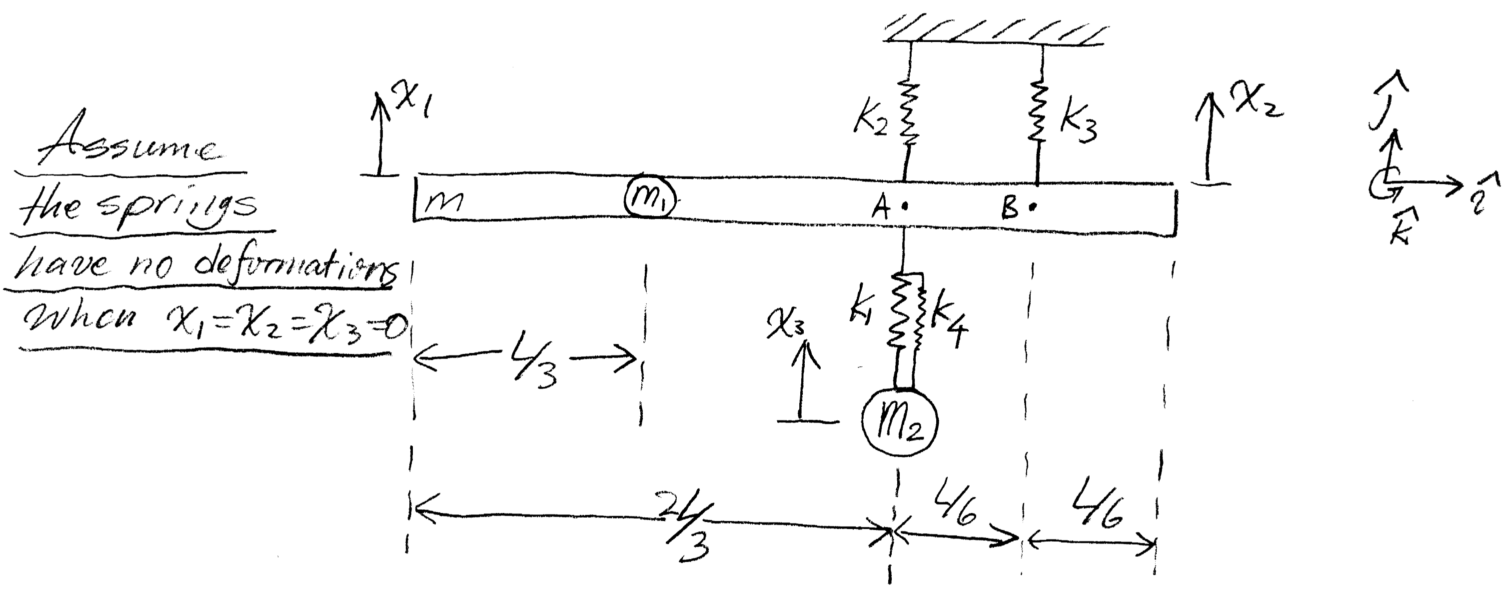
ENGRD / TAM 203

HW 26 (Assigned on April 27, due on May 4)

Solution by Dennis Yang

7.3.25 Find the equation of motion for the illustrated system in terms of the given coordinates. Assume m_2 does NOT rotate and the two springs are very close to each other.

