Krugman's 'who's zoomin[1] who' traffic paradox

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(If you are reading on a computer, clicking on blue text links you to a reference.)

OK, it's not[3] Krugman's. But that's where I saw it.

On July 26, 2013 Paul Krugman used his political and economic blog[4] to muse about traffic. This caused a flurry of confused responses which, in the end, left a puzzle that has nothing to do with world affairs, and even not much to real traffic. Here's the interesting puzzle implicit in Krugman's blog.

> **Puzzle**: In stop-and-go traffic on a two-lane-in-each-direction road, where all cars in both lanes move at the same average speed, it seems like you spend more time being passed than passing. Is that really so? And, can you figure this out from the driving pattern of one typical car?

Krugman complained that, even though he was doing no worse than anyone else, it felt unfair that he seemed to spend less time passing than being passed:

> "Consider the following thought experiment: you are driving on a road — let's arbitrarily call it Interstate 91 —and must choose a lane. Traffic is so heavy that you can't really change lanes thereafter. But there are many bad patches along the road; half of the distance can be covered at 60 miles an hour, but the other half only at 15.
>
> "You might imagine that your average speed is halfway between 15 and 60, but a little thought shows that this isnt true: your average speed is only 24 miles an hour. Also, the lanes aren't perfectly correlated: sometimes your lane is going 60 while the next is going 15, sometimes its the reverse. Again, you might think that this means you spend equal amounts of time watching the other lane whiz by and whizzing by yourself, but not so: you spend four times as much time watching the other guys race past.
>
> "And this creates intense frustration and anger, a sense that its grossly unfair that you are in the wrong lane. This sense persists even though (a) you have worked out the analysis above, and realize that in principle the lanes are equally good or bad and (b) you have in fact been playing leapfrog with the same Boltbus the whole way, so that you know that in fact neither lane is better. No matter; you are angry, frazzled, and late for your family event ..."

As a moral tale this is right in line with one of Krugman's themes: when calculations and reality conspire against your intuition you just have to take it.

This blog post led to about 80 comments, some of which, like one of mine, disagreed with Krugman's analysis. These, in turn, provoked an impatient lesson from Krugman 2 days later, explaining his math and ending with 'You're welcome'.

In the 200+ responses to this second post I didn't notice any sincere 'Thanks'. People either already got it already, didn't get it but trusted what they read, or still didn't get it. Some responses again critiqued Krugman's math. And some, echoing Krugman, ridiculed the doubters. One wrote, "May I suggest a Nobel in Chutzpah to those who challenged PK's math.....". Another wrote,

> "Surprised so few readers are aware of DeLong's Laws:

1. Paul Krugman is always right.
2. If your analysis leads you to a different conclusion than PK, see rule #1."

And, indeed, experiments on **drivers**[3] watching videos show that drivers think the other lane is going faster even when it is going a bit slower. Why? Because, apparently, in real traffic more time is spent being passed than passing.

The Krugman blog and responses are reminiscent of the **Monty Hall problem**[5]. Marilyn vos Savant, a Parade columnist, posed a game-show probability problem and solution. Lots of people who should have known better ridiculed her. Persi Diaconis, who knows probability well, **said**[6]

> "I can't remember what my first reaction to it was, because I've known about it for so many years. I'm one of the many people who have written papers about it. But I do know that my first reaction has been wrong time after time on similar problems. Our brains are just not wired to do probability problems very well, so I'm not surprised there were mistakes."

Krugman's traffic problem has more to do with reciprocals than with probability. And, in that realm, it's a bit trickier than Rabbi Jeff Glickman's miles-per-gallon **puzzle**[7] about reciprocals:

> *Which saves more gas, changing a 10 miles per gallon car to 11, or changing a 100 mpg car to 200?*

Answer: 10 to 11 saves more gas (assuming both cars go the same distance), the opposite of most people's first thought. What Diaconis said about people being bad at probabilities seems to be true for reciprocals too (**mpg *vs* liters per 100 km**[8]). Even maybe for **Krugman** sometimes.

The Krugman puzzle

As with the Monty Hall puzzle, part of the problem is in the details of the problem statement. So I'll rephrase it a bit.

> A long road has two lanes going the same direction. Cars travel in both lanes and don't change lanes. At every instant each car is either going
>
> - fast, say $v_f = 60$ mph (miles per hour, or just 60 for short); or
> - slow, say $v_s = 15$
>
> and nothing inbetween. The two lanes are statistically the same and may or may not have correlated motions. At the end of a long trip every car has driven
>
> - half the total distance at 15 and
> - half at 60.

Depending on how you read the blog, you may or may not see this as Krugman's original problem statement. Here are three questions Krugman answered in his first blog:

1. What is your average trip speed v_{ave}?
2. What fraction of the time do you drive fast?
3. What is the passing to passed ratio:

$$r = \frac{\text{time you spend passing cars in the other lane}}{\text{time you spend being passed by cars in the other lane}} \quad ?$$

The answers given by Krugman are:

1. Average speed? Implicit in the word average is the 'time-weighted' average, or total distance divided by total time so:

$$v_{ave} = \frac{\int v\,dt}{\int dt} = \frac{\int dx}{\int \frac{1}{v}dx} = \frac{\text{total distance}}{\text{total time}} = \frac{1}{(1/v_s + 1/v_f)/2} = \text{, say, } \frac{2}{1/15 + 1/60} = 24.$$

This differs from the distance-weighted average speed, the sloppy quick answer, of

$$v_{\text{dist ave}} = \frac{\int v\,dx}{\int dx} = \frac{v_s + v_f}{2} = \text{, say, } \frac{15 + 60}{2} = 37.5.$$

2. How much time going fast compared to slow? The ratio of time going fast to time going slow is

$$\frac{\text{fast driving time}}{\text{flow driving time}} = \frac{t_f}{t_s} = \frac{d_f/v_f}{d_s/v_s} = \frac{v_s}{v_f} = \text{, say, } 1/4.$$

These might be unintuitive at first, but no-one reading Krugman's column had any argument with this much. Some people were, however, confused about the discussion and thus vehemently defended these two answers, against attack from no-one.

The substantial (if you could call anything in this whole discussion substantial) confusions were about the 3rd question which Krugman answered

3. The passing to passed ratio is $r = 1/4$: "you spend four times as much time watching the other guys race past".

The first responders

Perhaps typical of the initial doubters was, say, me.

