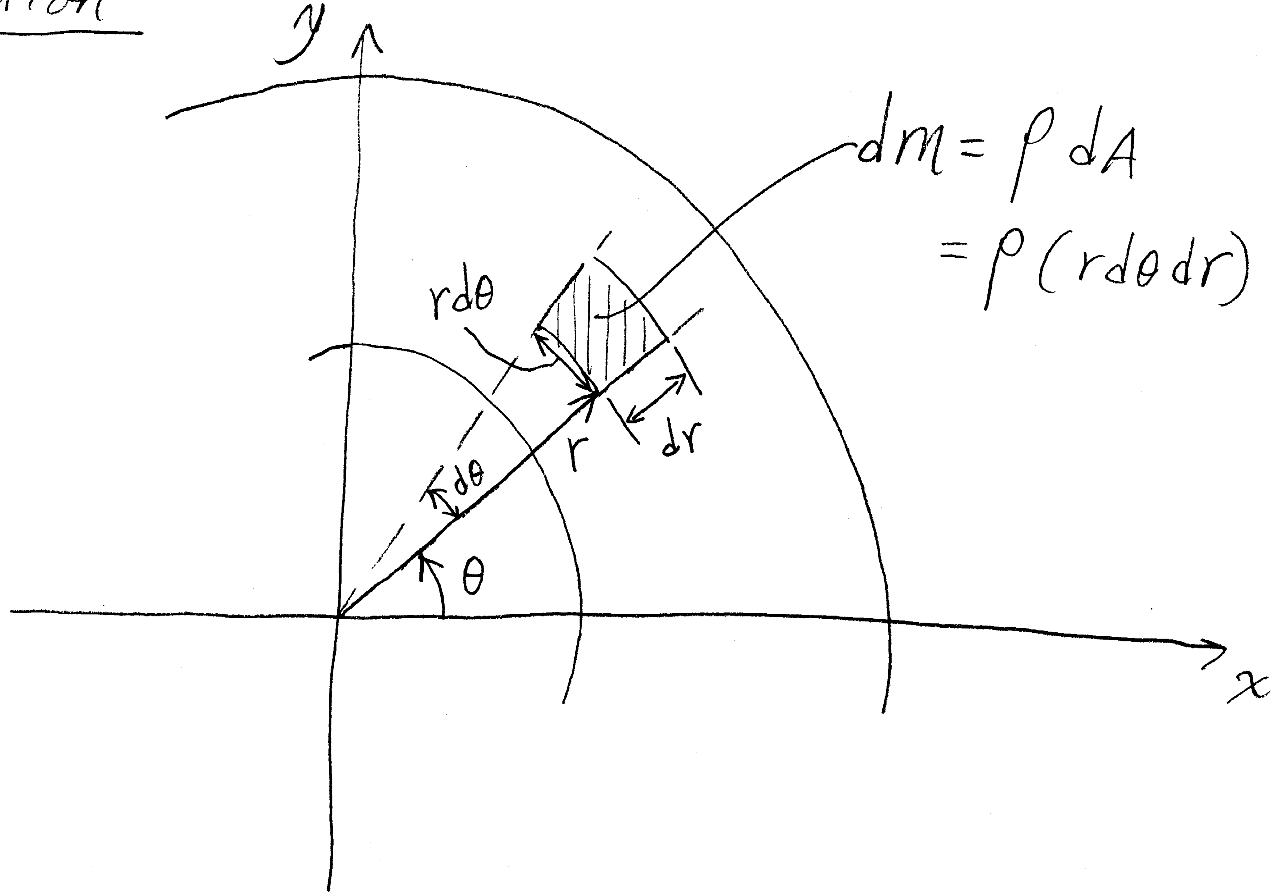
$\square$

7.2.17

The illustrated shape has areal density ρ. Find its mass moment of inertia about the out-of-plane z axis and also about the y axis. Express the answer in terms of ρ and in terms of m (the body's mass)



