ENGRD/TAM 203
HW27 (Assigned on May 2)
Solution by Dennis Yang

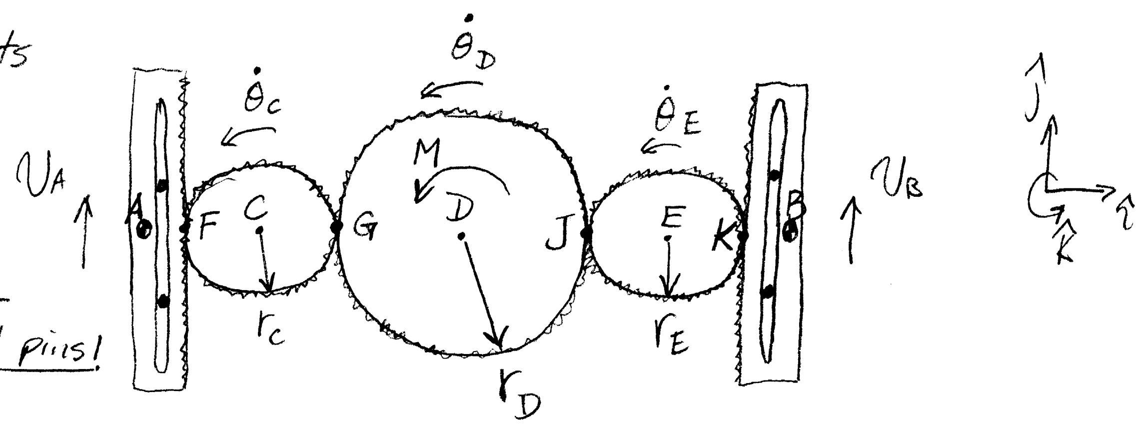
7.4.5

$m_A = 10kg$, $m_B = 20\ kg$, $m_C = 6\ kg$, $m_D = 12\ kg$. $m_E = 6kg$. $r_C = r_E = 0.2\ m$, $r_D = 0.4\ m$.

Neglect gravity!!!

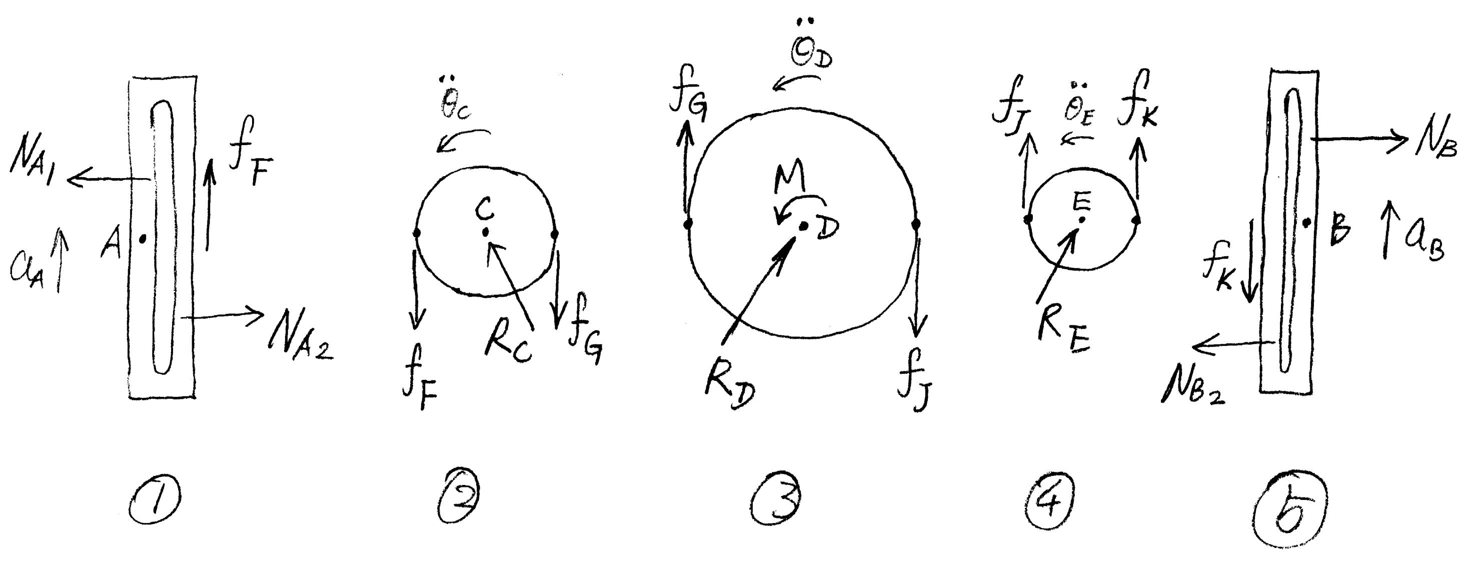
The system is initially at rest, and then a constant moment of 200 N·m is applied. find $\vec{v}_A$, $\vec{v}_B$ after 0.5 s