I thought each car spends 4/5 of the time driving slowly. Same with the cars in the other lane but with a different pattern. Because 1/5 of the time the other lane is driving fast, they are driving fast 1/5 of the time that I am driving slowly. That is, I am being passed 4/5×1/5 = 4/25 of the total time. On the other hand, 1/5 fifth of the total time I am driving fast, and of that, 4/5 of the time the other cars are driving slowly. So I am passing 1/5 × 4/5 = 4/25 of the total time. 4/25=4/25 so I spend as much time passing as being passed.

This is still reasonable sounding to me. But it isn't Krugman's answer, that you are passing only a quarter as much time as you are being passed. Hence the kerfuffle. What was Krugman's arithmetic?

Kruman's explanation

Krugman's reasoning, at least his instructive version, is in his 2nd blog post (July 28, 2013).

Life In The Slow Lane (Trivial)

"I see that quite a few readers think the math in my last Friday Night Music post was wrong. Since whining about traffic is very important, I guess I need to set everyone straight.

"So, let's be concrete (which is better than asphalt). Imagine that our journey is 120 miles. Half of the 120 miles is good distance, which you can cover at 60 miles an hour; the other half

is bad distance, which you cover at only 15 miles an hour. In the figure below, I assume that good and bad stretches — indicated by blue and red respectively — come in 30-mile blocks, and are perfectly uncorrelated:
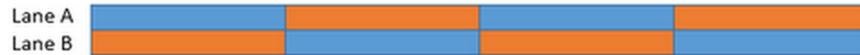
"If you think about if for a minute, youll see the following:

"1. It will take just 1/2 hour to traverse each blue block, but it will take 2 hours to traverse each red block. In total, therefore, although half the distance is red, half blue, you will spend only 1 hour in the blue and 4 hours in the red.
"2. Whenever you are in a blue block, you will be whizzing past the other lane. Whenever you are in a red block, the other lane will be whizzing past you.
"3. Therefore, you will spend 4 times as much time watching the other lane whiz past as you will being the one doing the whizzing.

"You're welcome."

In his first traffic post he had thought he was winking at us in our abilities to, with him, see past our bad initial intuitions. Given the (what he saw, mostly correctly, as inept) reactions to the first blog, he then had to explain it all to us, impatiently, as if we were all conservatives making primitive economics errors.

Never mind that some responses picked on Krugman's sloppy uses of the phrases, 'perfectly uncorrelated' (meaning perfectly negatively correlated) and 'red' (meaning orange); it's really not about probabilities or color wheels. The main arguments against Krugman came in two flavors: 1) bad reasoning (the majority, including some of mine), and 2) different implicit assumptions (which contradict Krugman's).

Both the "4/25=4/25" reasoning and Krugman's 2nd blog are reasonable. The difference is in implicit assumptions, that were not given in my re-statement of the question.

What are 'bad patches along the road'?

Whether Krugman originally meant that the road is literally bad in certain spots, or not, his reasoning depends on half the length of each lane is in construction or something. Independent of the traffic density or dynamics, every car goes slow in bad patches and fast in good patches. Maybe Rte 91 is so bad in some places that Krugman would drive 15 there no matter how fast was the car in front of him. Maybe there were construction speed-limit signs. But then why would he then used the word "traffic" in his lesson? If it's really about the roads, he could have said "You spend four times as much time on gravel, with clean pavement next to you, than the other way around." So maybe it was a mixture: the high traffic density had jamming sections that were spatially pinned by variations in pavement quality. The tone of Krugman's 2nd post makes it is clear that he doesn't recognize that there is even a question about this interpretation.

What else could 'a bad patch along the road' mean? When I get through a stop-and-go traffic jam I might say, 'there were some bad patches along the road' meaning that there were stretches where the going was slow due to bad things, like slow traffic in front of my car. These happened at certain places along the road, but are not attributes of the road, per se, but rather of the whole

traffic congestion and dynamics. Thus I might think of the bad patches as things which would vary in space *and* time.

If you think that "bad stretches along the road" unambiguously means places *where* the traffic was slow, then Krugman's answer of $r = 1/4$ is simply right. If you think it means times when the traffic was slow, then the other answer, $r = 1$, is right. And if you think it means you drove slow in some places and times, and other people did too, and that's all you know, then, as we will show, any answer is possible.

I might think it possible that Krugman thought his locked-in-concrete reasoning applied to more general traffic jams, but for DeLong's laws, to which I basically subscribe.

If you think that my restatement of Krugman's puzzle is what Krugman meant, then this is a demonstration that the problem is ill-defined: knowing the pattern of fast and slow distances of each car is not enough to determine the passing ratio, you need to know how the motions of the cars are correlated with each other. Or, if you think of Krugman's puzzle as assuming a spatially-fixed pattern at the get-go, then the rest of this paper is about a generalization of that puzzle to more general traffic patterns.

First let's do a slightly more thorough analysis of Krugman's model.

Slight generalization of Krugman's 2nd blog post

Start by using Krugman's spatially-fixed model I. But let's generalize by dropping his detailed perfectly-negatively-correlated cartoon of the lanes. Instead let's say the road pattern looks random in both lanes, something like this:
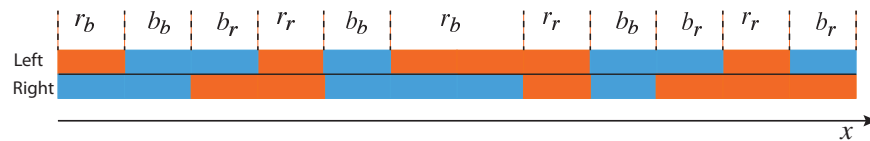
In each lane half the distance is red. The patterns in each lane are otherwise arbitrary. x_y is a region where the left lane is color x, the right lane y. As pointed out in a few blog responses, simple algebra shows that Krugman's result holds for all (fixed spatial) patterns.

Why? The total length of red on the left is $1/2$ and is made up of parts with red on the right, r_r, and parts with blue, r_b. Similarly with the total length of red on the right. So

$$r_b + r_r = 1/2 \quad \text{and} \quad b_r + r_r = 1/2. \quad \text{Subtracting} \quad \Rightarrow \quad r_b = b_r.$$

That is, no matter what the layout there is as much blue to the left of red as there is blue to the right of red. You can also see this without algebra by starting with Krugman's regular pattern, then switch any pair of blocks in one lane and see that the equality is never broken.

