Updated on February 24

ENGRD/TAM 203: Dynamics (Spring 2006)

Solution of Homework 3 (assigned on Jan. 31, due on Feb. 7)

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1. Problem 2.3.2.

Statement. Car B is driving strait toward the point O at a constant speed v. An observer, located at A, tracks the car with a radar gun (see Figure 1). What is the speed $|\dot{r}_{B/A}|$ that the observer at A records? **Note that $r_{B/A}$ means the magnitude (same as the "length") of vector $\vec{r}_{B/A}$ while $\dot{r}_{B/A}$ means the time derivative of this magnitude.**

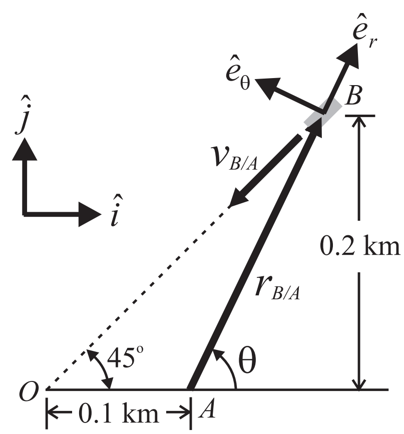

Figure 1

Solution. Take point A to be the origin of our polar coordinate system (as shown in Figure 1). Then $\|\vec{r}_{B/A}\| = r$ and $\dot{r}_{B/A} = \dot{r}$. In terms $\hat{e}_r$ and $\hat{e}_\theta$, $\vec{v}_{B/A} = \frac{d}{dt}\vec{r}_{B/A}$ can be given as

$$\vec{v}_{B/A} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \tag{1.1}$$

while in terms of $\hat{\imath}$ and $\hat{\jmath}$, we have

$$\vec{v}_B = v\cos 45^\circ(-\hat{\imath}) + v\sin 45^\circ(-\hat{\jmath})\,. \tag{1.2}$$

Since $\vec{v}_A = 0$ (A is fixed), we have that $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_{B/A}$. It follows from (1.1) and (1.2) that

$$\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \;=\; v\cos 45^\circ(-\hat{\imath}) + v\sin 45^\circ(-\hat{\jmath})\,. \tag{1.3}$$

We dot product the both sides of (1.3) with $\hat{e}_r$

$$\Big(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\Big) \bullet \hat{e}_r = \Big(v\cos 45^\circ(-\hat{\imath}) + v\sin 45^\circ(-\hat{\jmath})\Big) \bullet \hat{e}_r$$

$$\Longrightarrow \dot{r}\,\hat{e}_r \bullet \hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \bullet \hat{e}_r = -v\cos 45^\circ\,\hat{\imath} \bullet \hat{e}_r - v\sin 45^\circ\,\hat{\jmath} \bullet \hat{e}_r$$

$$\Longrightarrow \dot{r} = -v\cos 45^\circ(\cos\theta) - v\sin 45^\circ(\sin\theta)$$

$$\Longrightarrow \boxed{\dot{r} = -v(\cos 45^\circ\cos\theta + \sin 45^\circ\sin\theta)\,.} \tag{1.4}$$

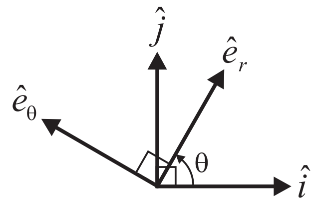

Figure 2

From the given geometry, we know that

$$\cos\theta = \frac{0.1\,\mathrm{km}}{r} = \frac{0.1\,\mathrm{km}}{\sqrt{(0.1\,\mathrm{km})^2 + (0.2\,\mathrm{km})^2}} \approx 0.4472\,; \tag{1.5}$$

$$\sin\theta = \frac{0.2\,\mathrm{km}}{r} = \frac{0.2\,\mathrm{km}}{\sqrt{(0.1\,\mathrm{km})^2 + (0.2\,\mathrm{km})^2}} \approx 0.8944\,. \tag{1.6}$$

The substitution of (1.5) and (1.6) into (1.4) gives

$$\begin{aligned} \dot{r} &= -v(\cos 45^\circ \cdot 0.4472 + \sin 45^\circ \cdot 0.8944) \\ &\approx -0.9487v\,. \end{aligned}$$

Thus, $\underline{|\dot{r}_{B/A}| = |\dot{r}| = 0.9487v\,.}$ □

2. Problem 2.3.3.

Statement. Explain why $\|\frac{d}{dt}\vec{r}(t)\|$ is not in general equal to $\frac{d}{dt}\|\vec{r}(t)\|$.

