A Derivation of the Transisition Rule at Heelstrike which appears in the paper "The Simplest Walking Model: Stability, Complexity, and Scaling" by Garcia et al.

Mario W. Gomes

October 4, 1999

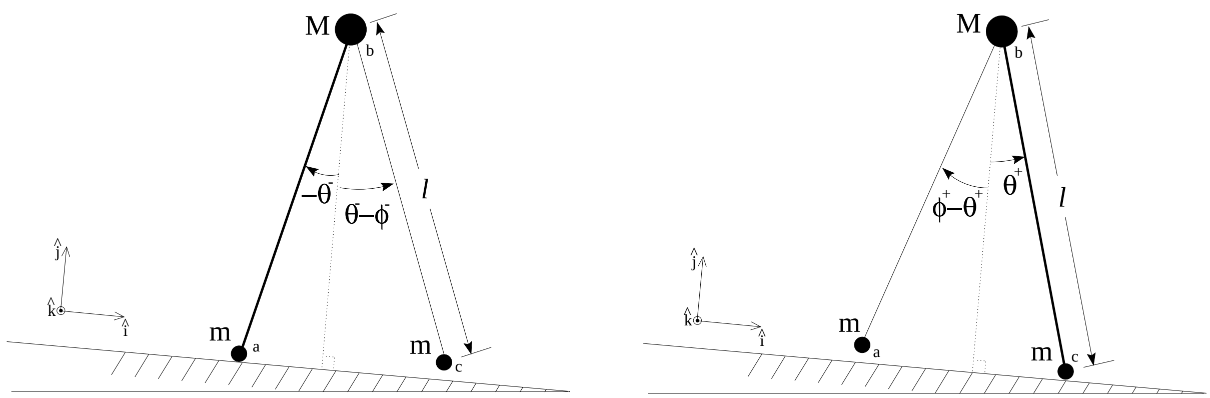

Figure 1: After heelstrike collision (whole walker)

From Geometry

$$\begin{aligned}
\theta^+ &= \theta^- - \phi^- \\
\phi^- &= 2\theta^- \quad \text{(at time of interest)} \\
\theta^+ &= -\theta^- && (1) \\
\phi^+ - \theta^+ &= -\theta^- \\
\phi^+ &= -\theta^- + \theta^+ \\
\phi^+ &= -2\theta^- && (2)
\end{aligned}$$

Trigonometric Identities

$$\begin{aligned}
\cos(A+B) &= \cos A \cos B - \sin A \sin B && (3) \\
1 &= \sin^2 A + \cos^2 A && (4)
\end{aligned}$$

At heelstrike a collision occurs at point "c". We assume (1) that there is only an impact on the walker at "c" and that no impact (impulse) occurs at point "a" and (2) that the impulsive force is much larger than the non-impulsive forces (i.e. gravity) for the duration of the impact. With these two assumptions angular momentum about the impact site (point "c" in Fig. and Fig. 1) is conserved through the heelstrike collision. The impact site for the whole walker is located at the swing foot contact point before collision and at the stance foot contact point after the collision. In reality the impact site remains at the same location, but due to the terminology that we've used in describing stance and swing legs, our name for the impact site has changed (i.e. the leg we call the "swing" leg gets renamed to the "stance" leg after the collision takes place) The diagrams for the situation before and after the collision are shown in Fig. and Fig. 1.

If we use a coordinate system which is tilted with the slope we can write the expressions for angular momentum of the walker both before and after the collision.

Before Collision (Figure)

$$\mathbf{H}^{-}_{/c} = \mathbf{r}^{-}_{b/c} \times M\mathbf{v}^{-}_{b} + \mathbf{r}^{-}_{a/c} \times m\mathbf{v}^{-}_{a} \tag{5}$$