So you spend equal distances passing as being passed. But you cover them in unequal times, and the passing ratio is

$$r = \frac{\text{time you spend passing cars to your right}}{\text{time you spend being passed by cars on your right}} = \frac{d_{\text{passing}}/v_f}{d_{\text{being passed}}/v_s} = \frac{v_s}{v_r} = \text{, say,} \frac{1}{4}.$$

You might spend some time, maybe a lot of time, at the same speed as your neighbor. That doesn't count because you're not passing or being passed. The answer holds for any pattern but for the degenerate one of total left-right matching which yields $r = 0/0$. Despite my early doubts and the doubts of others, Krugman's answer of $1/4$ is solid. At least for his model where speed varies only with position along the road.

And the answer is a bit more general still. What if the fraction of distance going slow wasn't 50% but was some other number, say, 17%. Then the reasoning above still shows that you still spend equal distance passing as being passed and the passing ratio is still $r = v_s/v_f$ (=1/4 with Krugman's speeds).

So Krugman's answer, $r = 1/4$, holds for any *spatially fixed* pattern of fast and slow, so long as the fraction of fast and slow distances are the same in the two lanes.

How to get the other answer, in more detail.

None of Krugman's contrary first responders noticed, at least not at first, that their answer ($r = 1$) was in some sense no more right or wrong than Krugman's ($r = 1/4$). It just used, implicitly, different assumptions. If Krugman's answer was wrong by virtue of not acknowledging assumptions (which it might not have been, depending on how your read 'bad patches') then theirs (mine included) was just as wrong. Let's make the implicit explicit.

Let's allow the pattern of fast and slow to vary with time rather than place. This does not change my restricted problem statement. But it does violate Krugman's implicit locked-in-the-concrete assumptions.

The Temporally Synchronous model II. The whole left lane goes fast and slow together, 80% of the time going slowly. Same with the right, but in a different temporal pattern. Any different temporal pattern, correlated, uncorrelated or negatively correlated. This was presumably what was in the backs of the minds of the people (like me) who disagreed with Krugman. Here's an example.
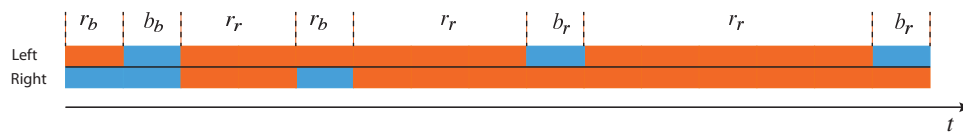
The axis is now time t, not space x. We still get the basic results: the average speed is still 24, and you still spend 80% of your time going 15 through half the distance.

Both the left and right lanes spend 80% of their time going slow, so:

$$r_b + r_r = .8 \qquad \text{and} \qquad b_r + r_r = .8. \quad \text{Subtracting} \quad \Rightarrow \quad r_b = b_r.$$

That is, you spend equal *time* going fast next to slow as the other way around: You can also get that $r_b = b_r$ by drawing pictures and adding and moving colored blocks around. Thus, for this model

$$r = \frac{\text{time you spend passing cars in the other lane}}{\text{time you spend being passed by cars in the other lane}} = 1$$

which is not the same as Krugman's answer of 1/4. And, for this temporally-synchronous traffic model where the whole long right lane speeds up and slows down together, as does the left lane, you get the symmetry that most people, at least at first, naively expect.

Slight further generalization. You don't need quite such an absurd model to get $r = 1$. If you divide the whole length of road into chunks, and for each chunk apply temporal patterns to the right and left lanes, you still get $r = 1$. And with chunks like that, you don't have any long range distance correlations that are so obviously implausible.

Just as for model I, the answer for model II doesn't depend on what fraction of the time you drive fast, so long as it is the same in both lanes. Unlike for model I, the model II answer doesn't depend on the speeds themselves, you still spend as much time passing as being passed. So the passing ratio is $r = 1$, independent of the two speeds or of the average speed.

A more general model

Stop and go traffic might generally have patches of fast and slow that vary in time and space in a complicated way. For example the pattern of fast and slow in one lane might look like this.
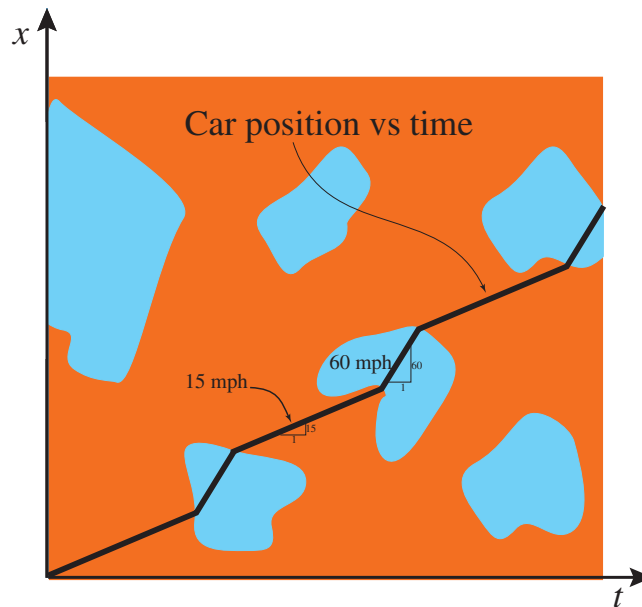
Patches of fast and slow in right lane

The cars going fast motion are not fixed at certain places or at certain times, but occur as regions in space-time. The path of your car car is shown, position x vs t. The blobs need to be shaped and sized so that the average transit speed is, say, 24. And that has to work for cars starting at different times so the structure needs to be fine compared to the time and distance of the whole trip. Assume the left lane has a similar coloring, statistically, but not totally identical (else the problem is boring because you would then always drive next to the same car).

You, as someone driving through this turbulence, don't know its source. You might not even be able to tell if it has a time-varying nature or not. On some stretches of road you go fast and on others you go slow and you don't know what is going on with the cars way ahead of or way behind you or the one's who visit that spot on the road before you or after you. Given this possible patchiness in speed, but with the restriction that the average speed is 24, can we say anything about how much time you are passing compared to being passed?