Solution



For the calculation of $I_{/z}$, the distance from "dm" to z axis is "r". Thus,

$$I_{/z} = \iint_{body} r^2 \, dm$$

$$= \iint_{body} r^2 (\rho \, r \, d\theta \, dr)$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=r_1}^{r=r_2} \rho r^3 \, dr \, d\theta$$

$$= 2\pi \frac{\rho}{4} r^4 \Big|_{r=r_1}^{r=r_2}$$

$$\boxed{I_{/z} = \frac{\rho}{2} \pi (r_2^4 - r_1^4)}$$

$$= \frac{\rho}{2} \pi (r_2^2 - r_1^2)(r_2^2 + r_1^2)$$

this is the area of the body

$$\boxed{I_{/z} = \frac{1}{2} m (r_2^2 + r_1^2)}$$

For calculation of $I_{/y}$, the distance from "dm" to y axis is $|r\cos\theta|$. Thus,

$$I_{/y} = \iint_{body} |r\cos\theta|^2 dm$$

$$= \iint_{body} r^2\cos^2\theta \,(\rho\, r\, d\theta\, dr)$$

$$= \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \left[\int_{r=r_1}^{r=r_2} \rho\, r^3 dr\right] d\theta$$

$$= \left[\int_{\theta=0}^{\theta=2\pi} \cos^2\theta\, d\theta\right]\left[\int_{r=r_1}^{r=r_2} \rho\, r^3 dr\right]$$

$$= \pi \cdot \rho \frac{r^4}{4}\Big|_{r=r_1}^{r=r_2}$$

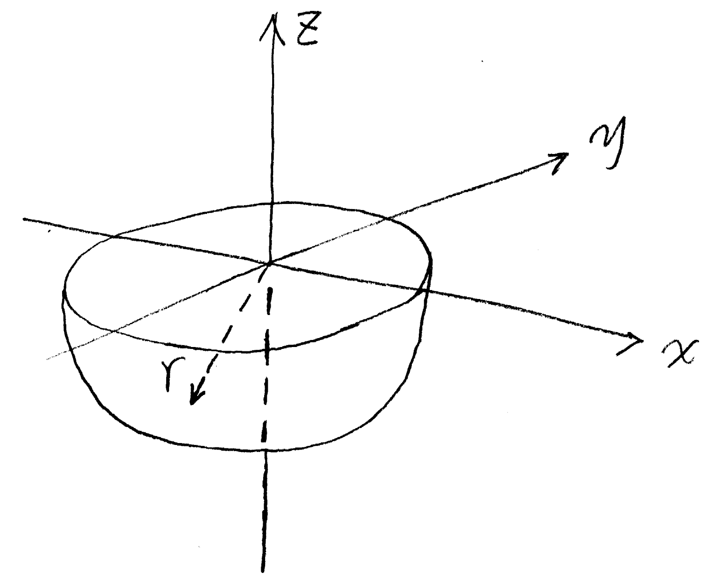
$$\boxed{I_{/y} = \frac{\pi}{4}\rho(r_2^4 - r_1^4)}$$