Solution. $\vec{r}(t)$ is a vector. $\frac{d}{dt}\vec{r}(t)$, which is also a vector, is the time derivative of $\vec{r}(t)$. $\|\frac{d}{dt}\vec{r}(t)\|$, which is always positive, is the magnitude of vector $\frac{d}{dt}\vec{r}(t)$.

On the other hand, $\|\vec{r}(t)\|$ is the magnitude of vector $\vec{r}(t)$. $\frac{d}{dt}\|\vec{r}(t)\|$ is the time derivative of this magnitude. Thus, $\frac{d}{dt}\|\vec{r}(t)\|$ can be positive, negative, or zero as the magnitude of $\vec{r}(t)$ can increase, decrease, or be constant, with respect to time.

In terms of polar coordinates, we have

$$\frac{d}{dt}\vec{r}(t) = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \Longrightarrow \left\|\frac{d}{dt}\vec{r}(t)\right\| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

while

$$\vec{r}(t) = r\,\hat{e}_r \Longrightarrow \|\vec{r}(t)\| = r \Longrightarrow \frac{d}{dt}\|\vec{r}(t)\| = \dot{r}\,.$$

It is obvious that $\dot{r} \neq \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$ in general since $\dot{\theta} \neq 0$ in general.

Consider a particle going in circles at speed v. Let the radius of the circle be r. Then $\|\frac{d}{dt}\vec{r}(t)\| = v \neq 0$ while $\frac{d}{dt}\|\vec{r}(t)\| = \dot{r} = 0$ since r is constant for the circle. □

3. Problem 2.3.19.

Statement. Assume that the distance between the hill and the cloud layer is 150 ft. Neglect the dimensions of the hill. Initially, the light from the headlights makes a 20° angle with the horizontal, and the light beam rotates around at the constant rate of 20°/s (clockwise) as the car moves up the hill. What are the initial speed and acceleration of the light spot on the cloud's underside? What are they when the light beam makes a 5° angle with the horizontal?

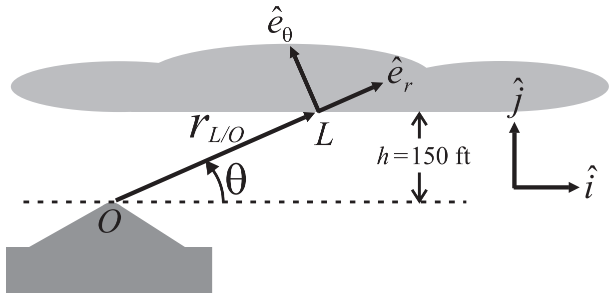

Figure 3

Solution. As depicted in Figure 3, using polar coordinates, we can describe the motion of the light spot by

$$\vec{r}_{L/O} = r\,\hat{e}_r\,; \tag{3.1}$$
$$\vec{v}_L = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\,; \tag{3.2}$$
$$\vec{a}_L = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta\,. \tag{3.3}$$

Now we express $\hat{e}_r$ and $\hat{e}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ (see Figure 4)

$$\begin{aligned}\hat{e}_r &= (\hat{e}_r \bullet \hat{\imath})\,\hat{\imath} + (\hat{e}_r \bullet \hat{\jmath})\,\hat{\jmath}\\ &= \cos\theta\,\hat{\imath} + \cos(90^\circ - \theta)\,\hat{\jmath}\\ &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\,; \end{aligned}\tag{3.4}$$
$$\begin{aligned}\hat{e}_\theta &= (\hat{e}_\theta \bullet \hat{\imath})\,\hat{\imath} + (\hat{e}_\theta \bullet \hat{\jmath})\,\hat{\jmath}\\ &= \cos(90^\circ + \theta)\,\hat{\imath} + \cos\theta\,\hat{\jmath}\\ &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\,. \end{aligned}\tag{3.5}$$