The answer: no. Why? Because we can get any answer by looking at a special subset of traffic patterns. First lets revisit models I and II.

Models I and II as strips

First look at models I and II on the $t - x$ plane for the left lane. To keep things simple use a regular pattern. The jagged curve below is the position of your car vs time. Note that the car is going fast and slow in the same patches of road, and at the same times, for both models. If you only look at your position vs time you can't tell which of these models applies. From your point of view both have the same 'bad patches along the road'. What is different between the models? What the other cars are doing. Not what each one is doing on average, because that is also the same between the two models, but what the other cars are doing at each position and time.

Your car and the traffic in your lane

Colors show the speeds of other cars in your lane
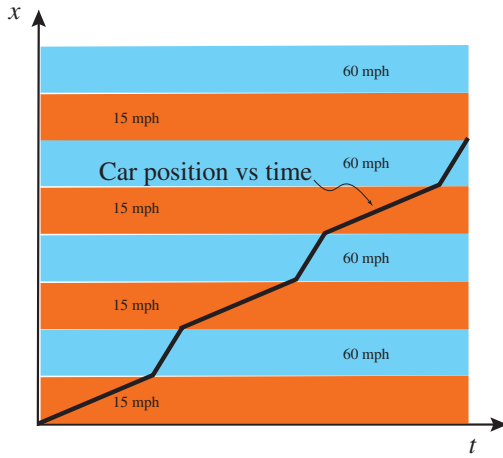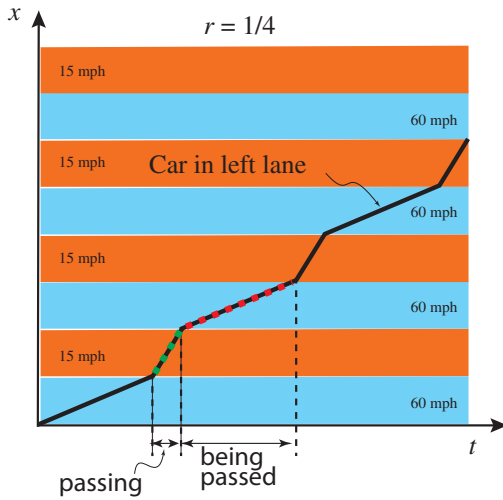

Krugman Model I in space time
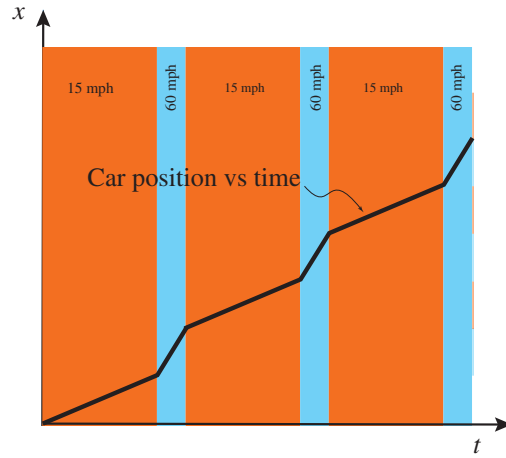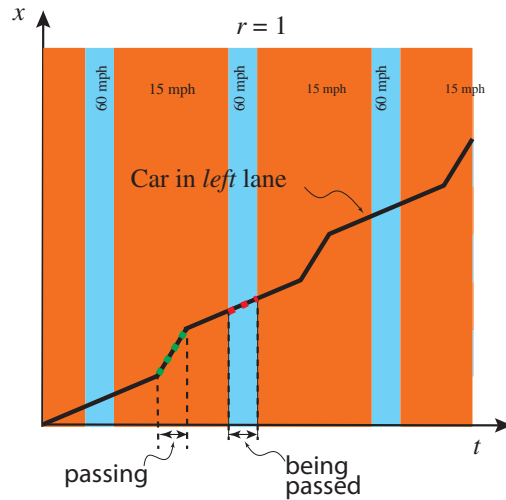

Synchronous Model II in space time

To figure out when you are passing and when you are being passed the next graphs show what the right lane is doing as you drive in the left. Following Krugman, in the figures below we just offset the left lane pattern and put it on the right lane.

Your car and the other lane's traffic

Colors show speeds in *right* lane, the same as the left lane (above) but shifted

Krugman Model I in space time

Synchronous Model II in space time

The pictures above show that for the spatially-locked model I the time spent being passed is bigger than the time spent passing ($r = 1/4$). And for the synchronous model II the time spent passing matches the time spent being passed ($r = 1$). As already shown, for model I the answer depends only on the speeds ($r = v_s/v_f$) associated with red and blue stripes, not on their exact spacing nor on the left lane and right lane having the same spacing pattern. But it's easier to picture with a regular pattern and simple offset. For model II, not even the speeds matter.
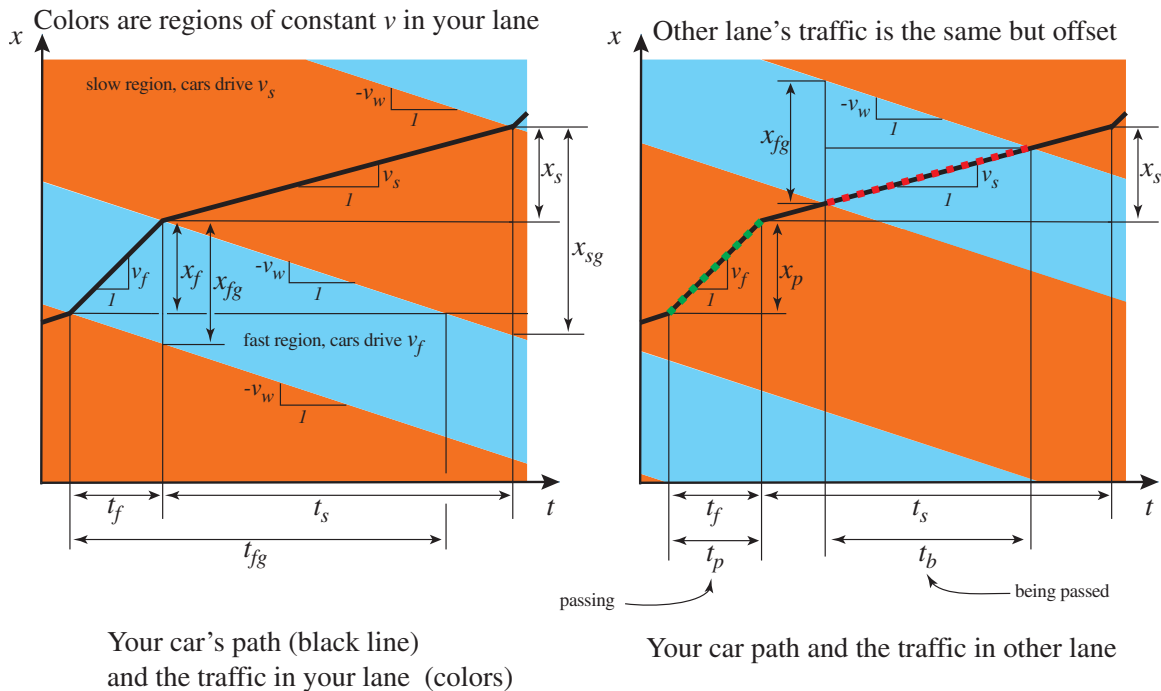
Travelling waves

