ENGRD/TAM 203 Spring 2006
HW7 (Assigned on Feb 14, due on Feb 21)

Solution by Dennis Yang

3.1.37 A test sled has a mass of 400 kg. A constant force of 200,000 N acts on it, causing it to accelerate. A viscous drag with magnitude $1600\ N\cdot s/m\,|\dot{x}|$ acts to retard the motion.

a) If the sled starts from rest, how long must the force act before it travels 2000 m?

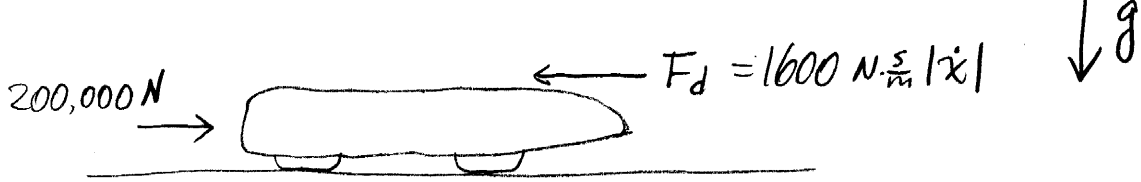
b) How fast is it when $x = 2000$ m?

c) What is the terminal velocity?

d) How long does it take to travel to $x = 2000$ m if we assume it is massless?

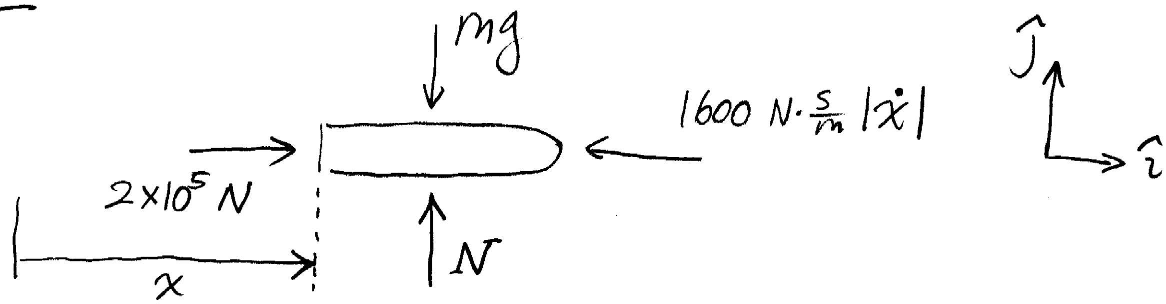
e) Did the time computed under the massless assumption come close to the answer of (a)

f) Is there a distance over which neglecting the sled's mass loads to a poor approximation of the elapsed time?



Solution

FBD



$$\sum_i \vec{F}_i = m\vec{a} \Rightarrow 2\times10^5 N\,\hat{\imath} + 1600\,N\cdot\tfrac{s}{m}|\dot{x}|(-\hat{\imath}) + mg(-\hat{\jmath}) + N\hat{\jmath}$$
$$= m\ddot{x}\,\hat{\imath} \quad (*)$$

$$(*)\cdot\hat{\imath} \Rightarrow 2\times10^5 N\,\hat{\imath}\cdot\hat{\imath} + 1600\,N\cdot\tfrac{s}{m}|\dot{x}|(-\hat{\imath})\cdot\hat{\imath} + mg(-\hat{\jmath})\cdot\hat{\imath} + N\hat{\jmath}\cdot\hat{\imath} = m\ddot{x}\,\hat{\imath}\cdot\hat{\imath}$$

$$\Rightarrow 2\times10^5 N - 1600\,N\cdot\tfrac{s}{m}|\dot{x}| = m\ddot{x} \quad (**)$$

Since the sled travels to the right, $\dot{x} = |\dot{x}| > 0$.
Thus $(**)$ becomes

$$2\times10^5 N - 1600\,N\cdot\tfrac{s}{m}\,\dot{x} = m\ddot{x}$$

$$\Rightarrow m\ddot{x} = -1600\,N\cdot\tfrac{s}{m}\,\dot{x} + 2\times10^5 N$$

$$\Rightarrow m\ddot{x} = -1600\,N\cdot\tfrac{s}{m}\left(\dot{x} + \frac{2\times10^5 N}{-1600\,N\cdot\frac{s}{m}}\right)$$

$$\Rightarrow m\ddot{x} = -1600\,N\cdot\tfrac{s}{m}\left(\dot{x} - 125\ m/s\right)$$

$$\boxed{N\cdot\tfrac{s}{m} = \left(kg\cdot\tfrac{m}{s^2}\right)\cdot\tfrac{s}{m} = kg/s}$$

$$\Rightarrow \ddot{x} = -\frac{1600\ kg/s}{m}\left(\dot{x} - 125\ m/s\right) \quad (***)$$

Let $\beta = \dot{x} - 125\ m/s$, then $\dot{\beta} = \ddot{x}$. This makes
$(***)$ become
$$\dot{\beta} = -\frac{1600\ kg/s}{m}\,\beta$$

$$\Rightarrow \beta(t) = C e^{-\left(\frac{1600\ kg/s}{m}\right)t},$$ with "C" being a constant of integration.