Pins are fixed to provide constraints to racks A & B, so that they move vertically only. Assume NO friction between A, B and pins!



Solution

FBDs:



FBD ①: $\vec{a}_A = \frac{d}{dt}(\vec{v}_A) = \frac{d}{dt}(v_A \hat{j}) = \dot{v}_A \hat{j}$

$$\sum_i \vec{F}_i = m\vec{a} \Longrightarrow N_{A2}\hat{i} + N_{A1}(-\hat{i}) + f_F \hat{j} = m_A \vec{a}_A \quad (1)$$

$$(1)\cdot\hat{j} \Longrightarrow \underline{f_F = m_A \dot{v}_A \quad (2)}$$

FBD ②: $\sum_i \vec{M}_{i/C} = I_C \ddot{\theta}_C \hat{k}$

$$\Longrightarrow f_F r_C \hat{k} + f_G r_C(-\hat{k}) = I_C \ddot{\theta}_C \hat{k} \quad (3)$$

$$(3)\cdot\hat{k} \Longrightarrow \underline{f_F r_C - f_G r_C = I_C \ddot{\theta}_C \quad (4)}$$

FBD ③: $\sum_i \vec{M}_{i/D} = I_D \ddot{\theta}_D \hat{k}$

$$\Longrightarrow f_G r_D(-\hat{k}) + f_J r_D(-\hat{k}) + M\hat{k} = I_D \ddot{\theta}_D \hat{k} \quad (5)$$

$$(5)\cdot\hat{k} \Longrightarrow \underline{-f_G r_D - f_J r_D + M = I_D \ddot{\theta}_D \quad (6)}$$

FBD ④: $\sum_i \vec{M}_{i/E} = I_E \ddot{\theta}_E \hat{k}$

$$\Longrightarrow f_J r_E(-\hat{k}) + f_K r_E \hat{k} = I_E \ddot{\theta}_E \hat{k} \quad (7)$$

$$(7)\cdot\hat{k} \Longrightarrow \underline{-f_J r_E + f_K r_E = I_E \ddot{\theta}_E \quad (8)}$$

FBD ⑤: $\vec{a}_B = \frac{d}{dt}(\vec{v}_B) = \frac{d}{dt}(v_B \hat{j}) = \dot{v}_B \hat{j}$

$$\sum_i \vec{F}_i = m\vec{a} \Longrightarrow N_{B1}\hat{i} + N_{B2}(-\hat{i}) + f_K(-\hat{j}) = m_B \dot{v}_B \hat{j} \quad (9)$$

$$(9)\cdot\hat{j} \Longrightarrow \quad -f_K = m_B \dot{v}_B \quad (10)$$

$$(2) \Longrightarrow \qquad f_F = m_A \dot{v}_A$$

$$-(4)/r_C \Longrightarrow \qquad -f_F + f_G = -\frac{I_C}{r_C}\ddot{\theta}_C$$

$$(6)/r_D \Longrightarrow \qquad -f_G - f_J + \frac{M}{r_D} = \frac{I_D}{r_D}\ddot{\theta}_D$$