One way to find the range of possible values of r, given the fast and slow car speeds, and the average speed of each car, is to explore more general traffic patterns. What is the simplest generalization that includes the spatial model I and temporal model II, above, as special cases? Strips of fast and

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slow, but now think of them as tipped in space-time. The slope of the strip is in some sense a wave speed v_w because the position of a region of, say, fast cars, progresses at this speed. In models I and II above we had wave speeds of 0 and ∞, respectively. The construction looks like this.

i. Draw the car position vs time that pleases you. A jagged line with just two slopes, v_s and v_f, distributed so that the average speed is a v_{ave} that pleases you. This jagged line could come from another traffic model, like I or II above.

ii. Pick a wave speed v_w that you like. The strips, regions of constant v, have slope of v_w.

iii. At every place the car changes speed, draw a line with slope v_w through that bend in the graph.

iv. Color the regions between these parallel lines according to the speed of the car in that region, red for slow regions, blue for fast.

v. Redraw the car path on a new plot and superpose the colors of the other plot, displaced. This shows your car position vs time, with the color showing the other lane's speed.

vi. Find where you are passing as a steep (fast) part of your curve over a red (slow) region. Find where you are being passed as a flatter (slow) part of your curve over a blue (fast) region.

Here is the same driving pattern we've been using , but blown up. Unless you are into the math details (Appeendix II) ignore the little arrows, lines and letters with subscripts. This example uses a backwards going wave ($v_w < 0$).

Generic traffic wave

Your car's path (black line)
and the traffic in your lane (colors)

Your car path and the traffic in other lane

By this means, one can cook up a one parameter (v_w) family of traffic-speed distributions all of which give the same speeds for one given car, at the same places and at the same times. In Appendix II we use this picture to find the passing ratio r in terms of the slow car speed v_s, the fast car speed v_s and the wave speed v_w:

$$r = \frac{v_s/v_f - v_w/v_f}{1 - v_w/v_f}.$$

Some examples

Let's look at some special cases.

Wave speed is zero (Krugman's model): $v_w = 0 \;\Rightarrow\; r = v_s/v_f$, as Krugman got with his block diagram, that's $r = 1/4$ with Krugman's numbers. This holds up if he had gone 17% of the trip at 15mph and 83% at 60 mph (Krugman's 50-50 explanation would have to be a bit to explain this, but his answer still works).

Wave speed is infinity: $v_w = \infty \;\Rightarrow\; r = 1$. This is the synchronous model II and answer.

Wave speed is just over the fast speed: $v_w = v_f(1 + \epsilon) \;\Rightarrow\; r = (1 - v_s/v_f)/\epsilon$ (to leading order). That's big when ϵ is small. As big you want. The picture also shows that if the waves are just faster than 60 mph then the passing ratio is then huge.

$$\text{Waves just faster than } 60: \quad v_w = v_f(1 + \varepsilon)$$

Your car and your lane's traffic
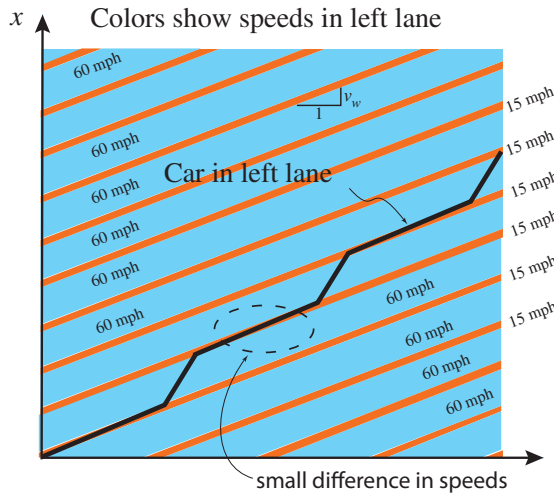
Your car and the *other* lane's traffic
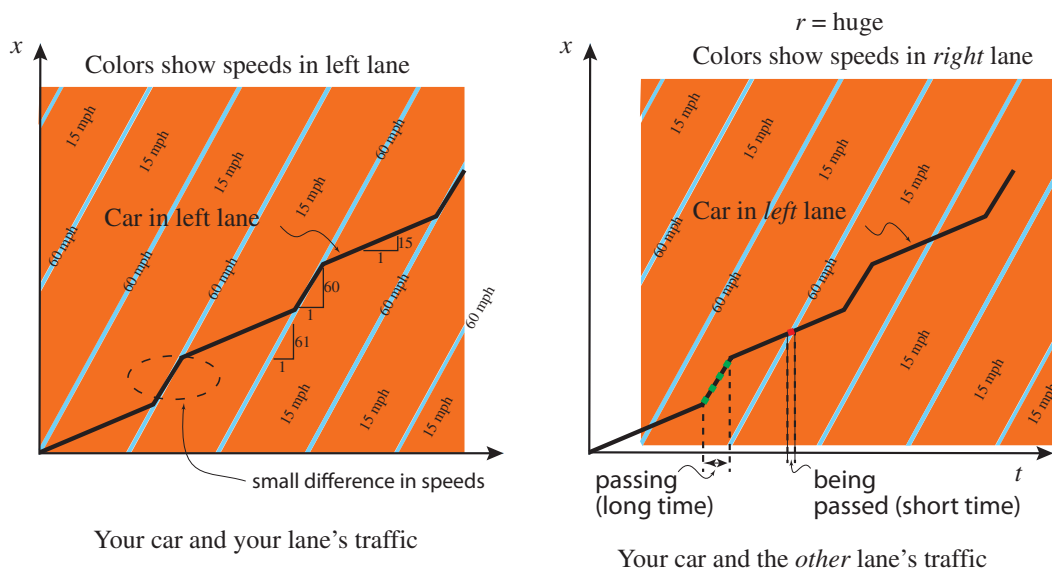
In this wave-speed $= 60^+$ case you still spend 80% of your time going slowly. Yet you spend *much* more time passing than you spend being passed. Arbitrarily much more. With the right tempero-spatial distribution of car speeds, your passing ratio can approach infinity, even as you and all of the other cars drive just as Krugman described.