Therefore, $\dot{x}(t) = \beta(t) + 125 \text{ m/s}$

$$= Ce^{-\left(\frac{1600 \text{ kg/s}}{m}\right)t} + 125 \text{ m/s}$$

The sled started at rest $\Longrightarrow$ $\dot{x}(0_s) = 0 \text{ m/s}$

$$\Longrightarrow \quad C + 125 \text{ m/s} = 0 \text{ m/s}$$

$$\Longrightarrow \quad C = -125 \text{ m/s}$$

$$\Longrightarrow \quad \dot{x}(t) = -125 \text{ m/s}\, e^{-\left(\frac{1600 \text{ kg/s}}{m}\right)t} + 125 \text{ m/s}$$

$$\Longrightarrow \quad \boxed{\dot{x}(t) = 125 \text{ m/s}\left(1 - e^{-\left(\frac{1600 \text{ kg/s}}{m}\right)t}\right)}$$

Note: we are NOT in a rush to substitute $m = 400$ kg into the formular since we want to know how the mass affects the solution

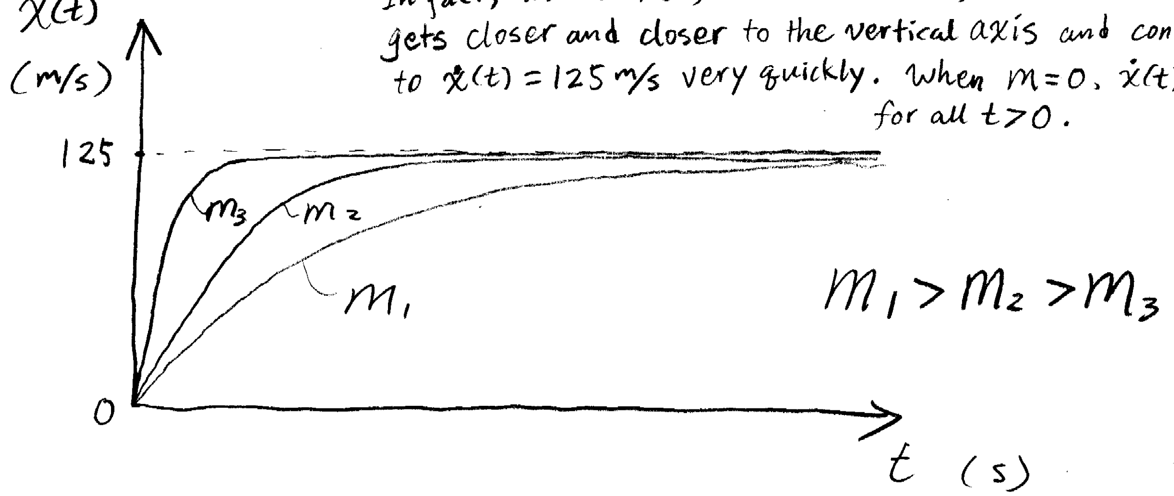
Regardless the value of m (we know $m > 0$ though), we have

$$\lim_{t\to\infty} \dot{x}(t)\hat{\imath} = \lim_{t\to\infty} 125 \text{ m/s}\left(1 - e^{-\left(\frac{1600 \text{ kg/s}}{m}\right)t}\right)\hat{\imath}$$

$$= \underline{125 \text{ m/s}\ \hat{\imath}}$$ ◀ <u>this is the terminal velocity (part C)</u>

Now, we want to see how mass affects the solution.

In fact, as $m \to 0^+$, the curve $\dot{x}(t)$ gets closer and closer to the vertical axis and converges to $\dot{x}(t) = 125$ m/s very quickly. When $m=0$, $\dot{x}(t) = 125$ m/s for all $t > 0$.



Again, we look at (***), and evaluate it at $t = 0_s$

$$\ddot{x}(0_s) = -\frac{1600 \text{ kg/s}}{m}\left(\underbrace{\dot{x}(0_s)}_{=0\,\text{m/s}} - 125 \text{ m/s}\right)$$

$$\Rightarrow \ddot{x}(0_s) = \frac{2 \times 10^5 \text{ kg} \cdot \text{m/s}^2}{m}$$

Thus, the smaller m is, the larger the slope of $\dot{x}(t)$ at $t = 0_s$ will be. Consequently, $\dot{x}(t)$ will converge faster to its terminal value (see the plot)