$$= \frac{1}{4}\rho\pi(r_2^2 - r_1^2)(r_2^2 + r_1^2)$$

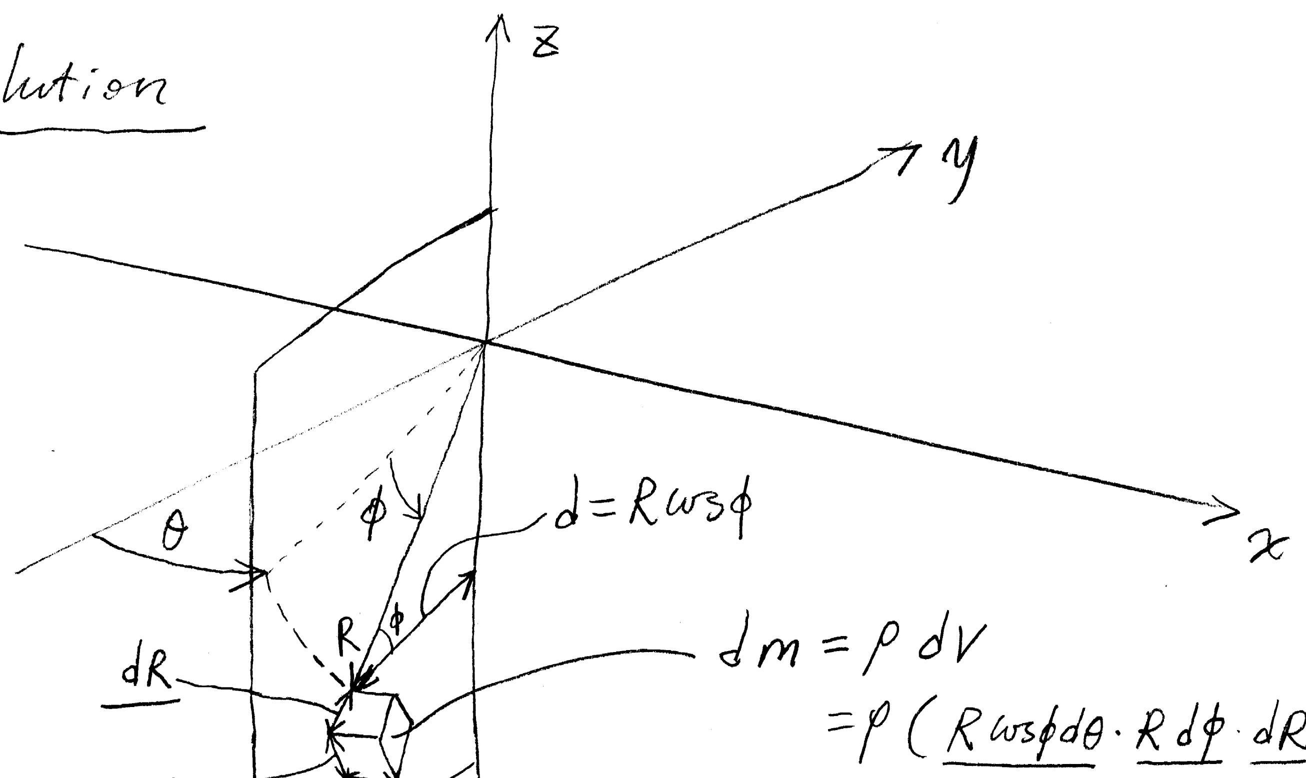
$$\boxed{I_{/y} = \frac{1}{4} m (r_2^2 + r_1^2)}$$

$\square$

7.2.18 A solid half-sphere is shown, with radius "r". Find its mass moment of inertia about z axis and express it in terms of density "ρ" and in terms mass "m".



Solution



$$dm = \rho\, dV$$
$$= \rho\,(\underline{R\cos\phi\, d\theta} \cdot \underline{R\, d\phi} \cdot \underline{dR})$$
$$= \rho R^2 \cos\phi\, dR\, d\theta\, d\phi$$

The distance from "dm" to z axis is $|R\cos\phi|$. Thus,

$$I_{/z} = \iiint_{body} |R\cos\phi|^2 \, dm$$

$$= \iiint_{body} R^2\cos^2\phi \,(\rho R^2 \cos\phi \, dR \, d\theta \, d\phi)$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} \left[\int_{R=0}^{R=r} \rho R^4 \, dR \right] \cos^3\phi \, d\phi \right] d\theta$$

$$= \left[\int_{\theta=0}^{\theta=2\pi} d\theta \right] \cdot \left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} \cos^3\phi \, d\phi \right] \cdot \left[\int_{R=0}^{R=r} \rho R^4 \, dR \right]$$

$$= 2\pi \cdot \frac{2}{3} \cdot \rho \frac{r^5}{5}$$

$$\boxed{I_{/z} = \frac{4}{3}\pi r^3 \rho \cdot \frac{r^2}{5}}$$

or

$$\boxed{I_{/z} = 2m \cdot \frac{r^2}{5}}$$

$\square$