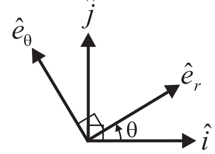

Figure 4

With (3.4) and (3.5), (3.2) and (3.3) become

$$\begin{aligned} \vec{v}_L &= \dot{r}\Big(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\Big) + r\dot{\theta}\Big(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\Big) \\ &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\,\hat{\imath} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\,\hat{\jmath}\,; \end{aligned} \tag{3.6}$$

$$\begin{aligned} \vec{a}_L &= (\ddot{r} - r\dot{\theta}^2)\Big(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\Big) + (2\dot{r}\dot{\theta} + r\ddot{\theta})\Big(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\Big) \\ &= \Big((\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta\Big)\,\hat{\imath} + \Big((\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta\Big)\,\hat{\jmath}\,. \end{aligned} \tag{3.7}$$

Since the light spot is constrained to move along the cloud's underside, which is parallel to the $\hat{\imath}$ direction, it follows that $\vec{v}_L \bullet \hat{\jmath} = 0\,\mathrm{m/s}$ and $\vec{a}_L \bullet \hat{\jmath} = 0\,\mathrm{m/s}^2$. Then we have

$$\begin{aligned} \vec{v}_L \bullet \hat{\jmath} = 0\,\mathrm{m/s} &\implies (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\,\hat{\imath}\bullet\hat{\jmath} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\,\hat{\jmath}\bullet\hat{\jmath} = 0\,\mathrm{m/s} \\ &\implies \dot{r}\sin\theta + r\dot{\theta}\cos\theta = 0\,\mathrm{m/s} \\ &\implies \dot{r} = -\frac{r\dot{\theta}\cos\theta}{\sin\theta}\,; \end{aligned} \tag{3.8}$$

$$\begin{aligned} \vec{a}_L \bullet \hat{\jmath} = 0\,\mathrm{m/s}^2 &\implies \Big((\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta\Big)\,\hat{\imath}\bullet\hat{\jmath} \\ &\qquad + \Big((\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta\Big)\,\hat{\jmath}\bullet\hat{\jmath} = 0\,\mathrm{m/s}^2 \\ &\implies (\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta = 0\,\mathrm{m/s}^2 \\ &\implies \ddot{r} = -\frac{(2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta}{\sin\theta} + r\dot{\theta}^2\,. \end{aligned} \tag{3.9}$$

Since $\dot{\theta} \equiv -20^\circ/\mathrm{s}$, we have $\ddot{\theta} = 0^\circ/\mathrm{s}^2$. By this fact and (3.8), (3.9) becomes

$$\begin{aligned} \ddot{r} &= -\frac{2\dot{r}\dot{\theta}\cos\theta}{\sin\theta} + r\dot{\theta}^2 \\ &= -\frac{2\left(-\frac{r\dot{\theta}\cos\theta}{\sin\theta}\right)\dot{\theta}\cos\theta}{\sin\theta} + r\dot{\theta}^2 \\ &= \frac{2r\dot{\theta}^2\cos^2\theta}{\sin^2\theta} + r\dot{\theta}^2\,. \end{aligned} \tag{3.10}$$

The substitution of (3.8) into (3.6) yields

$$\begin{aligned} \vec{v}_L &= \left[\left(-\frac{r\dot{\theta}\cos\theta}{\sin\theta}\right)\cos\theta - r\dot{\theta}\sin\theta\right]\hat{\imath} + 0\,\mathrm{m/s}\,\hat{\jmath} \\ &= \left[-\frac{r\dot{\theta}\cos^2\theta}{\sin\theta} - r\dot{\theta}\sin\theta\right]\hat{\imath} \\ &= -\frac{r\dot{\theta}}{\sin\theta}\,\hat{\imath} \\ &= -\frac{(r\sin\theta)\,\dot{\theta}}{\sin^2\theta}\,\hat{\imath} \\ &= -\frac{h\dot{\theta}}{\sin^2\theta}\,\hat{\imath}\,, \end{aligned} \tag{3.11}$$

where the $\hat{\jmath}$ component is automatically zero since that is how we have derived (3.8). Similarly, the substitution of (3.8) and (3.10) into (3.7) yields (note that $\ddot{\theta} = 0°/\text{s}^2$)