Now, take $m = 400$ kg, we have

$$\dot{x}(t) = 125 \text{ m/s}\left(1 - e^{-\frac{1600 \text{ kg/s} \cdot t}{400 \text{ kg}}}\right)$$

$$\Rightarrow \boxed{\dot{x}(t) = 125 \text{ m/s} \cdot \left(1 - e^{-\frac{t}{0.25\,\text{s}}}\right)}$$

Substitute it to $x(t) = \underbrace{x(0_s)}_{=0\,\text{m}} + \int_{0_s}^{t} \dot{x}(t)\, dt$, we obtain

$$x(t) = 0\,m + \int_{0s}^{t} \dot{x}(\tau)\,d\tau$$
$$= \int_{0s}^{t} 125\,m/s \cdot \left(1 - e^{-\frac{\tau}{0.25s}}\right) d\tau$$
$$= 125\,m/s \cdot \tau \Big|_{0s}^{t} + 125\,m/s \cdot 0.25s\, e^{-\frac{\tau}{0.25s}} \Big|_{0s}^{t}$$
$$\Rightarrow \boxed{x(t) = 125\,m/s \cdot t + \frac{125}{4}\,m \cdot \left(e^{-\frac{t}{0.25s}} - 1\right)} \quad (1)$$

"how long does it take to get to $x = 2000\,m$, starting at rest, with $m = 400\,kg$"

Part a) $x(t^*) = 2000\,m = 125\,m/s \cdot t^* + \frac{125}{4}\,m \cdot \left(e^{-\frac{t^*}{0.25s}} - 1\right)$

$$\Rightarrow \quad t^* = 16.25\,s$$

solved by Maple® :

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[> solve(2000=125*t+125/4*(exp(-t/0.25)-1),t);
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"How fast is it when $x = 2000\,m$"

Part b) $\dot{x}(t^*) = \dot{x}(16.25s) = 125\,m/s \left(1 - e^{-\frac{16.25s}{0.25s}}\right)$

$$\simeq 125.0\,m/s$$

Note : $e^{-\frac{16.25s}{0.25s}} \simeq 0.59 \times 10^{-28}$!

"terminal velocity"

Part c) As previously solved, the terminal velocity is $125\,m/s\,\hat{\imath}$.

"how long does it take to get to $x = 2000\,m$, assuming zore mass"

Part d) consider (**) for $m = 0\,kg$, i.e.,

$$2\times10^5\,N - 1600\,N\cdot s/m \cdot \dot{x} = \cancel{m}^{\,0\,kg}\ddot{x} = 0$$
$$\Rightarrow \quad \dot{x} \equiv \frac{2\times10^5\,N}{1600\,N\cdot s/m} = 125\,m/s \quad (\text{for all } t > 0)$$

$$\Longrightarrow x(t) = \cancelto{0\,m}{x(0_s)} + \int_{0s}^{t} \dot{x}(\tau)\, d\tau$$

$$= \int_{0s}^{t} 125\ m/s\ d\tau$$

$$\Longrightarrow \boxed{x(t) = 125\ m/s \cdot t} \quad (2)$$

By (2), $x(t^*) = 2000\,m \Longrightarrow t^* = \frac{2000m}{125\ m/s} = \underline{16\,s}$

Part e) "comparison with the answer of a)"

~~compare~~ with the result from part a),

we have the relative error

$$\frac{16.25\,s - 16\,s}{16.25\,s} \simeq 1.5\% \text{ (a pretty good approximation)}$$

Part f) "when can we use the zero mass assumption"

for "t" sufficiently large,

(1) becomes $x(t) \approx 125\,m/s \cdot t - \frac{125}{4}m \Longrightarrow \boxed{t = \frac{x(t) + \frac{125}{4}m}{125\ m/s}}$ (3)

while (2) is always $x(t) = 125\ m/s \cdot t \Longrightarrow \boxed{t = \frac{x(t)}{125\,m/s}}$ (4)

In order to have (4) as a good approximation of (3)

we need that $\underline{x(t) \gg \frac{125}{4}\,m}$.

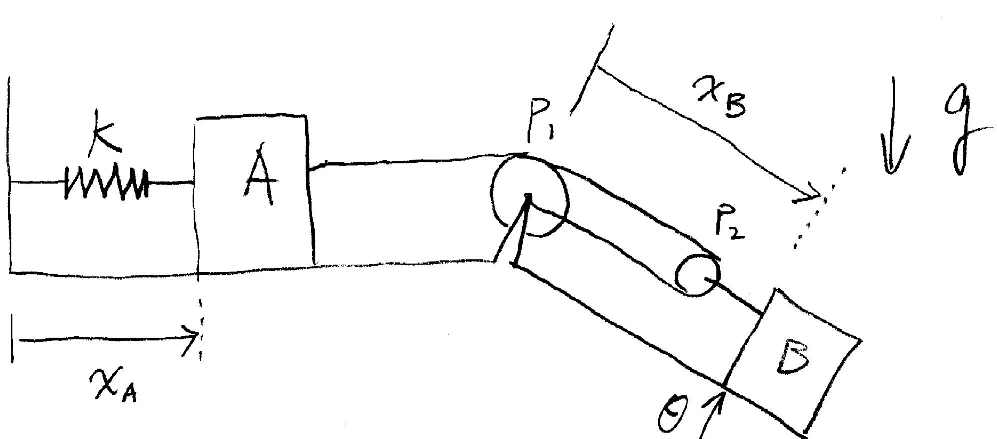
$\square$

3.1.40 Two masses are coupled by a pulley and restrained by a string. The string has an unstreched length of L and is stretched by an amount δ and then attached to block A, as shown. The pulleys and ropes are massless. In addition, the pulleys are perfectly round and frictionless.