$$\begin{aligned}
\mathbf{r}^{-}_{b/c} &= -l\sin(\theta^{-} - \phi^{-})\hat{i} + l\cos(\theta^{-} - \phi^{-})\hat{j} \\
\mathbf{r}^{-}_{b/c} &= l\sin(\phi^{-} - \theta^{-})\hat{i} + l\cos(\phi^{-} - \theta^{-})\hat{j} \\
\mathbf{r}^{-}_{b/a} &= l\sin(-\theta^{-})\hat{i} + l\cos(-\theta^{-})\hat{j} \\
\mathbf{r}^{-}_{b/a} &= -l\sin(\theta^{-})\hat{i} + l\cos(\theta^{-})\hat{j} && (6) \\
\mathbf{v}^{-}_{b} &= \dot{\mathbf{r}}^{-}_{b/a} \\
&= -l\dot{\theta}^{-}\cos(\theta^{-})\hat{i} - l\dot{\theta}^{-}\sin(\theta^{-})\hat{j} && (7) \\
\mathbf{v}^{-}_{a} &= \mathbf{0} \quad \text{(point "a" is in contact with the ground)} && (8) \\
\mathbf{H}^{-}_{/c} &= -Ml^2\dot{\theta}^{-}\sin(\theta^{-})\sin(\phi^{-} - \theta^{-})\hat{k} + \\
&\quad Ml^2\dot{\theta}^{-}\cos(\theta^{-})\cos(\phi^{-} - \theta^{-})\hat{k} \\
\mathbf{H}^{-}_{/c} &= Ml^2\dot{\theta}^{-}\cos(\phi^{-})\hat{k} && (9)
\end{aligned}$$

After Collision (Figure 1)

$$\mathbf{H}^{+}_{/c} = \mathbf{r}^{+}_{b/c} \times M\mathbf{v}^{+}_{b} + \mathbf{r}^{+}_{a/c} \times \mathbf{v}^{+}_{a} \tag{10}$$

$$\begin{aligned}
\mathbf{r}^{+}_{b/c} &= -l\sin(\theta^{+})\hat{i} + l\cos(\theta^{+})\hat{j} && (11) \\
\mathbf{v}^{+}_{b} &= \dot{\mathbf{r}}^{+}_{b/c} = -l\dot{\theta}^{+}\cos(\theta^{+})\hat{i} - l\dot{\theta}^{+}\sin(\theta^{+})\hat{j} && (12)
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}^{+}_{a/c} &= (-l\sin(\theta^{+}) - l\sin(\phi^{+} - \theta^{+}))\hat{i} + \\
&\quad (l\cos(\theta^{+}) - l\cos(\phi^{+} - \theta^{+}))\hat{j} && (13) \\
\mathbf{v}^{+}_{a} &= \dot{\mathbf{r}}^{+}_{a/c}
\end{aligned}$$

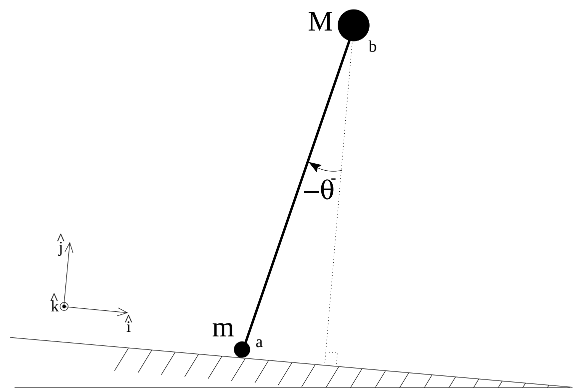

Figure 2: Before heelstrike collision (old stance leg)

$$\begin{aligned}
&= (-l\dot{\theta}^{+}\cos(\theta^{+}) - \\
&\quad l(\dot{\phi}^{+} - \dot{\theta}^{+})\cos(\phi^{+} - \theta^{+}))\hat{i} + \\
&\quad (-l\dot{\theta}^{+}\sin(\theta^{+}) + \\
&\quad l(\dot{\phi}^{+} - \dot{\theta}^{+})\sin(\phi^{+} - \theta^{+}))\hat{j} && (14)
\end{aligned}$$

$$\begin{aligned}
\lim_{\frac{m}{M}\to 0} \mathbf{H}^{+}_{/c} &= \mathbf{r}^{+}_{b/c} \times M\mathbf{v}^{+}_{b} && (15) \\
\mathbf{H}^{+}_{/c} &= Ml^2\dot{\theta}^{+}\hat{k} && (16) \\
\mathbf{H}^{-}_{/c} &= \mathbf{H}^{+}_{/c} && (17)
\end{aligned}$$

Using eqn. (9) and (16)

$$\begin{aligned}
Ml^2\dot{\theta}^{-}\cos(\phi^{-})\hat{k} &= Ml^2\dot{\theta}^{+}\hat{k} && (18) \\
\dot{\theta}^{+} &= \cos(2\theta^{-})\dot{\theta}^{-} && (19)
\end{aligned}$$

If we look at the leg "ab" in isolation from leg "bc" we now **assume** that the impact that leg "bc" feels when heelstrike occurs is felt by leg "ab" through the hip joint at "b". Thus if we isolate leg "ab" we assume that the impact that occurs on leg "ab" due to heelstrike **only** occurs at point "b". **With that assumption**, the angular momentum of leg "ab" about point "b" will be conserved throughout the impact.