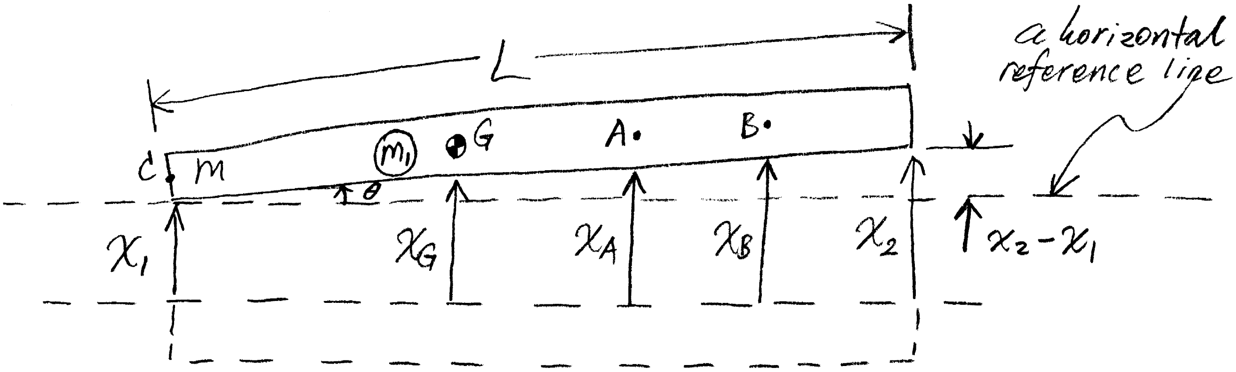


Neglect gravity and nonlinear terms, consider small angle of rotation for m.

Solution

Let "G" denote the center of mass of the system of "m" and "m_1".

In addition, define x_G, x_A, and x_B as drawn in the figure on the next page :

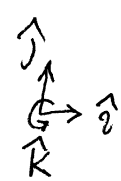


$$\frac{x_2 - x_1}{L} = \sin\theta \approx \theta$$

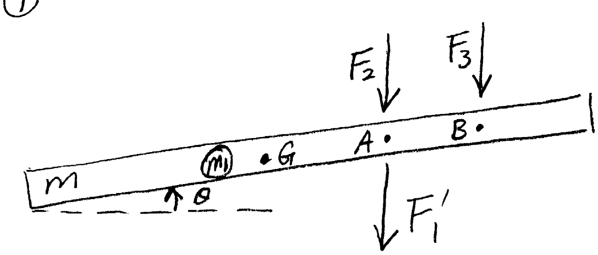
assume "θ" is small

Thus, $\boxed{\theta = \frac{x_2 - x_1}{L} \text{ and } \dot{\theta} = \frac{\dot{x}_2 - \dot{x}_1}{L},\ \ddot{\theta} = \frac{\ddot{x}_2 - \ddot{x}_1}{L}}$

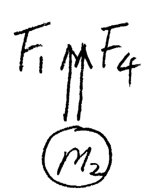
F.B.Ds



①



②



FBD ①

$$\vec{F_1'} = (K_1 + K_4)(x_A - x_3)(-\hat{j})$$

$$\vec{F_2} = K_2 x_A (-\hat{j})$$

$$\vec{F_3} = K_3 x_B (-\hat{j})$$

$$\boxed{\sum_i \vec{F}_i = m\vec{a}_G} \Longrightarrow \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (m+m_1)\ddot{x}_G \hat{j}$$

$$\Longrightarrow \left[(k_1+k_4)(x_A - x_3) + k_2 x_A + k_3 x_B\right](-\hat{j}) = (m+m_1)\ddot{x}_G \hat{j} \quad (a1)$$

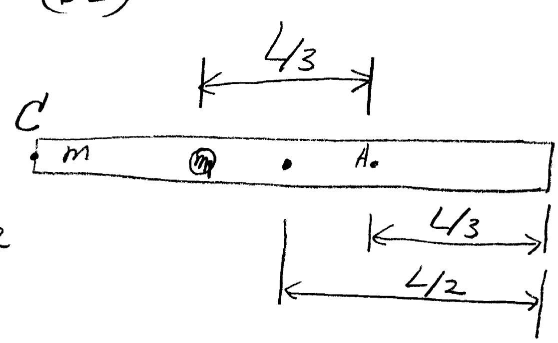
$$(a1)\cdot\hat{j} \Longrightarrow \boxed{-\left((k_1+k_4)(x_A - x_3) + k_2 x_A + k_3 x_B\right) = (m+m_1)\ddot{x}_G} \quad (a2)$$

$$\boxed{\sum_i \vec{M}_{i/A} = I_{/A}\ddot{\theta}\hat{k}} \Longrightarrow \vec{r}_{B/A} \times \vec{F}_3 = I_{/A}\ddot{\theta}\hat{k}$$

$$\Longrightarrow \left(\frac{L}{6}\hat{i}\right) \times k_3 x_B(-\hat{j}) = I_{/A}\ddot{\theta}\hat{k}$$

$$\Longrightarrow \frac{L k_3 x_B}{6}(-\hat{k}) = I_{/A}\ddot{\theta}\hat{k} \quad (b1)$$