a) Determine the minimum mass m_B^* needed for block A to move to the right and for block B to move down the inclined surface.

b) What is the maximum coefficient of static friction that allow there exists a m_B^* for part a).

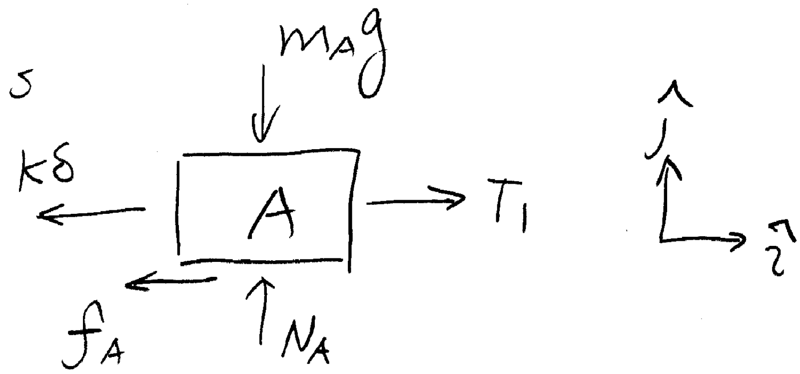
c) When such a m_B^* does exist, letting $m_B > m_B^*$, find expressions for the accelerations of blocks A and B.

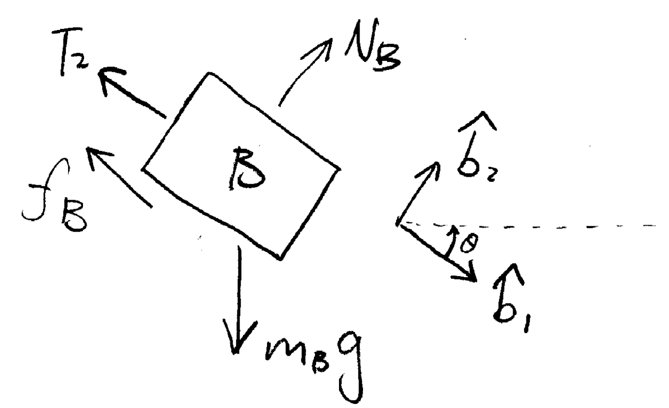


$\mu_d < \mu_s$

Solution

Part a) FBDs



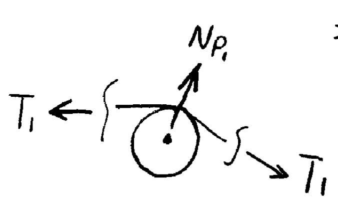


Notes: ① at the critical equilibrium state, A tends to move to the right while B tends to move down along the inclined surface although both A & B are still at rest. At this state, $f_A = N_A \mu_s$ and $f_B = N_B \mu_s$. Their directions are opposite to the directions A and B tend to move.

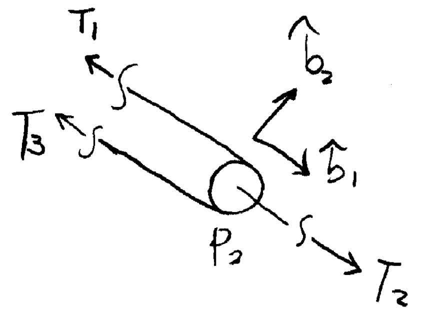
② the pulleys are round, massless, and frictionless

$\Longrightarrow$ the tensions in the rope on both sides of Pulley P_1 are the same, i.e., both are T_1

(this is also true for part c).)



(3)



Pulley P_2 is massless, round, and frictionless

$\Longrightarrow$ the tensions in the rope on both sides of the pulley are the same, i.e., $T_1 = T_3$

In addition, $m_{P_2} = 0 \Longrightarrow \sum_i \vec{F}_i = \cancel{m_{P_2}} \vec{a}_{P_2} = \vec{0}$, regardless if $\vec{a}_{P_2}$ is zero or not.

Thus, $T_1(-\hat{b}_1) + T_3(-\hat{b}_1) + T_2(\hat{b}_1) = \vec{0}$ (*)

$(*) \cdot \hat{b}_1 \Longrightarrow -T_1 - T_3 + T_2 = 0$

$$\xRightarrow[T_1 = T_3]{} \boxed{T_2 = 2T_1}$$

This is also true for <u>part c)</u>.