$$-(8)/r_E \Longrightarrow \qquad f_J - f_K = -\frac{I_E}{r_E}\ddot{\theta}_E$$

$$-(10) \Longrightarrow \qquad f_K = -m_B \dot{v}_B$$

$$(2) - \frac{(4)}{r_C} + \frac{(6)}{r_D} - \frac{(8)}{r_E} - (10) \Longrightarrow \frac{M}{r_D} = m_A\dot{v}_A - \frac{I_C}{r_C}\ddot{\theta}_C + \frac{I_D}{r_D}\ddot{\theta}_D - \frac{I_E}{r_E}\ddot{\theta}_E - m_B\dot{v}_B \quad (11)$$

$$(11)\cdot r_D \Longrightarrow \boxed{M = m_A r_D \dot{v}_A - I_C \frac{r_D}{r_C}\ddot{\theta}_C + I_D\ddot{\theta}_D - I_E\frac{r_D}{r_E}\ddot{\theta}_E - m_B r_D \dot{v}_B} \quad (12)$$

Next, we will use the kinematic constraints to derive how $\dot{v}_A$, $\ddot{\theta}_C$, $\ddot{\theta}_E$, and $\dot{v}_B$ are related to $\ddot{\theta}_D$

Now, consider kinematic constraints:

contact point "G"

$$(\dot{\theta}_C \hat{k}) \times \overbrace{(r_C \hat{\imath})}^{\vec{r}_{G/C}} = (\dot{\theta}_D \hat{k}) \times \overbrace{(r_D(-\hat{\imath}))}^{\vec{r}_{G/D}} = \vec{v}_G$$

$$\Longrightarrow \quad \dot{\theta}_C r_C \hat{\jmath} = \dot{\theta}_D r_D (-\hat{\jmath})$$

$$\Longrightarrow \quad \boxed{\dot{\theta}_C = -\frac{r_D}{r_C}\dot{\theta}_D} \quad (13a) \qquad \text{thus also, } \boxed{\ddot{\theta}_C = -\frac{r_D}{r_C}\ddot{\theta}_D} \quad (13b)$$

contact point "J"

$$(\dot{\theta}_E \hat{k}) \times \overbrace{(r_E(-\hat{\imath}))}^{\vec{r}_{J/E}} = (\dot{\theta}_D \hat{k}) \times \overbrace{(r_D \hat{\imath})}^{\vec{r}_{J/D}} = \vec{v}_J$$

$$\Longrightarrow \quad \dot{\theta}_E r_E (-\hat{\jmath}) = \dot{\theta}_D r_D \hat{\jmath}$$

$$\Longrightarrow \quad \boxed{\dot{\theta}_E = -\frac{r_D}{r_E}\dot{\theta}_D} \quad (14a) \qquad \text{thus also, } \boxed{\ddot{\theta}_E = -\frac{r_D}{r_E}\ddot{\theta}_D} \quad (14b)$$

for rack "A"

$$\vec{v}_A = v_A \hat{\jmath} = (\dot{\theta}_C \hat{k}) \times \overbrace{(r_C(-\hat{\imath}))}^{\vec{r}_{F/C}}$$

$$\Longrightarrow \quad v_A \hat{\jmath} = \dot{\theta}_C r_C (-\hat{\jmath})$$

$$\Longrightarrow \quad v_A = -\dot{\theta}_C r_C = -\left(-\frac{r_D}{r_C}\dot{\theta}_D\right) r_C$$

$$\Longrightarrow \quad \boxed{v_A = r_D \dot{\theta}_D} \quad (15a) \qquad \text{thus also, } \boxed{\dot{v}_A = r_D \ddot{\theta}_D} \quad (15b)$$