$$(b1)\cdot\hat{k} \Longrightarrow -\frac{L k_3 x_B}{6} = I_{/A}\ddot{\theta} \quad (b2)$$



$$I_{/A} = I_{m/A} + I_{m_1/A}$$

$$= \left[\frac{mL^2}{12} + m\left(\frac{L}{2} - \frac{L}{3}\right)^2\right] + m_1\left(\frac{L}{3}\right)^2$$

$$= \frac{1}{9}(m+m_1)L^2 \quad (b3)$$

$$(b2),(b3) \Longrightarrow -\frac{L k_3 x_B}{6} = \frac{1}{9}(m+m_1)L^2\ddot{\theta}$$

$$\Longrightarrow \boxed{-\frac{3 k_3 x_B}{2} = (m+m_1)L\ddot{\theta}} \quad (b4)$$

FBD②

$$\boxed{\sum_i \vec{F}_i = m\vec{a}} \Longrightarrow \vec{F}_1 + \vec{F}_4 = m_2 \ddot{x}_3 \hat{j}$$

$$\Longrightarrow k_1(x_A - x_3)\hat{j} + k_4(x_A - x_3)\hat{j} = m_2 \ddot{x}_3 \hat{j}$$

$$\Longrightarrow (k_1 + k_4)(x_A - x_3)\hat{j} = m_2 \ddot{x}_3 \hat{j} \quad (C1)$$

$$(C1)\cdot\hat{j} \Longrightarrow \boxed{(k_1 + k_4)(x_A - x_3) = m_2 \ddot{x}_3} \quad (C2)$$

Now, we need to locate "G":

$$(m + m_1)\vec{r}_{G/C} = m\frac{L}{2}\hat{i} + m_1\frac{L}{3}\hat{i}$$

$$\Longrightarrow (m + m_1)\vec{r}_{G/C} = \frac{3mL + 2m_1L}{6}\hat{i}$$

$$\Longrightarrow \vec{r}_{G/C} = \frac{3mL + 2m_1L}{6(m + m_1)}\hat{i}$$

Thus, $$\frac{x_G - x_1}{\|\vec{r}_{G/C}\|} = \frac{x_2 - x_1}{L} \quad (= \sin\theta)$$

$$\Longrightarrow x_G = \frac{\|\vec{r}_{G/C}\|}{L}(x_2 - x_1) + x_1$$

$$\Longrightarrow x_G = \frac{3m + 2m_1}{6(m + m_1)}(x_2 - x_1) + x_1$$

$$\Longrightarrow \quad x_G = \frac{3m+2m_1}{6(m+m_1)} x_2 + \frac{3m+4m_1}{6(m+m_1)} x_1$$

$$\Longrightarrow \quad \ddot{x}_G = \frac{3m+2m_1}{6(m+m_1)} \ddot{x}_2 + \frac{3m+4m_1}{6(m+m_1)} \ddot{x}_1 \qquad (d1)$$

$$(d1), (a2) \Longrightarrow -\left((k_1+k_4)(x_A - x_3) + k_2 x_A + k_3 x_B\right)$$

$$= \left(\frac{m}{2} + \frac{m_1}{3}\right) \ddot{x}_2 + \left(\frac{m}{2} + \frac{2m_1}{3}\right) \ddot{x}_1 \qquad (e1)$$

$$(b4), \boxed{\ddot{\theta} = \frac{\ddot{x}_2 - \ddot{x}_1}{L}} \Longrightarrow -\frac{3k_3 x_B}{2} = (m+m_1)\ddot{x}_2 - (m+m_1)\ddot{x}_1$$

$$(e2)$$

$$(e1) + (e2)\left(\frac{\frac{m}{2} + \frac{2m_1}{3}}{m+m_1}\right) \Longrightarrow -\left((k_1+k_4)(x_A - x_3) + k_2 x_A + k_3 x_B\right)$$

$$-\frac{3k_3 x_B}{2} \cdot \frac{\frac{m}{2} + \frac{2m_1}{3}}{m+m_1} = (m+m_1)\ddot{x}_2$$

$$\Longrightarrow \boxed{(m+m_1)\ddot{x}_2 = -(k_1+k_2+k_4)x_A - \left(2 - \frac{m}{4(m+m_1)}\right)k_3 x_B + (k_1+k_4)x_3}$$