Now, for A, $\sum_i \vec{F}_i = \vec{0}$

$\Longrightarrow k\delta(-\hat{\imath}) + m_A g(-\hat{\jmath}) + f_A(-\hat{\imath}) + N_A\hat{\jmath} + T_1\hat{\imath} = \vec{0}$ (a1)

$(a1)\cdot\hat{\imath} \Longrightarrow k\delta(-\hat{\imath})\cdot\hat{\imath} + m_A g(-\hat{\jmath})\cdot\hat{\imath} + f_A(-\hat{\imath})\cdot\hat{\imath} + N_A\hat{\jmath}\cdot\hat{\imath} + T_1\hat{\imath}\cdot\hat{\imath} = \vec{0}\cdot\hat{\imath}$

$\Longrightarrow -k\delta - f_A + T_1 = 0$

$\Longrightarrow T_1 = k\delta + f_A$ (a2)

$(a1)\cdot\hat{\jmath} \Longrightarrow k\delta(-\hat{\imath})\cdot\hat{\jmath} + m_A g(-\hat{\jmath})\cdot\hat{\jmath} + f_A(-\hat{\imath})\cdot\hat{\jmath} + N_A\hat{\jmath}\cdot\hat{\jmath} + T_1\hat{\imath}\cdot\hat{\jmath} = \vec{0}\cdot\hat{\jmath}$

$\Longrightarrow -m_A g + N_A = 0$

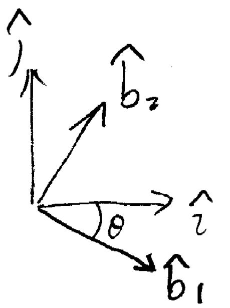
$\Longrightarrow N_A = m_A g$ (a3)

(a2), (a3), and $f_A = N_A \mu_s \Longrightarrow \boxed{T_1 = k\delta + m_A g \mu_s}$ (a4) for B, $\sum_i \vec{F}_i = \vec{0}$

$$\Longrightarrow T_2(-\hat{b}_1) + f_B(-\hat{b}_1) + N_B\hat{b}_2 + m_B^* g(-\hat{j}) = \vec{0} \quad (a5)$$

$$(a5)\cdot\hat{b}_1 \Longrightarrow T_2(-\hat{b}_1)\cdot\hat{b}_1 + f_B(-\hat{b}_1)\cdot\hat{b}_1 + N_B\hat{b}_2\cdot\hat{b}_1 + m_B^* g(-\hat{j})\cdot\hat{b}_1 = \vec{0}\cdot\hat{b}_1$$

$$\Longrightarrow -T_2 - f_B + m_B^* g\sin\theta = 0 \quad (a6)$$



$$\hat{b}_1\cdot\hat{j} = \cos(\theta+90^\circ) = -\sin\theta$$

$$\hat{b}_2\cdot\hat{j} = \cos\theta$$

$$(a5)\cdot\hat{b}_2 \Longrightarrow T_2(-\hat{b}_1)\cdot\hat{b}_2 + f_B(-\hat{b}_1)\cdot\hat{b}_2 + N_B\hat{b}_2\cdot\hat{b}_2 + m_B^* g(-\hat{j})\cdot\hat{b}_2 = \vec{0}\cdot\vec{b}_2$$

$$\Longrightarrow N_B - m_B^* g\cos\theta = 0$$

$$\Longrightarrow N_B = m_B^* g\cos\theta$$

$$\Longrightarrow f_B = N_B\mu_s = m_B^* g\cos\theta\,\mu_s$$

substitute f_B to (a6),

$$\Longrightarrow -T_2 - m_B^* g\cos\theta\,\mu_s + m_B^* g\sin\theta = 0$$

$$\Longrightarrow \boxed{T_2 = m_B^* g(\sin\theta - \cos\theta\,\mu_s)} \quad (a7)$$

(a6), (a7) and $T_2 = 2T_1$

$$\Longrightarrow m_B^* g(\sin\theta - \cos\theta\,\mu_s) = 2(k\delta + m_A g\mu_s)$$

$$\Longrightarrow \boxed{m_B^* = \frac{2(k\delta + m_A g\mu_s)}{(\sin\theta - \cos\theta\,\mu_s)g}}$$

Part b)

by part a), if exists, m_B^* is given by

$$m_B^* = \frac{2(k\delta + m_A g \mu_s)}{(\sin\theta - \cos\theta\,\mu_s)g}$$

In order to have $m_B^* > 0$, we need (and only need) that $\sin\theta - \cos\theta\,\mu_s > 0$. (b1)

since $0° < \theta < 90°$, we have $\sin\theta > 0$, $\cos\theta > 0$.

