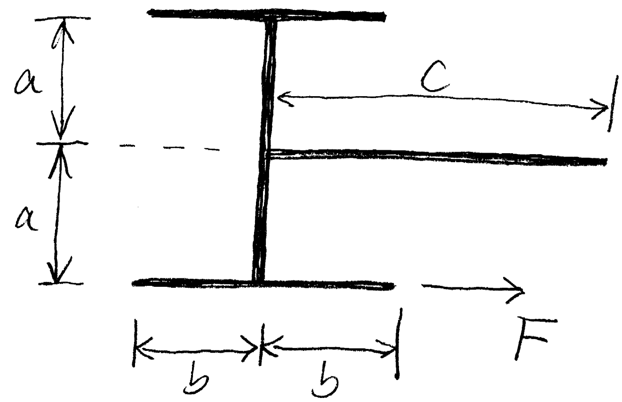
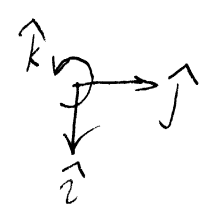
ENGRD/TAM203

HW25 (~~Assigned~~ ~~Assigned~~ on April 25, due on May 2)

Solution by Dennis Yang

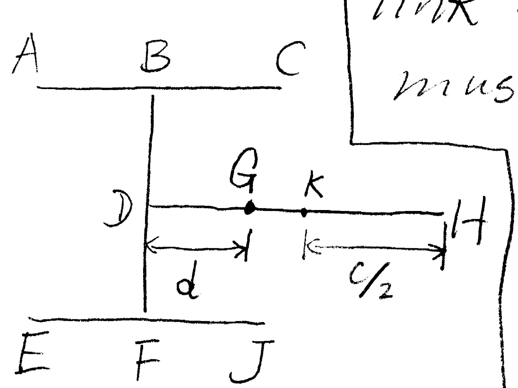
7.3.4. What is the body's acceleration (linear and angular) immediately following the application on the force? $a = 0.8\,m$, $b = 0.5\,m$, $c = 2.2\,m$, $m = 40\,kg$ (the whole body), and $\vec{F} = 10\,N\,\hat{j}$.





Solution

We are going to locate the center of mass first. Since the body is symmetric about link $\overline{DH}$, the center of mass of the body must lie on link $\overline{DH}$. Let the distance between D and G (the center of mass) be denoted as "d". Let "ρ" be the linear density of the body.



Then, we have $\rho = \dfrac{m}{4b+2a+c}$.

In addition,

(*) $$m\vec{r}_{G/D} = m_{AC}\vec{r}_{B/D} + m_{BF}\underbrace{\cancel{\vec{r}_{D/D}}}_{\vec{0}} + m_{EJ}\vec{r}_{F/D} + m_{DH}\left(\tfrac{1}{2}\vec{r}_{H/D}\right)$$

$$(*)\cdot\hat{j} \Rightarrow m(d\hat{j})\cdot\hat{j} = 2b\rho\, a(-\hat{i})\cdot\hat{j} + 2b\rho\, a\hat{i}\cdot\hat{j} + c\rho\cdot\tfrac{1}{2}c\hat{j}\cdot\hat{j}$$

$$\Rightarrow md = \tfrac{1}{2}\rho c^2$$

$$\Rightarrow d = \frac{1}{2}\frac{m}{(2a+4b+c)}c^2\cdot\frac{1}{m}$$

$$\Rightarrow \boxed{d = \frac{c^2}{2(2a+4b+c)}} \approx 0.417\ m$$

Now we compute the moment of inertia about point G.

$\boxed{\text{link } \overline{AC}:}$

$$I_{\overline{AC}/G} = I_{\overline{AC}/B} + \|\vec{r}_{B/G}\|^2\cdot m_{\overline{AC}}$$

$$= \frac{m_{\overline{AC}}(2b)^2}{12} + (a^2+d^2)\cdot m_{\overline{AC}}$$

$$= \frac{\left(\frac{2b}{2a+4b+c}\right)m\cdot 4b^2}{12} + (a^2+d^2)\cdot\frac{2b}{2a+4b+c}m$$

$$\approx 6{,}189\ kg\cdot m^2$$

link $\overline{EJ}$:

$$I_{\overline{EJ}/G} = I_{\overline{EJ}/F} + \|\vec{r}_{F/G}\|^2 \cdot m_{\overline{EJ}}$$