> Even if every car goes half the distance at 15 and half at 60, each car could spend 10 times *more* time passing than being passed. How? If the spatial-temporal map of speeds looked like the picture above.
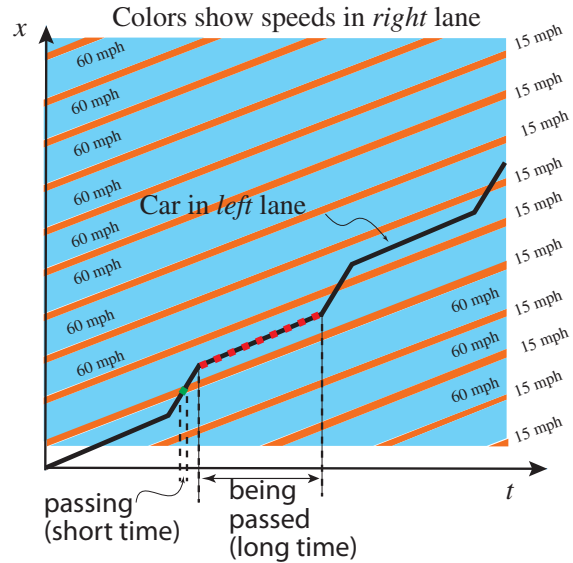
Wave speed is just under the slow speed: $v_w = v_s(1-\epsilon) \;\Rightarrow\; r = \epsilon\big((v_s/v_f)/(1 - v_x/v_f)\big)$ to leading order in ϵ, which is small, arbitrarily small (more on this below). For example, say the wave speed v_w is just under 15.

Waves just slower than 15: $v_w = v_s(1 - \epsilon)$

Colors show speeds in left lane

Car in left lane

small difference in speeds

Your car and *your* lane's traffic

$r =$ teany

Colors show speeds in *right* lane

Car in *left* lane

passing (short time)

being passed (long time)

Your car and the *other* lane's traffic

In the left picture you see the car driving slowly (small slope) in the slow red patches and driving quickly in the fast blue patches. At one point in space a small fraction of the time are cars going slowly. And at one point in time, a small spatial region is going slowly. Yet your car is going slowly 4/5 of the time.

In the right picture your left-lane car's position vs time is superposed on the velocities of the cars in the *right* lane. There you see that the time spent passing is teeny and the time spent being passed is huge. Thus r, the passing ratio is teeny, arbitrarily small (close to zero) if the stripes are arbitrarily close to 15 mph in wave speed.

> Even if every car goes half the distance at 15 and half at 60, each car could spend 10 times *less* time passing than being passed, much less than Krugman's $r = 1/4$.

The forbidden wave speeds: $v_s < v_w < v_f \Rightarrow r < 0$. The model doesn't work in this range. Drawing strips also shows that wave speeds between 15 and 60 are problematic. For $15 < v_w < 60$ the wave can't pass through the cars and the cars can't pass through the waves.

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Forbidden wave speeds: $v_s \leq v_w \leq v_f$
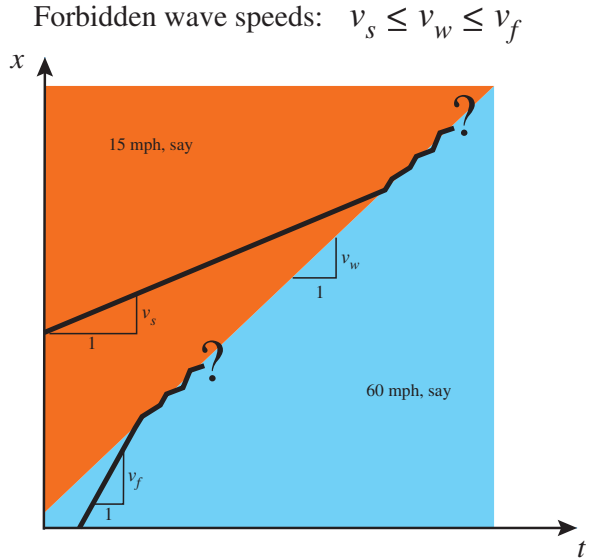
Car gets trapped at front edge of wave

The cars all get trapped at the leading edge of the fast (blue) region where they are stuck vibrating back and forth between 15 and 60 mph. In ODEs, I think this is called 'a sliding mode solution'. So r has no simple definition for wave speeds between $v_s \leq v_w \leq v_f$.

Note that if the slow speed v_s is very slow then the waves have to be even slower or backwards. Or they have to be unreasonably fast, $v_w > v_f$ (Maybe the special theory of traffic relativity forbids the wave velocity from exceeding the car speed limit). That is, for actual stop and go traffic the waves *have* to go backwards or really fast, that's a constraint independent of human psychology.

Wave speed is negative: $v_w < 0 \quad \Rightarrow \quad v_s/v_f < r < 1$, showing that, for the 'realistic models', the one's with negative wave speeds, the Krugman spatially fixed answer and the synchronous-lane answer are bounding cases.

Literally stop and go: $v_s = 0 \quad \Rightarrow \quad r = -(v_w/v_f)/(1-v_w/v_f) < 0$. Only backwards waves are possible.