$$\begin{aligned}
\vec{a}_L &= \left\{ \left[\left(\frac{2r\dot{\theta}^2 \cos^2\theta}{\sin^2\theta} + r\dot{\theta}^2 \right) - r\dot{\theta}^2 \right] \cos\theta - 2\left(-\frac{r\dot{\theta}\cos\theta}{\sin\theta} \right) \dot{\theta}\sin\theta \right\} \hat{\imath} + 0\,\text{m/s}^2\,\hat{\jmath} \\
&= \left\{ \frac{2r\dot{\theta}^2\cos^2\theta}{\sin^2\theta}\cos\theta + 2r\dot{\theta}^2\cos\theta \right\} \hat{\imath} \\
&= \frac{2r\dot{\theta}^2\cos\theta}{\sin^2\theta}\,\hat{\imath} \\
&= \frac{2(r\sin\theta)\dot{\theta}^2\cos\theta}{\sin^3\theta}\,\hat{\imath} \\
&= \frac{2h\dot{\theta}^2\cos\theta}{\sin^3\theta}\,\hat{\imath}\,,
\end{aligned} \tag{3.12}$$

where the $\hat{\jmath}$ component is again automatically zero since that is how we have derived (3.10).

Finally, evaluating (3.11) and (3.12) at $\theta = 20°$, $\dot{\theta} = -20°/\text{s} = -\frac{\pi}{9}\,\text{rad/s}$, and $h = 150\,\text{ft}$ gives

$$\begin{aligned}
\vec{v}_L &= -\frac{h\dot{\theta}}{\sin^2\theta}\,\hat{\imath} \\
&= -\frac{150\,\text{ft}\cdot(-\frac{\pi}{9}\,\text{rad/s})}{\sin^2 20°}\,\hat{\imath} \\
&\approx \underline{447.6\,\text{ft/s}\,\hat{\imath}\,;}
\end{aligned}$$

$$\begin{aligned}
\vec{a}_L &= \frac{2h\dot{\theta}^2\cos\theta}{\sin^3\theta}\,\hat{\imath} \\
&= \frac{2\cdot 150\,\text{ft}\cdot(-\frac{\pi}{9}\,\text{rad/s})^2\cos 20°}{\sin^3 20°}\,\hat{\imath} \\
&\approx \underline{858.6\,\text{ft/s}^2\,\hat{\imath}\,.}
\end{aligned}$$

Evaluating (3.11) and (3.12) at $\theta = 5°$, $\dot{\theta} = -20°/\text{s} = -\frac{\pi}{9}\,\text{rad/s}$, and $h = 150\,\text{ft}$ gives

$$\begin{aligned}
\vec{v}_L &= -\frac{h\dot{\theta}}{\sin^2\theta}\,\hat{\imath} \\
&= -\frac{150\,\text{ft}\cdot(-\frac{\pi}{9}\,\text{rad/s})}{\sin^2 5°}\,\hat{\imath} \\
&\approx \underline{6893\,\text{ft/s}\,\hat{\imath}\,;}
\end{aligned}$$

$$\begin{aligned}
\vec{a}_L &= \frac{2h\dot{\theta}^2\cos\theta}{\sin^3\theta}\,\hat{\imath} \\
&= \frac{2\cdot 150\,\text{ft}\cdot(-\frac{\pi}{9}\,\text{rad/s})^2\cos 5°}{\sin^3 5°}\,\hat{\imath} \\
&\approx \underline{5.500\times 10^4\,\text{ft/s}^2\,\hat{\imath}\,.}
\end{aligned}$$

Alternative Solution. Using unit vectors $\hat{\imath}$ and $\hat{\jmath}$, we have

$$\vec{r}_{L/O} = \frac{h}{\tan\theta}\,\hat{\imath} + h\,\hat{\jmath}\,.$$

Next, $\dot{h} = 0$ since h is constant. θ is a function of t with $\dot{\theta} = -20°/\mathrm{s} = -\frac{\pi}{9}\,\mathrm{rad/s}$ and $\ddot{\theta} = 0$. Then it follows that

$$\begin{aligned}
\vec{v}_L &= \frac{d}{dt}\vec{r}_{L/O} \\
&= \frac{d}{dt}\Big(\frac{h}{\tan\theta}\Big)\,\hat{\imath} + \frac{d}{dt}h\,\hat{\jmath} \\
&= -\frac{h\dot{\theta}}{\sin^2\theta}\,\hat{\imath}\,,
\end{aligned}$$

which is exactly the same as (3.11). In addition,

$$\begin{aligned}
\vec{a}_L &= \frac{d}{dt}\vec{v}_L \\
&= \frac{d}{dt}\Big(-\frac{h\dot{\theta}}{\sin^2\theta}\Big)\,\hat{\imath} \\
&= \frac{2h\dot{\theta}^2\cos\theta}{\sin^3\theta}\,\hat{\imath}\,,
\end{aligned}$$

which is exactly the same as (3.12). □