$$= \frac{m_{\overline{EJ}}(2b)^2}{12} + (a^2+d^2) \cdot m_{\overline{EJ}}$$

$$= \frac{\left(\frac{2b}{2a+4b+c}\right) m \cdot 4b^2}{12} + (a^2+d^2)\left(\frac{2b}{2a+4b+c}\right) m$$

$$\approx 6.189 \text{ kg} \cdot \text{m}^2$$

link $\overline{BF}$:

$$I_{\overline{BF}/G} = I_{\overline{BF}/D} + \|\vec{r}_{D/G}\|^2 \cdot m_{\overline{BF}}$$

$$= \frac{m_{\overline{BF}}(2a)^2}{12} + d^2 \cdot m_{\overline{BF}}$$

$$= \frac{\left(\frac{2a}{2a+4b+c}\right) m \cdot 4a^2}{12} + d^2 \cdot \left(\frac{2a}{2a+4b+c}\right) m$$

$$\approx 4.273 \text{ kg} \cdot \text{m}^2$$

link $\overline{DH}$:

$$I_{\overline{DH}/G} = I_{\overline{DH}/K} + \|\vec{r}_{K/G}\|^2 m_{\overline{DH}}$$

$$= \frac{m_{\overline{DH}} \cdot C^2}{12} + \left(\frac{C}{2} - d\right)^2 m_{\overline{DH}}$$

$$= \frac{\left(\frac{C}{2a+4b+c}\right) m \cdot c^2}{12} + \left(\frac{C}{2} - d\right)^2 \left(\frac{c}{2a+4b+c}\right) m$$

$$\approx 13.197 \text{ kg} \cdot \text{m}^2$$

$$I_G = I_{\overline{AC}/G} + I_{\overline{EJ}/G} + I_{\overline{BF}/G} + I_{\overline{DH}/G}$$

$$= 6.189\ kg\cdot m^2 + 6.189\ kg\cdot m^2 + 4.273\ kg\cdot m^2 + 13.197\ kg\cdot m^2$$

$$= 29.848\ kg\cdot m^2$$

$$\boxed{\vec{M}_G = I_G\vec{\alpha}} \Longrightarrow \vec{r}_{J/G} \times \vec{F} = I_G\vec{\alpha}$$

$$\Longrightarrow \vec{\alpha} = \frac{1}{I_G}\left(a\hat{i} + (b-d)\hat{j}\right) \times 10N\hat{j}$$

$$\Longrightarrow \vec{\alpha} = \frac{a\cdot 10N}{I_G}\hat{k}$$

$$\Longrightarrow \boxed{\vec{\alpha} \approx 0.268\ rad/s^2\ \hat{k}}$$

$$\boxed{\vec{F} = m\vec{a}_G} \Longrightarrow 10N\hat{j} = 40kg\cdot\vec{a}_G$$

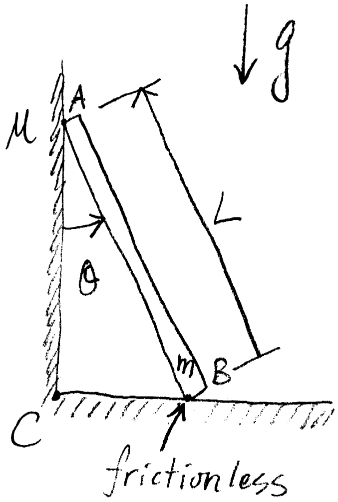
$$\Longrightarrow \boxed{\vec{a}_G = 0.25\ m/s^2\ \hat{j}}$$