thus (b1) $\Rightarrow$ $\boxed{\mu_s < \tan\theta}$ (b2)

In fact, $\lim\limits_{\mu_s \to \tan\theta^-} \dfrac{2(k\delta + m_A g \mu_s)}{(\sin\theta - \cos\theta\,\mu_s)g} \to +\infty$

Therefore, to have a positive and finite m_B^*, we need that $\mu_s < \tan\theta$

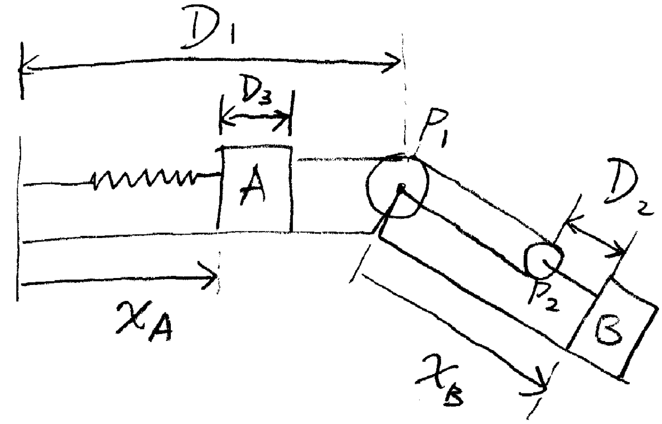
Part c)

Notes: ① Notes ② & ③ from part a) still apply.

② $f_A = N_A \mu_d$, $f_B = N_B \mu_d$, their directions are the same as those in part a).

③ FBDs are the same as those in part a).

④



Let l be the total length of the rope that connects A, P_1, and P_2

Then $l = (D_1 - x_A - D_3) + 2(x_B - D_2)$

$\Longrightarrow \cancel{\ddot{l}}^{\,0} = (\cancel{\ddot{D}_1}^{\,0} - \ddot{x}_A - \cancel{\ddot{D}_3}^{\,0}) + 2(\ddot{x}_B - \cancel{\ddot{D}_2}^{\,0})$

since l, D_1, D_2, D_3 are all constant.

$\Longrightarrow \quad 0 = -\ddot{x}_A + 2\ddot{x}_B$

$\Longrightarrow \quad \boxed{\ddot{x}_A = 2\ddot{x}_B}$

For A, $\sum_i \vec{F}_i = m_A \vec{a}_A$

$$\Longrightarrow \quad k\delta(-\hat{\imath}) + f_A(-\hat{\imath}) + N_A\hat{\jmath} + m_A g(-\hat{\jmath}) + T_1\hat{\imath} = m_A \ddot{x}_A \hat{\imath} \qquad (C1)$$

$$(C1)\cdot\hat{\imath} \Longrightarrow k\delta(-\hat{\imath})\cdot\hat{\imath} + f_A(-\hat{\imath})\cdot\hat{\imath} + N_A\hat{\jmath}\cdot\hat{\imath} + m_A g(-\hat{\jmath})\cdot\hat{\imath} + T_1\hat{\imath}\cdot\hat{\imath} = m_A\ddot{x}_A\hat{\imath}\cdot\hat{\imath}$$

$$\Longrightarrow -k\delta - f_A + T_1 = m_A\ddot{x}_A \qquad (C2)$$

$$(C1)\cdot\hat{\jmath} \Longrightarrow k\delta(-\hat{\imath})\cdot\hat{\jmath} + f(-\hat{\imath})\cdot\hat{\jmath} + N_A\hat{\jmath}\cdot\hat{\jmath} + m_A g(-\hat{\jmath})\cdot\hat{\jmath} + T_1\hat{\imath}\cdot\hat{\jmath} = m_A\ddot{x}_A\hat{\imath}\cdot\hat{\jmath}$$

$$\Longrightarrow \quad N_A - m_A g = 0$$

$$\Longrightarrow \quad N_A = m_A g \qquad (C3)$$

(C2), (C3), and $f_A = N_A \mu_d$

$$\Longrightarrow \boxed{-K\delta - m_A g \mu_d + T_1 = m_A \ddot{x}_A} \quad (C4)$$

For B, $\sum_i \vec{F}_i = m_B \vec{a}_B$

$$\Longrightarrow T_2(-\hat{b}_1) + f_B(-\hat{b}_1) + N_B \hat{b}_2 + m_B g(-\hat{j}) = m_B \ddot{x}_B \hat{b}_1 \quad (C5)$$

$$(C5)\cdot\hat{b}_1 \Longrightarrow T_2(-\hat{b}_1)\cdot\hat{b}_1 + f_B(-\hat{b}_1)\cdot\hat{b}_1 + N_B \hat{b}_2\cdot\hat{b}_1 + m_B g(-\hat{j})\cdot\hat{b}_1 = m_B \ddot{x}_B \hat{b}_1\cdot\hat{b}_1$$