for rack "B"

$$\vec{v}_B = v_B\hat{j} = (\dot{\theta}_E\hat{k})\times\overbrace{(r_E\hat{i})}^{\vec{r}_{K/E}}$$

$$\Longrightarrow \quad v_B\hat{j} = \dot{\theta}_E r_E \hat{j}$$

$$\Longrightarrow \quad v_B = \dot{\theta}_E r_E = \left(-\frac{r_D}{r_E}\dot{\theta}_D\right) r_E$$

$$\Longrightarrow \quad \boxed{v_B = -r_D\dot{\theta}_D} \qquad \text{thus also,} \quad \boxed{\dot{v}_B = -r_D\ddot{\theta}_D}$$

(16a) (16b)

Now, the substitution of (13b), (14b), (15b), and (16b) into (12) gives :

$$M = m_A r_D(r_D\ddot{\theta}_D) - \overbrace{I_C}^{\frac{1}{2}m_C r_C^2}\frac{r_D}{r_C}\left(-\frac{r_D}{r_C}\ddot{\theta}_D\right) + \overbrace{I_D}^{\frac{1}{2}m_D r_D^2}\ddot{\theta}_D - \overbrace{I_E}^{\frac{1}{2}m_E r_E^2}\frac{r_D}{r_E}\left(-\frac{r_D}{r_E}\ddot{\theta}_D\right) - m_B r_D(-r_D\ddot{\theta}_D)$$

$$\Longrightarrow M = \left[m_A r_D^2 + \tfrac{1}{2}m_C r_D^2 + \tfrac{1}{2}m_D r_D^2 + \tfrac{1}{2}m_E r_D^2 + m_B r_D^2\right]\ddot{\theta}_D$$

$$\Longrightarrow \quad \ddot{\theta}_D = \frac{M}{(m_A + m_B + \frac{1}{2}m_C + \frac{1}{2}m_D + \frac{1}{2}m_E)\, r_D^2}$$

$$\Longrightarrow \quad \dot{\theta}_D\Big|_{t=0.5s} - \cancelto{0 \text{ (initially at rest!)}}{\dot{\theta}_D\Big|_{t=0s}} = \ddot{\theta}_D \cdot \Delta t$$

$$\Longrightarrow \boxed{\dot{\theta}_D\Big|_{t=0.5s} = \frac{M\Delta t}{(m_A + m_B + \frac{1}{2}m_C + \frac{1}{2}m_D + \frac{1}{2}m_E)\, r_D^2}}$$