$$(f1)$$

$$(e1) - (e2)\cdot\left(\frac{\frac{m}{2} + \frac{m_1}{3}}{m+m_1}\right) \Longrightarrow -\left((k_1+k_4)(x_A - x_3) + k_2 x_A + k_3 x_B\right)$$

$$+ \frac{3k_3 x_B}{2} \cdot \frac{\frac{m}{2} + \frac{m_1}{3}}{m+m_1} = (m+m_1)\ddot{x}_1$$

$$\Longrightarrow \boxed{(m+m_1)\ddot{x}_1 = -(k_1+k_2+k_4)x_A - \left(\frac{1}{2} - \frac{m}{4(m+m_1)}\right)k_3 x_B + (k_1+k_4)x_3}$$

(f 2)

Finally, we have a set of equations:

(f 1), (f 2), and (C 2), where

the unknowns are: x_1, x_2, and x_3 and their 2^{nd} order time derivatives. In particular, we should realize that:

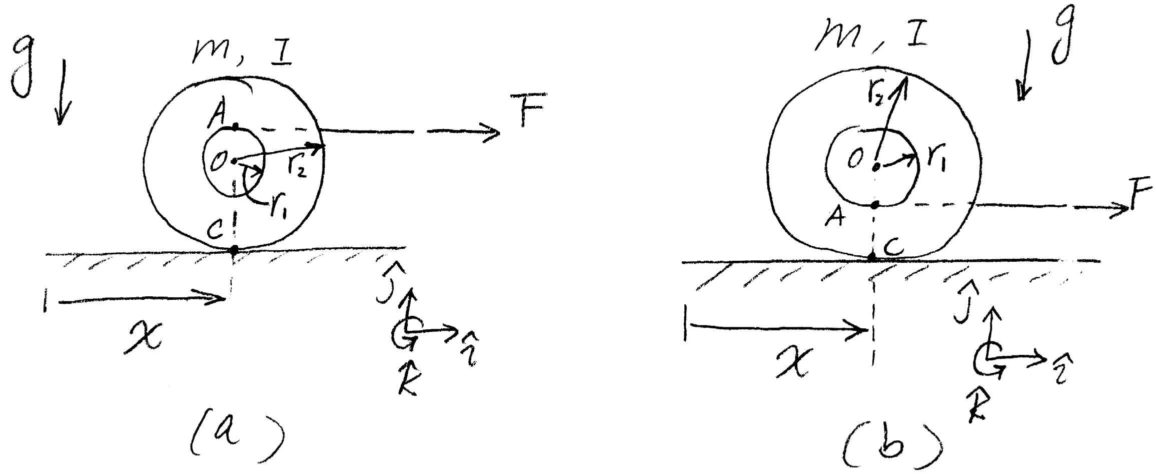
$$\sin\theta = \frac{x_A - x_1}{\frac{2L}{3}} = \frac{x_2 - x_1}{L} \Longrightarrow \underline{x_A = \frac{2}{3}(x_2 - x_1) + x_1}$$

$$\sin\theta = \frac{x_B - x_1}{\frac{5L}{6}} = \frac{x_2 - x_1}{L} \Longrightarrow \underline{x_B = \frac{5}{6}(x_2 - x_1) + x_1}$$

$\square$

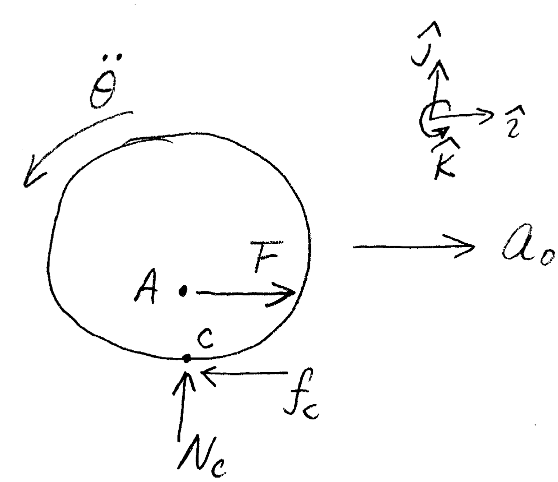
7.3.33

Assume pure rolling, find the acceleration for the center of the spool for both case (a) and (b).