Back to 'reality'

Of course serious people study these things. Not the Very Serious People Krugman ridicules (played in this moral tale by the people who criticized Krugman's math), but people who are really serious about traffic. And they note traveling waves that have **negative**[9] velocities $v_w < 0$ (Figs 19 and 20 **here**[10] are amazing). It is natural to think that this is because drivers react, with delay, to the drivers in front of them. But it's more than that. Our basic reasoning already shows that any positive wave speed has to be very slow or unreasonably fast, independent of driver psychology. In any case, more sophisticated models, and real traffic data, limit us to negative wave speeds.

With pictures like those above, but with negative slopes, you can see that the plausible range for values of r is bounded by the Krugman model I (horizontal strips, $v_w = 0$) result $r = 1/4$ and the synchronous model II (vertical strips, $v_w = \infty$) result of of $r = 1$ (see Appendix II).

According to Wikipedia, standing waves ($v_w = 0$) are also observed in traffic dynamics. This would then, again, correspond to Krugman's locked-in-concrete solution, but not because of the quality of the concrete. Putting this altogether we have:

Wave speed of traffic pattern = slope of strips = v_w
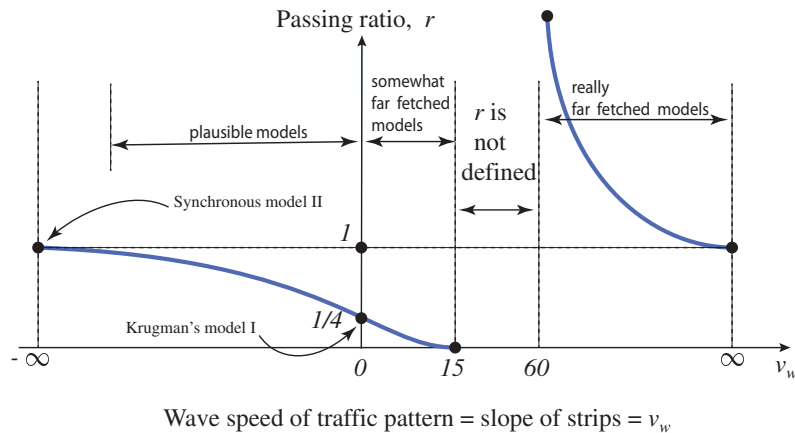
Conclusion

For traffic patterns that are fixed in space, Krugman said (and partially showed) that one spends less time passing than being passed. For other traffic patterns, all of which yield the same driving pattern for each car as in Krugman's example, the ratio of passing to being passed can range from 0 to ∞ depending on the spatial and temporal nature of the traffic pattern. That is, just knowing where one was going fast and slow, and even knowing that other cars had a similar fast-slow pattern, is not enough to figure out the passing ratio. For 'realistic' cases, ones with backwards going waves of motion, one extreme is Krugman's solution ($r = 1/4$). Another is the symmetric solution of equal time passing as being passed ($r = 1$).

So was Krugman right or wrong? If 'slow patches' means stretches fixed in space he was right, and his simple reasoning gives the right answer. If all one knows is that one's own car drives slowly in some stretches at some times, and that other cars drive equally erratically, but who knows when and where, then his answer is one of many and thus wrong if declared as unique. And then, the initial critics, like me, who might have had just as good a model, but a different answer, were at least just as wrong to think their answer was *the* right one.

Open questions. Obviously there is a world of questions if one wants to study **real traffic**[10]. But just sticking to the logic of this puzzle, here is one question. What are the generalizations of all of the results to more general, non-strip, traffic patterns $v(x, t)$? (See the ODE appendix for more questions).

Appendix I: symmetry, density and differential equations

There are other accounting schemes that are consistent with all of the results above.

- **Symmetry.** Whatever is true about your relation to the other lane is true about an individual in the other lane's relation to your lane. You pass as many cars as pass you. Any average quantity relating you and another individual car in the other lane has to be symmetric. However, some symmetries are not needed for the results above. For example, the densities of the cars can be arbitrary, and arbitrarily different between lanes, so long as the temporo-spatial speed variations are patterned the same way in the two lanes.

  And some symmetries are model dependent. Despite my, say, wrong initial thoughts, it is *not* true, generally, that because I and a stranger have equal passing times, that I spend as much time passing as being passed. And that is the whole point. It's the violation, at least in

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most traffic distributions, of that naive view of symmetry that made the problem interesting to Krugman to start with.

There don't seem to be any fundamental symmetry relations that transcend the symmetry of the patterns. For example, note that for the strip models the answered depended on your driving pattern and the other lanes spatio-temporal distribution. But both models used the same driving pattern. So if one lane was following spatial synchrony, and the other temporal synchrony, all cars in both lanes could be driving slow half the distance and fast the other half and all cars could have average speeds of 24. Yet one lane would have a passing ratio of $r = 1/4$ and the other $r = 1$.

- **Density.** As just mentioned, absolute car density can't be part of this problem or solution because none of the calculations used paid any notice of the irregular pattern by which cars may have entered the highway. Of course, in real traffic, density has an effect on the distribution of slow patches, that's what traffic modeling is usually all about, but here we assume that the distribution of speeds is given.

  With these caveats, what can we say about density? For the strip models, the slopes of the strips (the wave speeds v_w) do determine the change in density as the cars pass the wave front or wave back. For example, for the 14.9 mph wave there is a huge increase in density when a car crashes into a slow patch in front of it. That's why one spends so much time being passed, the fast cars are so sparse it takes a long time for them to get by you. For the 60.1 mph example there is a huge density increase when the fast patch shovels up the cars from behind.

  Here's the calculation: conservation of cars passing through the wave front is

  $$\dot{m} = \dot{m} \quad \Rightarrow \quad (v_s - v_w)\rho_f = (v_s - v_w)\rho_s \quad \Rightarrow \quad \frac{\rho_{\text{fast}}}{\rho_{\text{slow}}} = \frac{v_s/v_f - v_w/v_f}{1 - v_w/v_f}.$$

  In fluid mechanics this is the 1D continuity equation for a shock wave (Eqn 12 **here**[11]). It only applies to the density, assumed finite everywhere, on the two sides of the shock. It does not relate the average densities in the fast and slow regions unless they are assumed, and this assumption can't be derived from conservation-of-cars, that the densities are constants in these regions.