$\square$

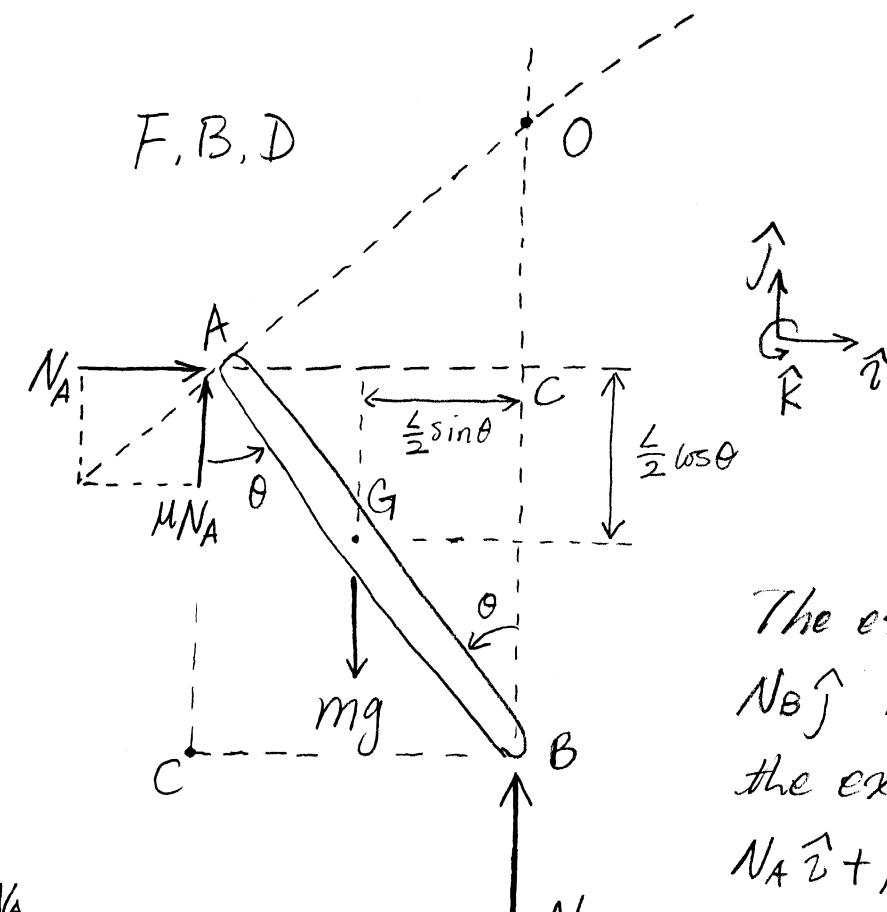
7.3.22

Derive the equation of motion for the system in which the interface between the end B and the floor is friction-free, while the interface at the end A and the wall has a coefficient of friction μ. Determine $t|_{\theta=45°}$, $\dot{\theta}|_{\theta=45°}$ for $\mu = 0, 0.5$, and 1. Plot $\dot{\theta}|_{\theta=45°}$ v.s. μ, $t|_{\theta=45°}$ v.s. μ, and $\dot{\theta}|_{\theta=45°}$ v.s. $t|_{\theta=45°}$.

Assume $t|_{\theta=15°} = 0$, $L = 2.0\,m$, $m = 10\,kg$.



Solution



The extension of $N_B\hat{\jmath}$ intersects with the extension of $N_A\hat{\imath} + \mu N_A\hat{\jmath}$ at point "O" (Note: this intersection exists as long as $N_A \neq 0$.)

$$\frac{\|\vec{r}_{O/C}\|}{\|\vec{r}_{A/C}\|} = \frac{\mu N_A}{N_A} = \mu$$

$$\Longrightarrow \|\vec{r}_{O/C}\| = \mu \|\vec{r}_{A/C}\| = \mu L \sin\theta$$

Since both $N_B\hat{j}$ and $N_A\hat{i} + \mu N_A\hat{j}$ go through point O, their moments about point O are both zero. Thus,

$$\boxed{\sum_i \vec{M}_{i/O} = I_G\ddot{\theta}\hat{k} + \vec{r}_{G/O} \times m\vec{a}_G}$$

$$\Longrightarrow \vec{r}_{G/O} \times mg(-\hat{j}) = I_G\ddot{\theta}\hat{k} + \vec{r}_{G/O} \times m\vec{a}_G \qquad (*)$$

In $(*)$, $\underline{\vec{r}_{G/O} = \frac{L}{2}\sin\theta(-\hat{i}) + (\mu L\sin\theta + \frac{L}{2}\cos\theta)(-\hat{j})}$

and $\vec{a}_G = \vec{a}_{G/C} + \cancelto{\vec{0}}{\vec{a}_C}$ (since the corner is fixed)

$$= \frac{d^2}{dt^2}(\vec{r}_{G/C})$$

$$= \frac{d^2}{dt^2}\left(\frac{L}{2}\sin\theta\,\hat{i} + \frac{L}{2}\cos\theta\,\hat{j}\right)$$

$$= \frac{L}{2}\frac{d^2}{dt^2}(\sin\theta)\,\hat{i} + \frac{L}{2}\frac{d^2}{dt^2}(\cos\theta)\,\hat{j} \qquad \text{(only "}\theta\text{" is varying w.r.t. "}t\text{")}$$

$$\underline{= \frac{L}{2}\left(-\sin\theta\cdot\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right)\hat{i} + \frac{L}{2}\left(-\cos\theta\cdot\dot{\theta}^2 - \sin\theta\,\ddot{\theta}\right)\hat{j}}$$