Solution

FBD



Kinematic constraint: $\boxed{\ddot{\theta} r_2 = -a_o}$

$$\boxed{\sum_i \vec{M}_{i/C} = I_o \ddot{\theta} \hat{k} + \vec{r}_{o/c} \times m \vec{a}_o}$$

$$\Longrightarrow \vec{r}_{A/C} \times \vec{F} = I_o \ddot{\theta} \hat{k} + r_2 \hat{j} \times m(-\ddot{\theta} r_2) \hat{i}$$

$$\Longrightarrow \vec{r}_{A/C} \times \vec{F} = I_o \ddot{\theta} \hat{k} + m r_2^2 \ddot{\theta} \hat{k} \quad (*)$$

$$(*)\cdot\hat{k} \Longrightarrow (\vec{r}_{A/C}\times\vec{F})\cdot\hat{k} = (I_0 + m r_2^2)\ddot{\theta}$$

$$\Longrightarrow \boxed{\ddot{\theta} = \frac{1}{I_0 + m r_2^2}(\vec{r}_{A/C}\times\vec{F})\cdot\hat{k}}$$

In case (a), $\vec{r}_{A/C} = (r_1 + r_2)\hat{j}$.

$$\Longrightarrow \ddot{\theta} = \frac{1}{I_0 + m r_2^2}\left((r_1+r_2)\hat{j}\times F\hat{i}\right)\cdot\hat{k}$$

$$= -\frac{(r_1+r_2)F}{I_0 + m r_2^2}$$

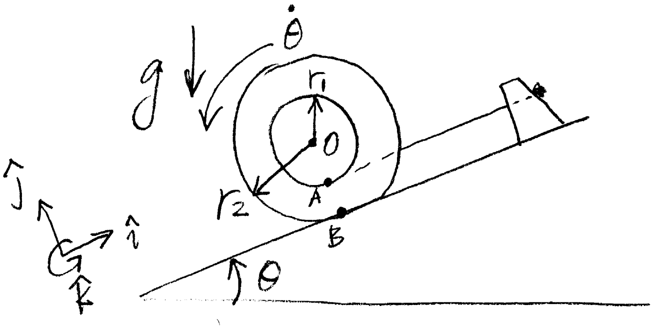
$$\Longrightarrow \vec{a}_0 = a_0\hat{i} = -\ddot{\theta} r_2\hat{i} = \boxed{\frac{(r_1+r_2)F}{I_0 + m r_2^2} r_2\hat{i}}$$

In case (b), $\vec{r}_{A/C} = (r_2 - r_1)\hat{j}$. (this is the only difference compared with case (a))

thus $\ddot{\theta} = -\dfrac{(r_2 - r_1)F}{I_0 + m r_2^2}$

and $\boxed{\vec{a}_0 = \dfrac{(r_2 - r_1)F}{I_0 + m r_2^2} r_2\hat{i}}$

7.3.41 What is the acceleration of the reel's center if it is released from rest? Assume roll without slip and the rope is inextensible



$r_2 > r_1$

Solution If the reel were able to "roll without slip", then

$$\vec{v}_O = \cancel{\vec{v}_A} + \dot{\theta}\hat{k} \times \vec{r}_{O/A} \quad \vec{0} \text{ (inextensible rope)}$$

$$= \dot{\theta}\hat{k} \times r_1\hat{j} = -r_1\dot{\theta}\hat{i}$$

it is also true that

$$\vec{v}_O = \cancel{\vec{v}_B} + \dot{\theta}\hat{k} \times \vec{r}_{O/B} \quad \vec{0} \text{ (roll without slip)}$$

$$= \dot{\theta}\hat{k} \times r_2\hat{j} = -r_2\dot{\theta}\hat{i} \neq -r_1\dot{\theta}\hat{i}$$

(for $r_2 > r_1$)

This contradiction indicates that the assumption of "rolling without slip" is WRONG!

That is, the reel either doesn't move at all, or it has to slip to move! In the former case, $\vec{a}_o = \vec{0}$, while for the latter case, we need the coefficient of dynamic friction to determine $\vec{a}_o$.