  Two cute things:

  1) The density ratio is the same as the passing ratio:

  $$\text{density ratio}\left(\frac{\rho_{\text{fast}}}{\rho_{\text{slow}}}\right) = \text{passing ratio}\left(r = \frac{t_{\text{passing}}}{t_{\text{being passed}}}\right)$$

  Why this matching? Look at flows where the densities in the fast and slow regions *are* the same constants (ρ_{fast} and ρ_{slow}) in the two lanes. When you are passing you pass $(v_{\text{fast}} - v_{\text{slow}})\rho_{\text{slow}}$ cars per unit time and thus $t_{\text{fast}}(v_{\text{fast}} - v_{\text{slow}})\rho_{\text{slow}}$ cars total. Likewise when you are being passed you are passed by $(v_{\text{fast}} - v_{\text{slow}})\rho_{\text{fast}}$ cars per unit time and thus $t_{\text{slow}}(v_{\text{fast}} - v_{\text{slow}})\rho_{\text{fast}}$ cars total. By symmetry, you pass as many cars as pass you. Equating, you get the identity above. Again, note that this relation between density and passing ratios depends on density assumptions that the rest of the paper does not depend on. But it helps intuition. When fast cars go by with a low density, it takes a long time for a group of them to pass.

  For the extreme case where the wave speed v_w is just faster than the fast driving speed v_s the density is extremely high in the fast lane associated with, as explained before, an

extremely high passing ratio. And when $v_w = v_s(1 - \epsilon)$, r is small and so is the density ρ_s in the slow lane.

2) Just like the passing ratio, if the wave speed is between the slow and fast car speeds, the calculated density ratio is also nonsense (a negative density) . Recall that this is the speed range where, if it existed, cars would pile up on top of each other at the front of the fast wave.

- **Differential equations.** This model is only part of a more general model that would include equations for, say, the relation between speed and density. And there are particle models too. It's a whole serious field. See the 75 page paper by Dirk Helbing (2000^+ citations in Google Scholar) if you want to be impressed. Helbing dismisses the whole problem here with the phrase "There are some effects that are not at all or only partly represented by the physical traffic models ... (vii) motivations for lane changing (Redelmeier and Tibshirani, 1999)...". For the models here we could say we are studying properties of the solutions of

$$dx/dt = f(t, x)$$

where f is the traffic speed and is a given function that always evaluates to 15 or 60 and has to be such that the average $\dot{x}_{ave}$ is 24. **Extra credit** (harder than anything here): Write that as a constraint on the shapes and sizes of the regions of high-v in the $x - t$ plane. More generally, assume you are given a traffic speed function $v(x, t)$. For simplicity assume it is periodic in x and t. What is a reasonable definition of average speed v_{ave}? For any such definition of v_{ave}, can you evaluate v_{ave} without the general solution of the ODE $\dot{x} = v(x, t)$?

Appendix II: Calculations

For arbitrary wave speed v_w and arbitrary distances, x_f and x_s, of driving fast and slow, v_f and v_s, one can cook up a formula for the passing ratio using the picture with slopes on page 9. I know it looks complicated but that's how it looks like to me. What's hard is that, for example, the length on the ground x_{fg} of the fast-moving patch is not the distance x_f people go in that patch. Same with the other key distances and times. We want the passing ratio

$$r = (\text{time passing})/(\text{time being passed}) = t_p/t_b.$$

The time spent passing is the time spent driving fast (at least in this picture)

$$t_p = t_f = x_f/v_f.$$

The time being passed obeys (look at the triangles in the top right of the right drawing),

$$t_b v_s - t_b v_w = x_{fg} \quad \Rightarrow \quad t_b = x_{fg}/(v_s - v_w),$$

where x_{fg} is the length of cars going fast at one instant in time (the height of the orange region). From the left picture,

$$x_{fg} = x_f - t_f v_w, \quad \text{where} \quad t_f = x_f/v_f.$$

Backsubstituting, we get

$$\boxed{r = \frac{v_s/v_f - v_w/v_f}{1 - v_w/v_f}.}$$

If god is good, the formula applies even if the picture changes in detail (e.g., with different ratios of fast to slow, with positive wave speeds, with less offset in the pattern between the lanes causing overlaps). That is, the answer really only depends on the fraction of blue being the same in both pictures, not on the average color (or average speed) or on any details of the patterning.

Appendix III: Technical caveats.

- This whole argument is about averages for a long road. For it to be accurate you need lots of speed switches while a car traverses the length. That is, the typical switching time needs to be much less than the total trip time. Yet the transitions between 15 and 60 also have to occur over a short enough distance, relative to the patch lengths, to be negligible.

- This is about a continuous stream of cars. We imagine the speed of a lane, when no car is next to you, is, say, the average of the speeds of the cars in front of and behind you. And you are being passed if that speed is greater than yours.

- As blog commenter 'pawelAZ' from Tucson commented: if the cars only drive at two speeds, if all of the cars are the same physical size, and if you only count passing time to be time when the physical cars overlap then, for for Krugman's traffic pattern (and all others with the same average speed on both lanes), the passing time is equal to the time being passed (Why? the same number of cars pass as get passed, and at the same pair of speeds, thus the same durations).

Thanks to: Steve Strogatz for sharing interest in this problem and for the Redelmeier reference; to Arend Schwab for help seeing the $t - x$ plane; and to Saskya van Nouhuys for pointing out that the word "traffic" generally concerns things other than just road conditions; to Les Schaffer for pointing me to Helbing; to Helbing for pointing me to Treiber; to Don Lewis for pointing me to Aretha Franklin; to Noah Cowan for help clarifying the density calculation; and to Jim Papadopoulos for editorial comments.

Notes/references

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b) Helbing pointed me to Martin Treiber who wrote this new book: *Traffic Flow Dynamics: Data, Models and Simulation*.
c) A responder to this news article: http://www.zeit.de/wissen/2013-08/Staumathematik pointed

to this paper: Lighthill and Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads*, Proc. Roy. Soc. A, vol 229, # 1178, pgs 317-345, 1955 (cited over 3000 times).

11. Wikipedia on shock waves, eqn 12 therein. http://en.wikipedia.org/wiki/Rankine-Hugoniot˙equation.