$$\boxed{\dot{\theta}_D \Big|_{t=0.5s} \approx 14.881 \text{ rad/s}}$$

By (15a), $\vec{v_A}\Big|_{t=0.5s} = v_A\Big|_{t=0.5s} \hat{j}$

$= r_D \cdot \dot{\theta}_D\Big|_{t=0.5s} \hat{j}$

$\approx 5.952 \text{ m/s } \hat{j}$

and by (16a) $\vec{v_B}\Big|_{t=0.5s} = v_B\Big|_{t=0.5s} \hat{j}$

$= -r_D \dot{\theta}_D\Big|_{t=0.5s} \hat{j}$

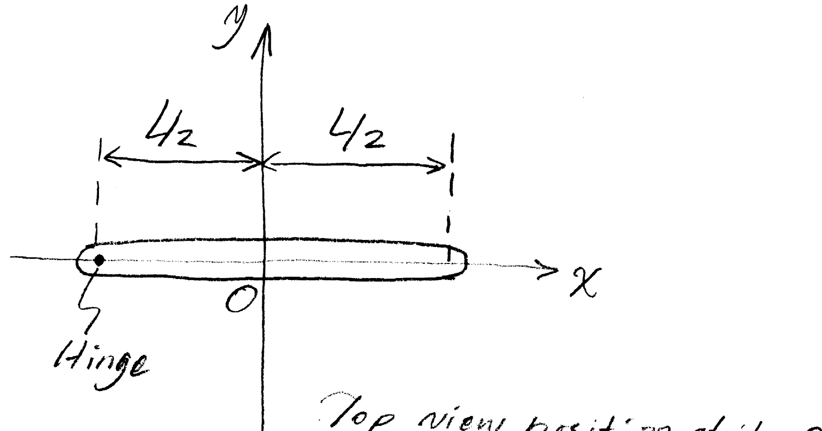
$\approx -5.952 \text{ m/s } \hat{j}$

□

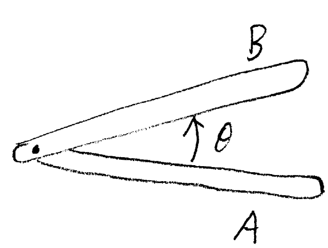
7.4.19

Two uniform bars (A & B) are joined together at their left ends by a hinge joint. A massless motor in A applies a moment to body B. Initially the system is at rest on a horizontal frictionless surface and aligned as shown. At $t=0$, the system is allowed to move as a result of the applied moment. How larger a circle, centered at $x=y=0$, must be drawn so that the entire system stays in the circle for all time $t>0$?

$L = 0.3\,m$, $m_A = 20\,kg$, $m_B = 60\,kg$.



initial configuration



general configuration

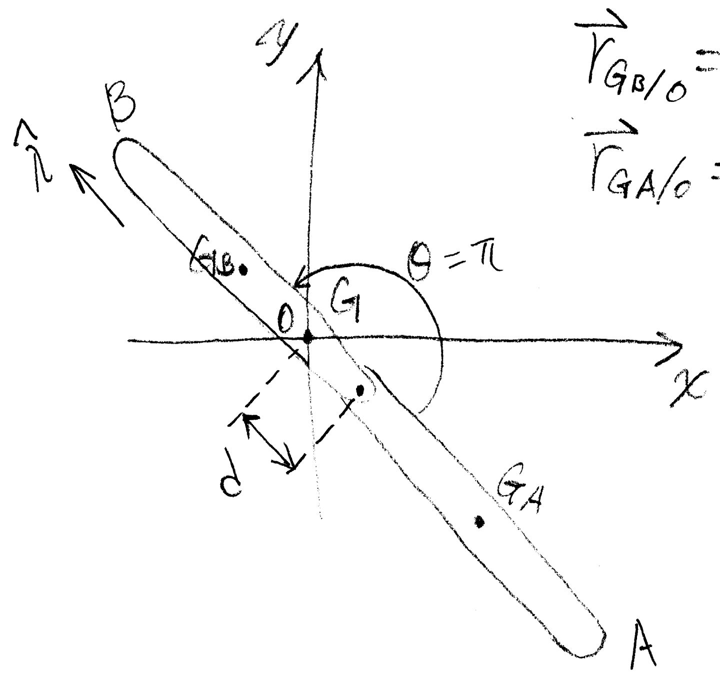
<u>Solution</u>

Let "G" be the center of mass of the 2-bar system and Let "G_A" and "G_B" be the centers of masses of bar A and bar B respectively.

Since the bars are both uniform, the initial configuration indicates that

$\vec{r}_{G/O} = \vec{0}$. In addition, the system is initially at rest and experiences No total external forces and moments, we have $\vec{v}_G = \vec{a}_G = \vec{0}$ throughout, i.e., point "G" stays at the origin all the time.