Now, plug $\vec{r}_{G/O}$ and $\vec{a}_G$ into $(*)$, we have

$$\left(\frac{L}{2}\sin\theta(-\hat{\imath})+\left(\mu L\sin\theta+\frac{L}{2}\cos\theta\right)(-\hat{\jmath})\right)\times mg(-\hat{\jmath})$$

$$= I_G\ddot{\theta}\hat{k}+\left(\frac{L}{2}\sin\theta(-\hat{\imath})+\left(\mu L\sin\theta+\frac{L}{2}\cos\theta\right)(-\hat{\jmath})\right)\times$$

$$m\frac{L}{2}\left(\left(-\sin\theta\,\dot{\theta}^2+\cos\theta\,\ddot{\theta}\right)\hat{\imath}+\left(-\cos\theta\,\dot{\theta}^2-\sin\theta\,\ddot{\theta}\right)\hat{\jmath}\right)$$

$$\Longrightarrow \frac{mgL\sin\theta}{2}\hat{k}=I_G\ddot{\theta}\hat{k}+\frac{mL^2}{2}\left(\mu\sin\theta+\frac{\cos\theta}{2}\right)\left(-\sin\theta\,\dot{\theta}^2+\cos\theta\,\ddot{\theta}\right)\hat{k}$$

$$+\frac{mL^2}{4}\sin\theta\left(-\cos\theta\,\dot{\theta}^2-\sin\theta\,\ddot{\theta}\right)(-\hat{k})$$

$$\Longrightarrow \frac{mgL\sin\theta}{2}\hat{k}=I_G\ddot{\theta}\hat{k}+\frac{mL^2}{2}\left(-\mu\sin^2\theta\,\dot{\theta}^2-\frac{\cos\theta\sin\theta}{2}\dot{\theta}^2+\mu\sin\theta\cos\theta\,\ddot{\theta}+\frac{\cos^2\theta}{2}\ddot{\theta}\right)\hat{k}$$

$$+\frac{mL^2}{2}\left(-\frac{\sin\theta\cos\theta}{2}\dot{\theta}^2-\frac{\sin^2\theta}{2}\ddot{\theta}\right)(-\hat{k})$$

$$\Longrightarrow \frac{mgL\sin\theta}{2}\hat{k}=\left(\frac{mL^2}{12}\ddot{\theta}+\frac{mL^2}{2}\left(-\mu\sin^2\theta\,\dot{\theta}^2+\mu\sin\theta\cos\theta\,\ddot{\theta}+\frac{1}{2}\ddot{\theta}\right)\right)\hat{k}$$

$$(**)$$

$$(**)\cdot\hat{k}\Longrightarrow \frac{mgL\sin\theta}{2}=\frac{mL^2}{12}\ddot{\theta}+\frac{mL^2}{2}\left(-\mu\sin^2\theta\,\dot{\theta}^2+\mu\sin\theta\cos\theta\,\ddot{\theta}+\frac{1}{2}\ddot{\theta}\right)$$

$$\Longrightarrow \left[\frac{mL^2}{12}+\frac{mL^2}{2}\left(\mu\sin\theta\cos\theta+\frac{1}{2}\right)\right]\ddot{\theta}=\frac{mgL\sin\theta}{2}+\frac{mL^2}{2}\mu\sin^2\theta\,\dot{\theta}^2$$

$$\Longrightarrow \boxed{\ddot{\theta} = \frac{\frac{g\sin\theta}{2L} + \frac{1}{2}\mu\sin^2\theta\,\dot{\theta}^2}{\frac{1}{3} + \frac{1}{2}\mu\sin\theta\cos\theta}}$$

need to convert this to a set of first order ODEs, so we can use "ode45", etc., to solve it numerically.

Let $\dot{\theta} = \omega$
then $\ddot{\theta} = \dot{\omega}$

$$\boxed{\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{\frac{g\sin\theta}{2L} + \frac{1}{2}\mu\sin^2\theta\,\omega^2}{\frac{1}{3} + \frac{1}{2}\mu\sin\theta\cos\theta} \end{aligned}}$$