$$\Longrightarrow -T_2 - f_B + m_B g \sin\theta = m_B \ddot{x}_B \quad (C6)$$

$$(C5)\cdot\hat{b}_2 \Longrightarrow T_2(-\hat{b}_1)\cdot\hat{b}_2 + f_B(-\hat{b}_1)\cdot\hat{b}_2 + N_B \hat{b}_2\cdot\hat{b}_2 + m_B g(-\hat{j})\cdot\hat{b}_2 = m_B \ddot{x}_B \hat{b}_1\cdot\hat{b}_2$$

$$\Longrightarrow N_B - m_B g \cos\theta = 0$$

$$\Longrightarrow N_B = m_B g \cos\theta \quad (C7)$$

(C6), (C7), and $f_B = N_B \mu_d$

$$\Longrightarrow -T_2 - m_B g \cos\theta \mu_d + m_B g \sin\theta = m_B \ddot{x}_B$$

$$\xrightarrow[\ddot{x}_A = 2\ddot{x}_B,\ T_2 = 2T_1]{} -2T_1 + m_B g(\sin\theta - \cos\theta \mu_d) = m_B \frac{\ddot{x}_A}{2}$$

$$\Longrightarrow \boxed{T_1 = \frac{1}{2}\left[m_B g(\sin\theta - \cos\theta \mu_d) - m_B \frac{\ddot{x}_A}{2}\right]} \quad (C8)$$

Substitute (C8) to (C4)

$$\Longrightarrow -K\delta - m_A g \mu_d + \frac{1}{2}\left[m_B g(\sin\theta - \cos\theta \mu_d) - m_B \frac{\ddot{x}_A}{2}\right] = m_A \ddot{x}_A$$

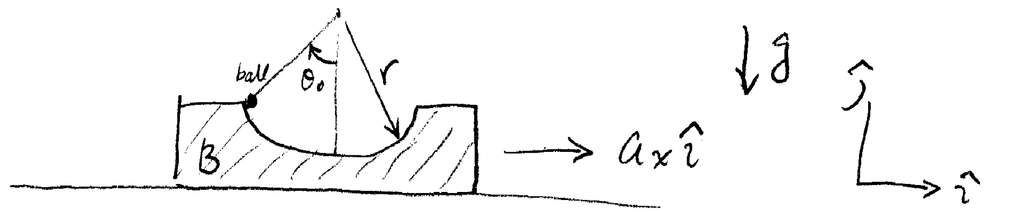
$$\Rightarrow \boxed{\ddot{x}_A = \frac{\frac{1}{2} m_B g (\sin\theta - \cos\theta \mu_d) - (k\delta + m_A g \mu_d)}{m_A + \frac{1}{4} m_B}}$$

Then $\vec{a}_A = \ddot{x}_A \hat{i}$

$$\vec{a}_B = \ddot{x}_B \hat{b}_1 = \frac{1}{2} \ddot{x}_A \hat{b}_1$$

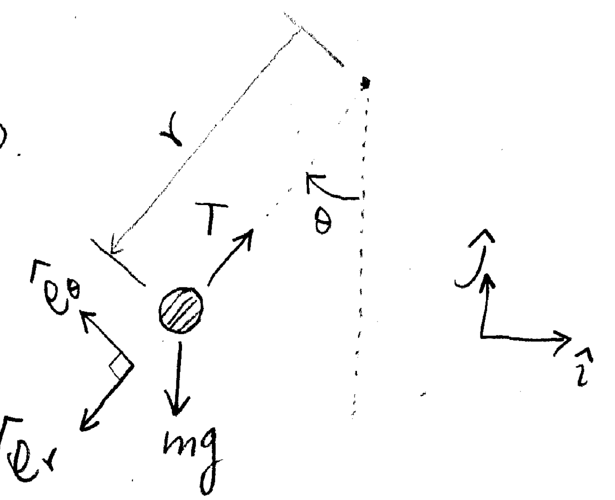
$\square$

3.2.2 A ball of mass m is placed in a bowl (as shown). The ball can slide in the bowl with no friction. The bowl accelerates to the right with a constant acceleration $a_x \hat{\imath}$ while the ball just reaches the edge of the bowl. Express a_x in terms of θ_0



Solution

FBD.