when $\theta = \pi$ rad, we have



$$\vec{r}_{G_B/O} = \left(\frac{L}{2} - d\right)\hat{\lambda}$$

$$\vec{r}_{G_A/O} = \left(\frac{L}{2} + d\right)(-\hat{\lambda})$$

and

$$\boxed{\begin{aligned} &m_B \vec{r}_{G_B/O} + m_A \vec{r}_{G_A/O} \\ &= (m_B + m_A)\cancelto{\vec{0}}{\vec{r}_{G/O}} = \vec{0} \end{aligned}}$$

$$\Downarrow$$

$$\Rightarrow m_B\left(\tfrac{L}{2}-d\right)\hat{\lambda}+m_A\left(\tfrac{L}{2}+d\right)(-\hat{\lambda})=\vec{0} \qquad (*)$$

$$(*)\cdot\hat{\lambda} \Longrightarrow m_B\left(\tfrac{L}{2}-d\right)-m_A\left(\tfrac{L}{2}+d\right)=0$$

$$\Longrightarrow (m_B-m_A)\tfrac{L}{2}=(m_B+m_A)d$$

$$\Longrightarrow \boxed{d=\frac{m_B-m_A}{m_B+m_A}\cdot\frac{L}{2}} \qquad (d>0 \text{ since } m_B>m_A)$$

Thus, $\|\vec{r}_{A/O}\|=d+L=\dfrac{m_B-m_A}{m_B+m_A}\dfrac{L}{2}+L$

$$=\frac{3m_B+m_A}{2(m_B+m_A)}L$$

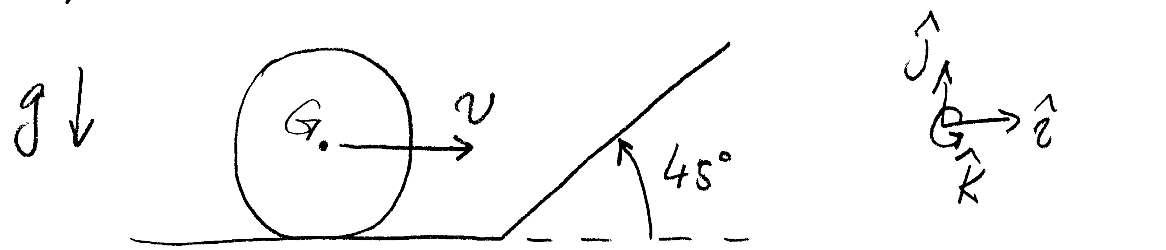
i.e., the minimum radius needed to enclose the whole system is $\|\vec{r}_{A/O}\|=\dfrac{3m_B+m_A}{2(m_B+m_A)}L$

for $m_A=20$ kg, $m_B=60$ kg,

$$\boxed{\|\vec{r}_{A/O}\|=\frac{5}{4}L}$$

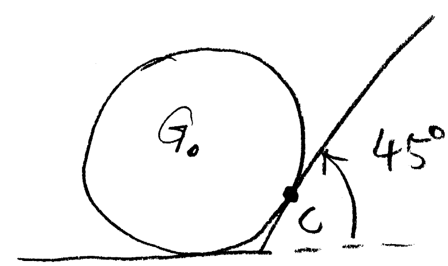
$\square$

7.4.23 A uniform disk A ($m_A = 10\,kg$, $r = 0.25\,m$) rolls without slip with velocity $v\hat{\imath}$. What is its angular velocity after striking a 45° sloped surface (Assume zero rebound).

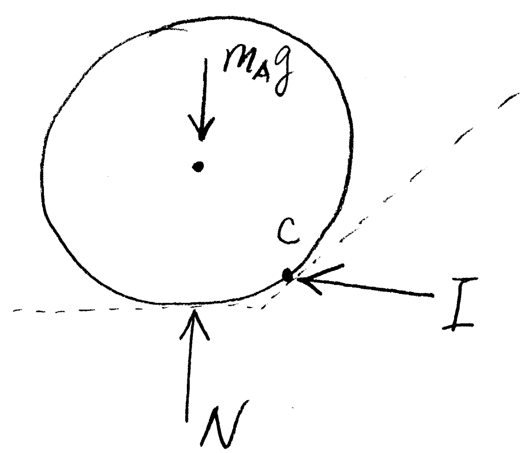


Solution

just at impact:



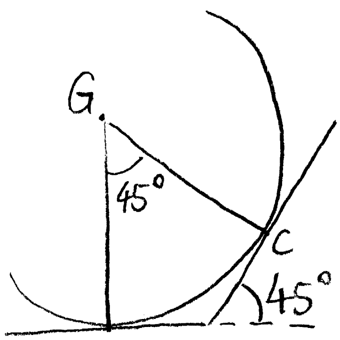
FBD



at the impact, the impulse goes through the impact point "C", thus provides no angular impulse about point "C". Then, the total angular momentum of the disk about point "C" is conserved.

The angular momentum about "C", before the impact, is given by

$$\vec{H}_{/C}\Big|_{before} = I_G\,\omega_1(-\hat{k}) + \vec{r}_{G/C}\times m_A\vec{v}$$



$$= I_G\,\omega_1(-\hat{k}) + \vec{r}_{G/C}\times m_A v\,\hat{\imath}$$

$$= \left(\tfrac{1}{2}m_A r^2\right)\omega_1(-\hat{k}) + \left[(\vec{r}_{G/C}\cdot\hat{\imath})\hat{\imath} + (\vec{r}_{G/C}\cdot\hat{\jmath})\hat{\jmath}\right]\times m_A v\,\hat{\imath}$$

$$= -\tfrac{1}{2}m_A r^2\omega_1\hat{k} + m_A v\,(\vec{r}_{G/C}\cdot\hat{\jmath})(-\hat{k})$$

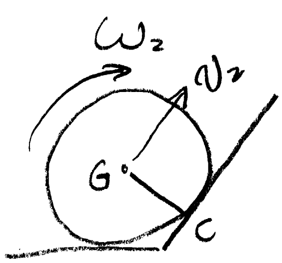
$$= -\tfrac{1}{2}m_A r^2\omega_1\hat{k} + m_A v\,r\cos 45^\circ(-\hat{k})$$

$$= -\left(\tfrac{1}{2}m_A r\,\underbrace{r\omega_1}_{=v} + m_A r\cos 45^\circ\,v\right)\hat{k}$$

$$\boxed{\vec{H}_{/C}\Big|_{before} = -\left(\tfrac{1}{2}+\cos 45^\circ\right)m_A r v\,\hat{k}}$$

The angular momentum about "C", right after the impact, is given by

$$\vec{H}_{/C}\Big|_{after} = I_G\,\omega_2(-\hat{k}) + \vec{r}_{G/C}\times m_A\vec{v}_2$$



$$= \tfrac{1}{2}m_A r^2\omega_2(-\hat{k}) + \vec{r}_{G/C}\times m_A\underbrace{\left(\omega_2(-\hat{k})\times\vec{r}_{G/C}\right)}_{\vec{v}_2}$$

$$= \tfrac{1}{2}m_A r^2\omega_2(-\hat{k}) + m_A\underbrace{\|\vec{r}_{G/C}\|^2}_{=r^2}\omega_2(-\hat{k})$$

$$= -\left(\frac{1}{2} m_A r^2 \omega_2 + m_A r^2 \omega_2\right) \hat{k}$$

$$\boxed{\vec{H}_{/C}\Big|_{after} = -\frac{3}{2} m_A r^2 \omega_2 \hat{k}}$$

$$\vec{H}_{/C}\Big|_{before} = \vec{H}_{/C}\Big|_{after} \Longrightarrow -\left(\frac{1}{2} + \cos 45°\right) m_A r v \hat{k}$$
$$= -\frac{3}{2} m_A r^2 \omega_2 \hat{k} \quad (*)$$

$$(*) \cdot \hat{k} \Longrightarrow -\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) m_A r v = -\frac{3}{2} m_A r^2 \omega_2$$

$$\Longrightarrow \boxed{\omega_2 = \frac{1+\sqrt{2}}{3} \frac{v}{r}}$$

$\square$