Before Collision (Figure 2)

$$\mathbf{H}^{-}_{/b} = \mathbf{r}^{-}_{a/b} \times m\mathbf{v}^{-}_{a} \tag{20}$$

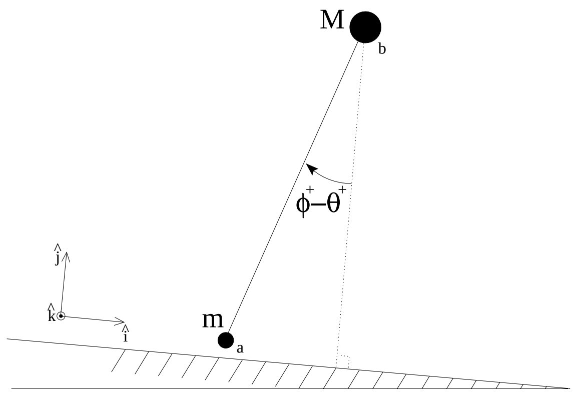

Figure 3: After heelstrike collision (new swing leg)

$$\begin{aligned} \mathbf{v}_a^- &= \mathbf{0} \text{ (point ``a'' is in contact with the ground)} & (21) \\ \mathbf{H}_{/b}^- &= \mathbf{0} & (22) \end{aligned}$$

After Collision (Figure 3)

$$\mathbf{H}_{/b}^+ = \mathbf{r}_{a/b}^+ \times m\mathbf{v}_a^+ \qquad (23)$$

$$\begin{aligned} \mathbf{r}_{a/b}^+ &= -l\sin(\phi^+ - \theta^+))\hat{i} + \\ &\quad -l\cos(\phi^+ - \theta^+))\hat{j} & (24) \end{aligned}$$

from eqn. (14)

$$\begin{aligned} \mathbf{v}_a^+ &= \dot{\mathbf{r}}_{a/c}^+ \\ &= (-l\dot{\theta}^+\cos(\theta^+) - \\ &\quad l(\dot{\phi}^+ - \dot{\theta}^+)\cos(\phi^+ - \theta^+))\hat{i} + \\ &\quad (-l\dot{\theta}^+\sin(\theta^+) + \\ &\quad l(\dot{\phi}^+ - \dot{\theta}^+)\sin(\phi^+ - \theta^+))\hat{j} \end{aligned}$$

using eqn. (4)

$$\begin{aligned} \mathbf{H}_{/b}^+ &= -ml^2\dot{\theta}^+\sin(\phi^+ - \theta^+)\sin(\theta^+) - \\ &\quad ml^2\dot{\theta}^+\cos(\phi^+ - \theta^+)\cos(\theta^+) - \\ &\quad ml^2(\dot{\theta}^+ - \dot{\phi}^+ \end{aligned}$$

using eqn. (3)

$$\begin{aligned} \mathbf{H}_{/b}^+ &= -ml^2(\dot{\theta}^+ - \dot{\phi}^+) - \\ &\quad ml^2\dot{\theta}^+cos(\phi^+) & (25) \\ \mathbf{H}_{/b}^- &= \mathbf{H}_{/b}^+ & (26) \end{aligned}$$

using eqn. (22)

and (25)

$$0 = \dot{\theta}^+ - \dot{\phi}^+ - \dot{\theta}^+cos(\phi^+)$$

using eqn. (2)

and (19)

$$\dot{\phi}^+ = \cos(2\theta^-)(1 - \cos(2\theta^-))\dot{\theta}^- \qquad (27)$$

putting eqns.(1),(2),(19),& (27) into matrix form:

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}^+ = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos(2\theta^-) & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & \cos 2\theta^-(1 - cos2\theta^-) & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}^-$$

Note that according to eqn.(1) we could replace every instance of θ^- with $-\theta^+$ in the above matrix equation. Since $\cos(-\theta) = \cos(\theta)$ the matrix equation would look exactly the same except for the superscripts in the matrix. Thus we define $\theta = \theta^-$.

Thus we have the same result for the transisiton rule at the heelstrike collision as Garcia et. al (1998) in “The Simplest Walking Model: Stability, Complexity, and Scaling” obained in eqn. (4).