The acceleration of the ball relative to the bowl

$$\vec{a}_{ball/B} = (\cancel{\ddot{r}}^{\,0} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\cancel{\dot{r}}^{\,0}\dot{\theta})\hat{e}_\theta \quad (r \equiv \text{constant})$$

$$= -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$

The acceleration of the bowl is

$$\vec{a}_B = a_x \hat{\imath}.$$

The acceleration of the ball: $\vec{a}_{ball} = \vec{a}_B + \vec{a}_{ball/B}$

Thus, $\vec{a}_{ball} = a_x \hat{\imath} + (-r\dot{\theta}^2)\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$

To the ball, $\sum_i \vec{F}_i = m\vec{a}_{ball}$

$$\Longrightarrow mg(-\hat{\jmath}) + T(-\hat{e}_r) = m(a_x\hat{\imath} + (-r\dot{\theta}^2)\hat{e}_r + r\ddot{\theta}\hat{e}_\theta)$$

$$\Longrightarrow g(-\hat{\jmath}) + \frac{T}{m}(-\hat{e}_r) = a_x\hat{\imath} + (-r\dot{\theta}^2)\hat{e}_r + r\ddot{\theta}\hat{e}_\theta \quad (*)$$

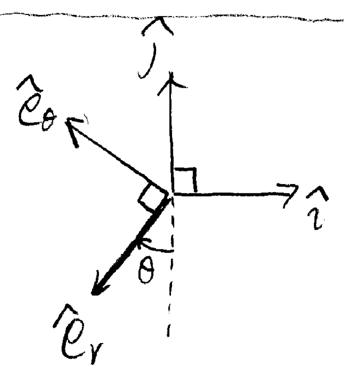
Since we only care about the motion along $\hat{e}_\theta$ direction (i.e., how will θ change with respect to time), we take dot product of $(*)$ and $\hat{e}_\theta$.

$$(*)\cdot\hat{e}_\theta \Longrightarrow g(-\hat{\jmath})\cdot\hat{e}_\theta + \frac{T}{m}(-\hat{e}_r)\cdot\hat{e}_\theta = a_x\hat{\imath}\cdot\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\cdot\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta\cdot\hat{e}_\theta$$

$$\Longrightarrow -g\sin\theta = a_x(-\cos\theta) + r\ddot{\theta}$$

$$\Longrightarrow \boxed{\ddot{\theta} = -\frac{g}{r}\sin\theta + \frac{a_x}{r}\cos\theta}$$

$(**)$



$\hat{\jmath}\cdot\hat{e}_\theta = \cos(180^\circ - 90^\circ - \theta)$, $\hat{\imath}\cdot\hat{e}_\theta = \cos[90^\circ + (180^\circ - 90^\circ - \theta)]$

$= \sin\theta \qquad\qquad = -\cos\theta$

$$(**) \Longrightarrow \frac{d\dot{\theta}}{dt} = -\frac{1}{r}(g\sin\theta - a_x\cos\theta)$$

$$\Longrightarrow \left(\frac{d\dot{\theta}}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = -\frac{1}{r}(g\sin\theta - a_x\cos\theta)$$

$$\Longrightarrow \left(\frac{d\dot{\theta}}{d\theta}\right)\dot{\theta} = -\frac{1}{r}(g\sin\theta - a_x\cos\theta)$$

$$\Longrightarrow \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = -\frac{1}{r}(g\sin\theta - a_x\cos\theta)$$

$$\Longrightarrow \frac{1}{2}\dot{\theta}^2 = \int\left[-\frac{1}{r}(g\sin\theta - a_x\cos\theta)\right]d\theta + \underbrace{C}_{\text{a constant of integration}}$$

$$\Longrightarrow \boxed{\frac{1}{2}\dot{\theta}^2 = \frac{g}{r}\cos\theta + \frac{a_x}{r}\sin\theta + C} \quad (***)$$

Initially, $\underline{\dot{\theta} = 0,\ \theta = 0}$

$$(***) \Longrightarrow 0 = \frac{g}{r} + C \Longrightarrow C = -\frac{g}{r}$$

When the ball reaches the edge of the bowl,

$\underline{\dot{\theta} = 0,\ \theta = \theta_0}$

$$(***) \Longrightarrow 0 = \frac{g}{r}\cos\theta_0 + \frac{a_x}{r}\sin\theta_0 - \frac{g}{r}$$

$$\Longrightarrow g\cos\theta_0 + a_x\sin\theta_0 = g$$

$$\Longrightarrow \frac{g}{\sqrt{g^2+a_x^2}}\cos\theta_0 + \frac{a_x}{\sqrt{g^2+a_x^2}}\sin\theta_0 = \frac{g}{\sqrt{g^2+a_x^2}}$$

Let $\cos\alpha = \frac{g}{\sqrt{g^2+a_x^2}}$

$\sin\alpha = \frac{a_x}{\sqrt{g^2+a_x^2}}$

$\tan\alpha = \frac{a_x}{g}$

$$\Longrightarrow \cos\alpha\cdot\cos\theta_0 + \sin\alpha\sin\theta_0 = \cos\alpha$$

$$\Longrightarrow \cos(\theta_0 - \alpha) = \cos\alpha \Longrightarrow \theta_0 - \alpha = \alpha$$

$\Longrightarrow \theta_0 = 2\alpha$, where $\tan\alpha = \dfrac{a_x}{g}$

$$\Longrightarrow \boxed{a_x = g \cdot \tan\left(\frac{\theta_0}{2}\right)}$$

$\square$